APPROACHES FOR SOLVING BIMATRIX INFORMATIONAL EXTENDED GAMES

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Different ways of solving bimatrix games in complete and perfect information (or over the set of informational extended strategies) are studied in the present paper. The Nash and Bayes-Nash solutions for informational extended games are discussed.

Keywords: non cooperative game, payoff function, set of strategies,informational extended games, Bayes-Nash equilibrium.

MODALITAŢI DE SOLUŢIONARE A JOCURILOR BIMATRICEALE INFORMAŢIONAL EXTINSE

În acest articol sunt analizate diferite moduri de soluţionare a jocurilor bimatriceale în informaţie completă şi perfectă. Informaţia perfectă permite jucătorilor să utilizeze strategii informaţional extinse. Se analizează soluţii de tip Nash şi Bayes-Nash pentru jocuri în strategii informaţional extinse.

Cuvinte cheie: jocuri noncooperatiste, functii de utilitate, multime de strategii, joc informational extins, echilibru de tip Bayes-Nash.

1 Bimatrix informational extended games

We consider the informational non extended bimatrix game in the strategic form

$$\Gamma = \langle I, J, A, B \rangle, \tag{1.1}$$

where $I = \{1, 2, .., n\}$ is the line index set (the set of strategies of the player 1), $J = \{1, 2, .., m\}$ is the column index set (the set of strategies of the player 2) and $A = ||a_{ij}||_{i\in I}^{j\in J}$, $B = ||b_{ij}||_{i\in I}^{j\in J}$ are the payoff matrices of player 1 and player 2, respectively. All players know exactly the payoff matrices and the sets of strategies. Players maximize their payoffs. So the game is in **complete information** (the players know exactly the normal form of the game). We assign to players an additional characteristic which we call *an informational type of the payer* [1,2]. More exactly, we say that the player 1 is of the "2 $\rightarrow$ 1 informational type" and respectively, the player 2 is of the "1 $\rightarrow$ 2 informational type" if the player 1 (respectively player 2) knows the precise value of the strategy which will be chosen by the player 2 (respectively by the player 1). These conditions stipulate that we can analyze the informational extension of the game generated by a double-sided informational flow, denoted by $1 \leftrightarrows 2$. It means the player 1 knows exactly the value of the strategy chosen by the player 2, as well as, simultaneously, the player 2 knows exactly the value of the strategy chosen by the player 1. So the game (1.1) is in **perfect information** over the sets of pure strategies.

The conditions described above stipulate that we can use the set of informational extended strategies of the player 1 (respectively 2) which is the set of the functions $\Theta_1 = \{\theta_1^\alpha : J \rightarrow I\}_{\alpha=1}^{\varkappa_1}$ and, respectively $\Theta_2 = \left\{\theta_2^\beta : I \rightarrow J\right\}_{\beta=1}^{\varkappa_2}$. It is easy to see that $\varkappa_1 = n^m$ and $\varkappa_2 = m^n$. Thus, the informational extended strategies of the player 1 are the functions θ_1^α such that, for all $j \in J$, there is $i_j^\alpha \in I$ such that $\theta_1^\alpha(j) = i_j^\alpha$ and it means the following: the player 1 will choose the line $i_j^\alpha \in I$ if the player 2 will choose the column $j \in J$. Respectively, the informational extended strategies of the player 2 are functions θ_2^β such that, for all $i \in I$, there is $j_i^\beta \in J$ such that $\theta_2^\beta(i) = j_i^\beta$ and it means the following: the player 2 will choose the column $j_i^\beta \in J$ if the player 1 will choose the line $i \in I$.

It should be mentioned that the players do not know the informational type of each other. In other words, the players do not know the informational extended strategies of each others and from this point of view we can consider that the game is in **imperfect information** structure over the sets of the informational extended strategies.

Denote by $Game\left(1\leftrightarrows 2\right)$ the bimatrix game in the informational extended strategies, described above. Remark that the notation $Game\left(1\overset{\inf}{\leftrightarrows}2\right)$ does not represent the normal form. This game is in imperfect information on the set of informational extended strategies, but because we do not know yet the normal form, we can not say if this game is in complete or incomplete information. The quantification of information in the games of type $Game\left(1\overset{\inf}{\leftrightarrows}2\right)$ is done by means of functions which represent informational extended strategies. We can use the following approach to solve the informational extended game $Game\left(1\leftrightarrows 2\right)$.

2 Solving the informational extended game by means of the normal form

Denote by

$$gr\theta_1^{\alpha}=\left\{(i,j):j\in J,i\equiv i_j^{\alpha}=\theta_1^{\alpha}\left(j\right)\right\},\quad gr\theta_2^{\beta}\left\{(i,j):i\in I,j\equiv j_i^{\beta}=\theta_2^{\beta}\left(i\right)\right\}$$

the graphs of the informational extended strategies θ_1^{α} and θ_2^{β}. It is clear that $gr\theta_1^{\alpha}$ (respectively $gr\theta_2^{\beta}$) is the set of the informational non extended strategy profiles generated by the informational extended strategy θ_1^{α} (respectively θ_2^{β}).

According to [3] we can construct the normal form of the informational extended game, denoted by

$$\Gamma\left(1\leftrightarrows 2\right)=\left\langle I,\Theta_1,\Theta_2,A\left(1\leftrightarrows 2\right),B\left(1\leftrightarrows 2\right)\right\rangle \tag{2.1}$$

where the payoff matrices of the player 1 is $A\left(1\leftrightarrows 2\right)=\left\|a_{\alpha\beta}\right\|_{\alpha=\overline{1,\varkappa_1}}^{\beta=\overline{1,\varkappa_2}}$, for

$$a_{\alpha\beta}=\begin{cases}\max\limits_{(i,j)\in\left[gr\theta_1^{\alpha}\bigcap gr\theta_2^{\beta}\right]}a_{ij} & if\ gr\theta_1^{\alpha}\bigcap gr\theta_2^{\beta}\neq\varnothing,\\ -\infty & if\ gr\theta_1\bigcap gr\theta_2=\varnothing,\end{cases} \tag{2.2}$$

and of the player 2 is $B\left(1\leftrightarrows 2\right)=\left\|b_{\alpha\beta}\right\|_{\alpha=\overline{1,\varkappa_1}}^{\beta=\overline{1,\varkappa_2}}$, for

$$b_{\alpha\beta}=\begin{cases}\max\limits_{(i,j)\in\left[gr\theta_1^{\alpha}\bigcap gr\theta_2^{\beta}\right]}b_{ij} & if\ gr\theta_1^{\alpha}\bigcap gr\theta_2^{\beta}\neq\varnothing,\\ -\infty & if\ gr\theta_1\bigcap gr\theta_2=\varnothing.\end{cases} \tag{2.3}$$

The game $\Gamma\left(1\leftrightarrows 2\right)$ is one in **complete** information because the players known exactly theirs payoff matrices and in **imperfect** information because the players do not know what kind of informational extended strategy will be chosen by each others.

Finally, to determine the Nash equilibrium profiles in the bimatrix informational extended game of type $\Gamma\left(1\leftrightarrows 2\right)$ we have to do the following steps:

- construct the sets of the informational extended strategies of the players, i.e. $\Theta_1=\left\{\theta_1^{\alpha}:J\to I\right\}_{\alpha=1}^{\varkappa_1}$ and $\Theta_2=\left\{\theta_2^{\beta}:I\to J\right\}_{\beta=1}^{\varkappa_2}$;
- determine the sets of all non informational extended strategy profiles generated by the informational extended strategies θ_1^{α} and θ_2^{β}, i.e. $gr\theta_1^{\alpha}$, $gr\theta_2^{\beta}$ and intersection $gr\theta_1^{\alpha}\cap gr\theta_2^{\beta}$;

- construct the payoff matrices $A\,(1 \leftrightarrows 2)$ and $B\,(1 \leftrightarrows 2)$ according to the relations (2.2)-(2.3);
- using existent algorithms, to determine the Nash equilibrium profile in the bimatrix game with the matrices $A\,(1 \leftrightarrows 2)$ and $B\,(1 \leftrightarrows 2)$ from (2.2)-(2.3).

In the following example we illustrate the described above methods (see [3]).

Example 2.1 *We construct the normal form of the "$1 \leftrightarrows 2$" informational extended game and determine the Nash equilibrium profiles in the following bimatrix game* $A = \begin{pmatrix} 3 & 5 & 4 \\ 6 & \mathbf{7} & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$.

Solution. The Nash equilibrium profile in the bimatrix game with informational non extended strategies is $(2,1)$. The set of the informational extended strategies of the player 1 is $\Theta_1 = \left\{\theta_1^{\alpha}(j) = i_j^{\alpha}, j = \overline{1,3}, i_j^{\alpha} \in I\right\}_{\alpha=\overline{1,8}}$ where $\theta_1^1(j) = 1\ \forall j = 1,2,3;$ $\theta_1^2(j) = 2\ \forall j = 1,2,3;$ $\theta_1^3(j) = \begin{cases} 1 \text{ if } j = 1,2, \\ 2 \text{ if } j = 3, \end{cases}$ $\theta_1^4(j) = \begin{cases} 1 \text{ if } j = 1,3, \\ 2 \text{ if } j = 2, \end{cases}$ $\theta_1^5(j) = \begin{cases} 1 \text{ if } j = 2,3, \\ 2 \text{ if } j = 1, \end{cases}$ $\theta_1^6(j) = \begin{cases} 1 \text{ if } j = 1, \\ 2 \text{ if } j = 2,3, \end{cases}$ $\theta_1^7(j) = \begin{cases} 1 \text{ if } j = 2, \\ 2 \text{ if } j = 1,3, \end{cases}$ $\theta_1^8(j) = \begin{cases} 1 \text{ if } j = 3, \\ 2 \text{ if } j = 1,2. \end{cases}$ The set of the informational extended strategies of the player 2 is $\Theta_2 = \left\{\theta_2^{\beta}(i) = j_i^{\beta}, i = \overline{1,2}, j_i^{\beta} \in J\right\}_{\beta=\overline{1,9}}$, where $\theta_2^1(i) = 1\ \forall i = 1,2;$ $\theta_2^2(i) = 2\ \forall i = 1,2;$ $\theta_2^3(i) = 3\ \forall i = 1,2;$ $\theta_2^4(i) = \begin{cases} 1 \text{ if } i = 1, \\ 2 \text{ if } i = 2, \end{cases}$ $\theta_2^5(i) = \begin{cases} 1 \text{ if } i = 2, \\ 2 \text{ if } i = 1, \end{cases}$ $\theta_2^6(i) = \begin{cases} 1 \text{ if } i = 1, \\ 3 \text{ if } i = 2, \end{cases}$ $\theta_2^7(i) = \begin{cases} 1 \text{ if } i = 2, \\ 3 \text{ if } i = 1, \end{cases}$ $\theta_2^8(i) = \begin{cases} 2 \text{ if } i = 1, \\ 3 \text{ if } i = 2, \end{cases}$ $\theta_2^9(i) = \begin{cases} 3 \text{ if } i = 1, \\ 2 \text{ if } i = 2. \end{cases}$ To determine payoffs of the players, in the following table we represent the graph intersections $gr\theta_1^{\alpha} \bigcap gr\theta_2^{\beta}$ for $\alpha = \overline{1,8}$ and $\beta = \overline{1,9}$:

$\bigcap$	$gr\theta_2^1$	$gr\theta_2^2$	$gr\theta_2^3$	$gr\theta_2^4$	$gr\theta_2^5$	$gr\theta_2^6$	$gr\theta_2^7$	$gr\theta_2^8$	$gr\theta_2^9$
$gr\theta_1^1$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,1)$	$(1,2)$	$(1,1)$	$(1,3)$	$(1,2)$	$(1,3)$
$gr\theta_1^2$	$(2,1)$	$(2,2)$	$(2,3)$	$(2,2)$	$(2,1)$	$(2,3)$	$(2,1)$	$(2,3)$	$(2,2)$
$gr\theta_1^3$	$(1,1)$	$(1,2)$	$(2,3)$	$(1,1)$	$(1,2)$	$(1,1)$ $(2,3)$	$\varnothing$	$(1,2)$ $(2,3)$	$\varnothing$
$gr\theta_1^4$	$(1,1)$	$(2,2)$	$(1,3)$	$(1,1)$ $(2,2)$	$\varnothing$	$(1,1)$	$(1,3)$	$\varnothing$	$(1,3)$ $(2,2)$
$gr\theta_1^5$	$(2,1)$	$(1,2)$	$(1,3)$	$\varnothing$	$(1,2)$ $(2,1)$	$\varnothing$	$(1,3)$ $(2,1)$	$(1,2)$	$(1,3)$
$gr\theta_1^6$	$(1,1)$	$(2,2)$	$(2,3)$	$(1,1)$ $(2,2)$	$\varnothing$	$(1,1)$ $(2,3)$	$\varnothing$	$(2,3)$	$(2,2)$
$gr\theta_1^7$	$(2,1)$	$(1,2)$	$(2,3)$	$\varnothing$	$(1,2)$ $(2,1)$	$(2,3)$	$(2,1)$	$(1,2)$ $(2,3)$	$\varnothing$
$gr\theta_1^8$	$(2,1)$	$(2,2)$	$(1,3)$	$(2,2)$	$(2,1)$	$\varnothing$	$(1,3)$ $(2,1)$	$\varnothing$	$(1,3)$ $(2,2)$

Using this table and relations (2.2)-(2.3) we can construct the payoff matrices of the player and finally obtain the following bimatrix game with elements type $(a_{\alpha\beta}, b_{\alpha\beta})$:

$$\begin{pmatrix}
(3,0) & (5,\underline{5}) & (\underline{4},1) & (3,0) & (5,\underline{5}) & (\underline{3},0) & (4,1) & (\underline{5},\underline{5}) & (4,1) \\
(\underline{6},\underline{4}) & (\underline{7},3) & (2,2) & (\underline{7},3) & (6,\underline{4}) & (2,2) & (\underline{6},\underline{4}) & (2,2) & (\underline{7},3) \\
(3,0) & (5,\underline{5}) & (2,2) & (3,0) & (5,\underline{5}) & (\underline{3},2) & (-\infty) & (3,\underline{5}) & (-\infty) \\
(3,0) & (\underline{7},\underline{3}) & (\underline{4},1) & (\underline{7},\underline{3}) & (-\infty) & (\underline{3},0) & (4,1) & (-\infty) & (\underline{7},\underline{3}) \\
(\underline{6},4) & (5,\underline{5}) & (\underline{4},1) & (-\infty) & (\underline{7},\underline{5}) & (-\infty) & (\underline{6},4) & (\underline{5},\underline{5}) & (4,1) \\
(3,0) & (\underline{7},\underline{3}) & (2,2) & (\underline{7},\underline{3}) & (-\infty) & (\underline{3},2) & (-\infty) & (2,2) & (\underline{7},\underline{3}) \\
(\underline{6},4) & (\underline{7},\underline{5}) & (\underline{4},2) & (-\infty) & (6,\underline{5}) & (2,2) & (\underline{6},4) & (\underline{5},\underline{5}) & (-\infty) \\
(\underline{6},\underline{4}) & (\underline{7},3) & (\underline{4},1) & (\underline{7},3) & (6,\underline{4}) & (-\infty) & (\underline{6},\underline{4}) & (-\infty) & (\underline{7},3)
\end{pmatrix} \tag{2.4}$$

where $(-\infty)$ denote $(-\infty, -\infty)$.

Below it is shown the accordance between Nash equilibrium profiles in the bimatrix game $\Gamma\left(1 \overset{\inf}{\leftrightarrows} 2\right)$ with matrices from (2.4) and, in the square brackets, profiles in the non informational extended game Γ from (1.1), that are generated by the respectively informational extended strategies:

$NE\left[\Gamma\left(1 \overset{\inf}{\leftrightarrows} 2\right)\right] = \{ (\theta_1^1, \theta_2^8)\ [(1,2)], (\theta_1^2, \theta_2^1)\ [(2,1)], (\theta_1^2, \theta_2^7)\ [(2,1)], (\theta_1^4, \theta_2^2)\ [(2,2)], (\theta_1^4, \theta_2^4)\ [(2,2)],$
$(\theta_1^4, \theta_2^9)\ [(2,2)], (\theta_1^5, \theta_2^5)\ [(2,1)], (\theta_1^5, \theta_2^8)\ [(1,2)], (\theta_1^6, \theta_2^2)\ [(2,2)], (\theta_1^6, \theta_2^4)\ [(2,2)], (\theta_1^7, \theta_2^2)\ [(1,2)],$
$(\theta_1^7, \theta_2^8)\ [(1,2)], (\theta_1^8, \theta_2^1)\ [(2,1)], (\theta_1^8, \theta_2^7)\ [(2,1)] \}$.

3 Solving the informational extended game by means of the informational non extended game

We can describe the informational extended strategies in bimatrix game as follows: to all informational extended strategies θ_1^α, respectively θ_2^β, we put in correspondence a set

$$I^\alpha = \left\{ i_j^\alpha : i_j^\alpha \in I, \forall j = \overline{1,m} \right\} \quad \text{and} \quad J^\beta = \left\{ j_i^\beta : j_i^\beta \in J, \forall i = \overline{1,n} \right\}.$$

So, for all $j \in J$, $\theta_1^\alpha(j) = i_j^\alpha \in I^\alpha$ and for all $i \in I$, $\theta_2^\beta(i) = j_i^\beta \in J^\beta$. Denote by $gr\theta_1^\alpha = \left\{ (j, i_j^\alpha) \equiv \left(i_j^\alpha, j\right) : j \in J, i_j^\alpha \in I^\alpha \right\}$ and $gr\theta_2^\beta = \left\{ (i, j_i^\beta) \equiv \left(j_i^\beta, i\right) : i \in I, j_i^\beta \in J^\beta \right\}$ the sets of the informational non extended strategy profiles of the player 1, respectively 2, generated by the informational extended strategies θ_1^α and θ_2^β, respectively. Denote by
$difI^\alpha = \left\{ i_j^\alpha \in I^\alpha : i_j^\alpha \neq i_k^\alpha, \forall j,k \in J, j \neq k \right\}$ and $difJ^\beta = \left\{ j_i^\beta \in J^\beta : j_i^\beta \neq j_r^\beta \forall i,r \in I, i \neq r \right\}$. Then the set $difI^\alpha$, respectively $difJ^\beta$, is the set of informational non extended strategies of the player 1, respectively 2, generated by the informational extended strategies θ_1^α, respectively θ_2^β. Here $\alpha = \overline{1,n^m}$ and $\beta = \overline{1,m^n}$. Using these notations, we can represent the informational extended strategies θ_1^α, respectively θ_2^β, by the cortege $\mathcal{I}^\alpha = \left(i_1^\alpha, i_2^\alpha, ..., i_j^\alpha, ...i_m^\alpha\right)$ where: $i_j^\alpha \in I^\alpha, \forall j = \overline{1,m}$, respectively $\mathcal{J}^\beta = \left(j_1^\beta, j_2^\beta, ..., j_i^\beta, ...j_n^\beta\right)$, where $j_i^\beta \in J^\beta, \forall i = \overline{1,n}$.

Now, according to [4], we can construct the normal form of the bimatrix game

$$\Gamma\left(\theta_1^\alpha, \theta_2^\beta\right) = \left\langle I, J, A^\alpha, B^\beta \right\rangle, \tag{3.1}$$

that is an informational non extended game generated by the informational extended strategies $\left(\theta_1^\alpha, \theta_2^\beta\right)$. Here $A^\alpha = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}$, $B^\beta = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}$, $i_j^\alpha \in I^\alpha$, $j_i^\beta \in J^\beta$. The game $\Gamma\left(\theta_1^\alpha, \theta_2^\beta\right)$ is played as follows: independently and simultaneously each player $k = \overline{1,2}$ chooses the informational non extended strategy $i \in I$, $j \in J$ after that players 1 and 2 calculate the value of the informational extended strategies $i_j^\alpha = \theta_1^\alpha(j)$ and $j_i^\beta = \theta_2^\beta(i)$, and further each player calculates the payoff values $a_{i_j^\alpha j_i^\beta}$, $b_{i_j^\alpha j_i^\beta}$, and with this the game is finished. It is clear that for all strategy profiles (i,j) in the game $\Gamma = \langle I, J, A, B \rangle$ from (1.1) the following realization $\left(i_j^\alpha = \theta_1^\alpha(j), j_i^\beta = \theta_2^\beta(i)\right)$ in terms of the informational extended strategies will correspond. The game (3.1) is the bimatrix game with complete and imperfect information over the set of informational non extended strategies I, J.

Finally, to determine the Nash equilibrium profiles in the bimatrix game of type $\Gamma\left(\theta_1^\alpha, \theta_2^\beta\right)$ defined in (3.1) we have to the following steps:

- using the "combinatorial algorithm" construct the corteges $\mathcal{I}^\alpha$, $\mathcal{J}^\beta$, for all α, β;

- for all fixed α, β, construct the payoff matrices $A^{\alpha} = ||a_{i_j^{\alpha} j_i^{\beta}}||_{i \in I}^{j \in J}$, $B^{\beta} = ||b_{i_j^{\alpha} j_i^{\beta}}||_{i \in I}^{j \in J}$;
- using existent algorithms determine the set $NE\left(A^{\alpha}, B^{\beta}\right)$ of Nash equilibrium profiles in the bimatrix game with the matrices A^{α} and B^{β}.

We illustrate the described above method in the following example:

Example 3.1 *We consider the bimatrix game* $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ *and construct the normal form of the game generated by the informational extended strategies.*

Solution. Consider the sets of the informational extended strategies from the Example 2.1. The corteges $\mathcal{I}^{\alpha}$ and $\mathcal{J}^{\beta}$ are:

- for player 1 : $\theta_1^1 \Rightarrow \mathcal{I}^1 = (1,1,1)$; $\theta_1^2 \Rightarrow \mathcal{I}^2 = (2,2,2)$; $\theta_1^3 \Rightarrow \mathcal{I}^3 = (1,1,2)$; $\theta_1^4 \Rightarrow \mathcal{I}^4 = (1,2,1)$; $\theta_1^5 \Rightarrow \mathcal{I}^5 = (2,1,1)$; $\theta_1^6 \Rightarrow \mathcal{I}^6 = (1,2,2)$; $\theta_1^7 \Rightarrow \mathcal{I}^7 = (2,1,2)$; $\theta_1^8 \Rightarrow \mathcal{I}^8 = (2,2,1)$;
- for player 2 : $\theta_2^1 \Rightarrow \mathcal{J}^1 = (1,1)$; $\theta_2^2 \Rightarrow \mathcal{J}^2 = (2,2)$; $\theta_2^3 \Rightarrow \mathcal{J}^3 = (3,3)$; $\theta_2^4 \Rightarrow \mathcal{J}^4 = (1,2)$; $\theta_2^5 \Rightarrow \mathcal{J}^5 = (2,1)$; $\theta_2^6 \Rightarrow \mathcal{J}^6 = (1,3)$; $\theta_2^7 \Rightarrow \mathcal{J}^7 = (3,1)$; $\theta_2^8 \Rightarrow \mathcal{J}^8 = (2,3)$; $\theta_2^9 \Rightarrow \mathcal{J}^9 = (3,2)$.

So, we can construct the all amount, equal to 72, of the informational non extended game generated by all informational extended strategy profile $\left(\theta_1^{\alpha}, \theta_2^{\beta}\right)$:

$$\Gamma\left(\theta_1^1, \theta_2^1\right) = \left\langle I, J, A^1 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix}, B^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle;$$

$$\Gamma\left(\theta_1^1, \theta_2^2\right) = \left\langle I, J, A^1 = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}, B^2 = \begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix} \right\rangle;$$

$$\Gamma\left(\theta_1^1, \theta_2^3\right) = \left\langle I, J, A^1 = \begin{pmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \end{pmatrix}, B^3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right\rangle;$$

$$\Gamma\left(\theta_1^3, \theta_2^4\right) = \left\langle I, J, A^3 = \begin{pmatrix} 3 & 3 & 6 \\ 5 & 5 & 7 \end{pmatrix}, B^4 = \begin{pmatrix} 0 & 0 & 4 \\ 5 & 5 & 3 \end{pmatrix} \right\rangle;$$

$$\Gamma\left(\theta_1^3, \theta_2^5\right) = \left\langle I, J, A^3 = \begin{pmatrix} 5 & 5 & 7 \\ 3 & 3 & 6 \end{pmatrix}, B^5 = \begin{pmatrix} 5 & 5 & 3 \\ 0 & 0 & 4 \end{pmatrix} \right\rangle;$$

$$\Gamma\left(\theta_1^4, \theta_2^4\right) = \left\langle I, J, A^4 = \begin{pmatrix} 3 & 6 & 3 \\ 5 & 7 & 5 \end{pmatrix}, B^4 = \begin{pmatrix} 0 & 4 & 0 \\ 5 & 3 & 5 \end{pmatrix} \right\rangle;$$

$$\Gamma\left(\theta_1^4, \theta_2^5\right) = \left\langle I, J, A^4 = \begin{pmatrix} 5 & 7 & 5 \\ 3 & 6 & 3 \end{pmatrix}, B^5 = \begin{pmatrix} 5 & 3 & 5 \\ 0 & 4 & 0 \end{pmatrix} \right\rangle;$$

$$\vdots$$

$$\Gamma\left(\theta_1^8, \theta_2^8\right) = \left\langle I, J, A^8 = \begin{pmatrix} 7 & 7 & 5 \\ 2 & 2 & 4 \end{pmatrix}, B^8 = \begin{pmatrix} 3 & 3 & 5 \\ 2 & 2 & 1 \end{pmatrix} \right\rangle;$$

$$\Gamma\left(\theta_1^8, \theta_2^9\right) = \left\langle I, J, A^8 = \begin{pmatrix} 2 & 2 & 4 \\ 7 & 7 & 5 \end{pmatrix}, B^9 = \begin{pmatrix} 2 & 2 & 1 \\ 3 & 3 & 5 \end{pmatrix} \right\rangle.$$

4 Bayes-Nash solutions in the bimatrix informational extended games

As was mentioned in the section "Solving the informational extended game by means of the informational non extended game" any strategy profile $(\theta_1^\alpha, \theta_2^\beta)$ in informational extended strategies generates a couple of matrices, which represent the utility of the players in informational non extended strategies

$$\left\{A(\alpha,\beta) = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, B(\alpha,\beta) = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}.$$

So as the players do not know what informational extended strategies are chosen by their partners, each player will have a possible set of utility matrices. This type of games is one in **incomplete information** because neither player 1 nor player 2 knows exactly which matrix from the mentioned set of matrices will be his utility.

Finally, the game $Game\,(1 \leftrightarrows 2)$ of imperfect information on the set of informational extended strategies, generates an incomplete information game on the set of informational non extended strategies. So we study the following two person game: the strategies of the player 1 are $I = \{1, 2, ..., n\}$ and of the player 2 are $J = \{1, 2, ..., m\}$; the payoff matrix of the player 1 is one of the matrices from the set $\left\{A(\alpha,\beta) = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}$ and the payoff matrix of the player 2 is one of the matrices from the set $\left\{B(\alpha,\beta) = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}$.

When, using the informational extended strategies, the matrices $A\left(\theta_1^\alpha, \theta_2^\beta\right)$ and $B\left(\theta_1^\alpha, \theta_2^\beta\right)$ were already built, we use the following notations: $\left\|a_{i_j^\alpha j_i^\beta}\right\|_{i\in I}^{j\in J} \equiv \left\|a_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$ and $\left\|b_{i_j^\alpha j_i^\beta}\right\|_{i\in I}^{j\in J} \equiv \left\|b_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}$ for all $\alpha = \overline{1,\kappa_1}$ and $\beta = \overline{1,\kappa_2}$, so we have a bimatrix game where the utility is determined by a set of matrices:

$$AB(\alpha,\beta) = \begin{pmatrix} \left(a_{11}^{\alpha\beta}, b_{11}^{\alpha\beta}\right) & \cdots & \left(a_{1j}^{\alpha\beta}, b_{1j}^{\alpha\beta}\right) & \cdots & \left(a_{1m}^{\alpha\beta}, b_{1m}^{\alpha\beta}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(a_{i1}^{\alpha\beta}, b_{i1}^{\alpha\beta}\right) & \cdots & \left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta}\right) & \cdots & \left(a_{im}^{\alpha\beta}, b_{im}^{\alpha\beta}\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left(a_{nm}^{\alpha\beta}, b_{nm}^{\alpha\beta}\right) & \cdots & \left(a_{nj}^{\alpha\beta}, b_{nj}^{\alpha\beta}\right) & \cdots & \left(a_{nm}^{\alpha\beta}, b_{nm}^{\alpha\beta}\right) \end{pmatrix}$$

for $\alpha = \overline{1,\kappa_1}$ and $\beta = \overline{1,\kappa_2}$ and the set of strategies are I and J. Every player knows that the utilities are determined by the set of matrices $\left\{AB(\alpha,\beta) = \left\|\left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta}\right)\right\|_{i\in I}^{j\in J}\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}$, but they do not know which matrix from this set will be used.

So, the game $Game\,(1 \leftrightarrows 2)$ of imperfect information on the set of informational extended strategies generates the following normal form incomplete information game

$$\widetilde{\mathbf{\Gamma}} = \left\langle \{1,2\}, I, J, \left\{AB(\alpha,\beta) = \left\|\left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta}\right)\right\|_{i\in I}^{j\in J}\right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}\right\rangle. \tag{4.1}$$

We call an agent Bayesian rational (or say that he has subjective expected utility preferences) if

(i) In settings with uncertainty he forms beliefs describing the probabilities of all relevant events;

(ii) When making decisions, he acts to maximize his expected utility given by his beliefs;

(iii) After receiving new information, he updates his beliefs by taking conditional probabilities whenever possible.

In the game theory, it is standard to begin analyses with the assumption that players are Bayesian rational.

The way to modelling this situation of **asymmetric** or **incomplete** informations by recurring to an idea generated by Harsanyi(1967). The key is to introduce a move by the Nature, which transforms the uncertainty by converting an **incomplete information** problem into an **imperfect information problem**. The idea is that the Nature moves determining player's types, a concept that embodies all the relevant private information about them (such as payoffs, preferences, beliefs about other players, etc.). Harsanyi described a game as having incomplete information when the players are uncertain about each other's types.

According to [6] we can construct the bimatrix Bayesian game for the bimatrix incomplete information game $\widetilde{\Gamma}$ from (4.1) that consists of the following.

1. A set of players $\{1, 2\}$;
2. A set of possible actions for each player: for player 1 is $I = \{1, 2, .., n\}$, the line index, and for player 2 is $J = \{1, 2, .., m\}$, the column index;
3. A set of possible types for each player that coincides with the set of informational extended strategies of that player, namely $\Theta_1 = \{\theta_1^\alpha : J \to I\}_{\alpha=1}^{\varkappa_1}$ for player 1 and respectively $\Theta_2 = \left\{\theta_2^\beta : I \to J\right\}_{\beta=1}^{\varkappa_2}$ for the player 2. So the types of the player 1 are $\Delta_1 = \{\alpha = 1, ..., \varkappa_1\}$ and of the player 2 are $\Delta_2 = \{\beta = 1, ..., \varkappa_2\}$. Only player 1(player 2) knows his type α (type β) when play begins.
4. A probability function that specifies, for each possible type of each player, a probability distribution over the other player's possible types, describing what each type of each player would believe about the other players' types $p : \Delta_1 \to \Omega(\Delta_2)$, $q : \Delta_2 \to \Omega(\Delta_1)$,where $\Omega(\Delta_2)$ (respectively $\Omega(\Delta_1)$) denotes the set of all probability distributions on a set Δ_1 (respectively Δ_2). The function p (respectively q) summarizes what player 1 (respectively player 2), given his type, believes about the types of the other players. So, $p(\beta|\alpha) = \dfrac{p(\beta \cap \alpha)}{p(\alpha)}$ (Bayes'Rule) (respectively $q(\alpha|\beta) = \dfrac{q(\alpha \cap \beta)}{q(\beta)}$) is the conditional probability assigned to the type $\beta \in \Delta_2$ (respectively $\alpha \in \Delta_1$) when the type of the player 1 is α (respectively of the player 2 is β).
5. Combining actions and types for each player it is possible to construct the strategies. Strategies will be given by a mapping from the type space to the action space. In other words, a strategy may assign different actions to different types. The sets of pure strategies of the players (line and columns) will depend on the type of the players (or, in other words, on what informational extended strategy will chose the players). So, in this way, we will construct the strategies of the players. If player 1 is of type $\alpha \in \Delta_1$ and player 1 knows that the type of the player 2 may be an element from the set $\Delta_2 = \{\beta = 1, ..., \varkappa_2\}$, and because the utility matrix elements also depend on the type β of player 2, then the set of matrices that represent his utility is $\left\{A(\alpha, \beta) = \left\|a_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}\right\}_{\beta=\overline{1,\kappa_2}}$. We will denote the pure strategy of player 1 by $\widetilde{\mathbf{i}} = i_1 i_2 ... i_\beta ... i_{\kappa_2}$ and it has the following meaning: the player will chose the line $i_1 \in I$ if $\beta = 1$, namely line i_1 from the utility matrix $A(\alpha, 1)$ and line $i_2 \in I$ if $\beta = 2$ and so on, line $i_{\kappa_2} \in I$ if $\beta = \kappa_2$. Then the set of all pure strategy of player 1 will be determined by the set of all corteges of type $i_1 i_2 ... i_\beta ... i_{\kappa_2}$ for all $i_\beta \in I$ and will be denoted by $\widetilde{\mathbf{I}}(\alpha)$. In his turn, if player 2 is of type $\beta \in \Delta_2$ and he knows that the type of player 1 may be an element from the set $\Delta_1 = \{\alpha = 1, ..., \varkappa_1\}$, and because the utility matrix elements depend also on the type α of player 1, then the set of matrices that represent his utility is $\left\{B(\alpha, \beta) = \left\|b_{ij}^{\alpha\beta}\right\|_{i\in I}^{j\in J}\right\}_{\alpha=\overline{1,\kappa_1}}$. By the same way we will denote the pure strategy of player 2 by $\widetilde{\mathbf{j}} = j_1 j_2 ... j_\alpha ... j_{\kappa_2}$ and it has the following meaning: the

player will chose column $j_1 \in J$ if $\alpha = 1$, namely column j_1 from utility matrix $B(1,\beta)$ and column $j_2 \in J$ if $\alpha = 2$ and so on he will chose column $j_{\kappa_1} \in J$ if $\alpha = \kappa_1$. Then the set of all pure strategy of player 2 will be determined by the set of all corteges of type $j_1 j_2 ... j_\alpha ... j_{\kappa_2}$ for all $j_\alpha \in J$ and will be denoted by $\widetilde{\mathbf{J}}(\beta)$.

6. A payoff function specifies each player's expected payoff matrices for every possible combination of all player's actions and types. Hence, if the player 1 of type α chooses the pure strategy $\widetilde{\mathbf{i}} \in \widetilde{\mathbf{I}}(\alpha)$, and the player 2 plays some strategy $\widetilde{\mathbf{j}} \in \widetilde{\mathbf{J}}(\beta)$ for all $\beta \in \Delta_2$, then expected payoffs of player 1 is the following matrix

$$\mathbf{A}(\alpha) = \left\| \mathbf{a}_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}} \right\|_{\widetilde{\mathbf{i}} \in \widetilde{\mathbf{I}}(\alpha)}^{\widetilde{\mathbf{j}} \in \widetilde{\mathbf{J}}(\beta)} \tag{4.2}$$

where $\mathbf{a}_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}} = \sum_{\beta \in \Delta_2} p(\beta/\alpha) a_{i_\beta j_\alpha}^{\alpha\beta}$. Similarly, if player 2 of type β chooses the pure strategy $\widetilde{\mathbf{j}} \in \widetilde{\mathbf{J}}(\beta)$ and the player 1 plays some strategy $\widetilde{\mathbf{i}} \in \widetilde{\mathbf{I}}(\alpha)$ for all $\alpha \in \Delta_1$, then expected payoffs of player 2 of type β is

$$\mathbf{B}(\beta) = \left\| \mathbf{b}_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}} \right\|_{\widetilde{\mathbf{i}} \in \widetilde{\mathbf{I}}(\alpha)}^{\widetilde{\mathbf{j}} \in \widetilde{\mathbf{J}}(\beta)} \tag{4.3}$$

where $\mathbf{b}_{\widetilde{\mathbf{i}}\widetilde{\mathbf{j}}} = \sum_{\alpha \in \Delta_1} q(\alpha|\beta) b_{i_\beta j_\alpha}^{\alpha\beta}$.

So we can introduce the following definition.

Definition 4.1 *For the incomplete information game $\widetilde{\mathbf{\Gamma}}$ from (4.1) the normal form game*

$$\Gamma_{Bayes} = \left\langle \{1,2\}, \widetilde{\mathbf{I}}, \widetilde{\mathbf{J}}, \mathcal{A}, \mathcal{B} \right\rangle, \tag{4.4}$$

where $\widetilde{\mathbf{I}} = \bigcup_{\alpha \in \Delta_1} \widetilde{\mathbf{I}}(\alpha)$, $\widetilde{\mathbf{J}} = \bigcup_{\beta \in \Delta_2} J(\alpha)$ *and the utility matrices are* $\mathcal{A} = \|\mathbf{A}(\alpha)\|_{\alpha \in \Delta_1}$ *and* $\mathcal{B} = \|\mathbf{B}(\beta)\|_{\beta \in \Delta_2}$, *is called the associated Bayesian game in the non informational extended strategies.*

It is important to discuss a little bit each part of the definition above. Players types contain all relevant information about certain player's private characteristics of the informational extended strategy to choose. The type α (respectively β) is only observed by player 1 (player 2), who uses this information both to make decisions and to update his beliefs about the likelihood of opponents types (using the conditional probability $p(\beta|\alpha)$ (respectively $q(\alpha|\beta)$). We still assume common knowledge of the 1)-6) items, but we allow uncertainty about players' preferences. Player's (α,β) type determines (α,β) payoffs matrices $(\mathbf{A}(\alpha), \mathbf{B}(\beta)$.

The games defined above are sometimes called *Bayesian normal form games*, since the drawing of types is followed by a simultaneous move game. One can also define *Bayesian extensive form games*, where the drawing of types is followed by an extensive form game.

Definition 4.2 *(Bayesian Nash Equilibrium) The strategy profiles* $(\mathbf{i}^*, \mathbf{j}^*)$, $\mathbf{i}^* \in \widetilde{\mathbf{I}}$, $\mathbf{j}^* \in \widetilde{\mathbf{J}}$ *is Bayes-Nash equilibrium if we have*

$$\begin{cases} \mathbf{a}_{\mathbf{i}^*\mathbf{j}^*} \geqslant \mathbf{a}_{\mathbf{i}\mathbf{j}^*} & \text{for all } \mathbf{i} \in \widetilde{\mathbf{I}}, \\ \mathbf{b}_{\mathbf{i}^*\mathbf{j}^*} \geqslant \mathbf{b}_{\mathbf{i}^*\mathbf{j}} & \text{for all } \mathbf{j} \in \widetilde{\mathbf{J}}. \end{cases}$$

Denote by $BE\left[\Gamma_{Bayes}\right]$ the set of all Bayes-Nash strategies profile of the game Γ_{Bayes} from (4.4).

Remark 4.1 *The Bayesian Game Γ_{Bayes} (4.4) for all $\alpha \in \Delta_1$ and $\beta \in \Delta_2$ is a bimatrix game where player 1 is of type α and player 2 is of type β. The Bayese-Nash equilibria profile following the Definition 4.2 will be found in the next way: we find the Nash equilibria profile for a bimatrix game where the sets of strategies are the "extended sets"* $\widetilde{\mathbf{I}} = \bigcup_{\alpha \in \Delta_1} \widetilde{\mathbf{I}}(\alpha)$, $\widetilde{\mathbf{J}} = \bigcup_{\beta \in \Delta_2} J(\alpha)$ *and the utility matrices are the "extended matrices"* $\mathcal{A} = \|\mathbf{A}(\alpha)\|_{\alpha \in \Delta_1}$ *and* $\mathcal{B} = \|\mathbf{B}(\beta)\|_{\beta \in \Delta_2}$.

We will introduce the next definition.

Definition 4.3 *For all fixed $\alpha \in \Delta_1$ and $\beta \in \Delta_2$ the game* $sub\Gamma_{Bayes} = \left\langle \{1,2\}, \widetilde{\mathbf{I}}(\alpha), \widetilde{\mathbf{J}}(\beta), \mathbf{A}(\alpha), \mathbf{B}(\beta) \right\rangle$ *will be called a subgame of the Bayesian game* Γ_{Bayes} *from (4.4).*

According to [5], using the notion of "type-players", the $sub\Gamma_{Bayes}$ is the bimatrix game of the type-player α and of the type-player β.

Example 4.1 *The construction of the Bayesian game for the 2×3 bimatrix games in incomplete information, generated by the informational extended strategies.*

Solution. Consider a bimatrix game in incomplete information for which the utilities are:

$$AB(\alpha,\beta) = \left(\begin{array}{ccc} \left(a_{11}^{\alpha\beta}, b_{11}^{\alpha\beta} \right) & \left(a_{12}^{\alpha\beta}, b_{12}^{\alpha\beta} \right) & \left(a_{13}^{\alpha\beta}, b_{13}^{\alpha\beta} \right) \\ \left(a_{21}^{\alpha\beta}, b_{21}^{\alpha\beta} \right) & \left(a_{22}^{\alpha\beta}, b_{22}^{\alpha\beta} \right) & \left(a_{23}^{\alpha\beta}, b_{23}^{\alpha\beta} \right) \end{array} \right).$$

The Bayesian game will contain the elements.

a) The set of players $\{1,2\}$.

b) The set of actions of the players $I = \{1,2\}$, $J = \{1,2,3\}$.

c) The set of types of the player 1 is $\Delta_1 = \{\alpha = \overline{1,8}\}$ and of the player 2 is $\Delta_2 = \{\beta = \overline{1,9}\}$.

d) denote the type probability for player 1 by $p(\beta|\alpha)$, respectively $q(\alpha|\beta)$ for player 2.

e) For any fixed α we introduce the notation $i_\beta i_\gamma$, for $\beta, \gamma \in \Delta_2$, $\beta \neq \gamma$, which satisfies the conditions: the player 1 will chose the line $i_\beta \in I$ in case if the player 2 is of type β, namely, the utility of the player is the matrix $\left\| a_{ij}^{\alpha\beta} \right\|_{i \in I}^{j \in J}$, and will chose the line $i_\gamma \in I$ if the player 2 is of type γ (namely, the utility of the player is the matrix $\left\| a_{ij}^{\alpha\gamma} \right\|_{i \in I}^{j \in J}$). Thus the set of pure strategies of the player 1 is $\widetilde{\mathbf{I}}(\alpha) = \{\widetilde{\mathbf{i}} = i_\beta i_\gamma : i_\beta \in I, i_\gamma \in I, \forall \beta, \gamma \in \Delta_2, \beta \neq \gamma\} = \{1_1 1_2, 1_1 2_2, 2_1 1_2, 2_1 2_2\}$. In the same way we will construct the strategies of the player 2. For any fixed β we will denote $j_\alpha j_\delta$ for $\alpha, \delta \in \Delta_1, \alpha \neq \delta$, which meaning is: the player 2 will chose the column $j_\alpha \in J$ if the player 1 is of type α, i.e. the utility of the player is the matrix $\left\| b_{ij}^{\alpha\beta} \right\|_{i \in I}^{j \in J}$, and will chose the column $j_\delta \in J$ if the player 1 is of type δ, i.e. the utility of the player is the matrix $\left\| b_{ij}^{\delta\beta} \right\|_{i \in I}^{j \in J}$. Thus the set of pure strategies of the player 2 is $\widetilde{\mathbf{J}}(\beta) = \{\widetilde{\mathbf{j}} = j_\alpha j_\delta : j_\alpha \in J, j_\delta \in J, \forall \alpha, \delta \in \Delta_1, \alpha \neq \delta\} = \{1_1 1_2, 1_1 2_2, 2_1 1_2, 2_1 2_2, 1_1 3_2, 3_1 1_2, 2_1 3_2, 3_1 2_2, 3_1 3_2\}$.

g) The players do not know the exact type of the partners and supply this lack of information by the belief probabilities. Let $\Delta_1 = \{\alpha = 1,2\}$ and $\Delta_2 = \{\beta = 1,2\}$. Thus the player 1, being of type α, will assume with the probability $p(\beta = 1|\alpha)$ that he has the payoff matrice $\left(\begin{array}{ccc} a_{11}^{\alpha 1} & a_{12}^{\alpha 1} & a_{13}^{\alpha 1} \\ a_{21}^{\alpha 1} & a_{22}^{\alpha 1} & a_{23}^{\alpha 1} \end{array} \right)$ and, with probabilities $p(\beta = 2|\alpha)$, the payoff matrice $\left(\begin{array}{ccc} a_{11}^{\alpha 2} & a_{12}^{\alpha 2} & a_{13}^{\alpha 2} \\ a_{21}^{\alpha 2} & a_{22}^{\alpha 2} & a_{23}^{\alpha 2} \end{array} \right)$. Respectively, the player 2, being of type β, will assume with the probability $q(\alpha = 1|\beta)$ that he has the payoff matrice $\left(\begin{array}{ccc} b_{11}^{1\beta} & b_{12}^{1\beta} & b_{13}^{1\beta} \\ b_{21}^{1\beta} & b_{22}^{1\beta} & b_{23}^{1\beta} \end{array} \right)$ and, with the probability $q(\alpha = 2|\beta)$, the payoff matrice $\left(\begin{array}{ccc} b_{11}^{2\beta} & b_{11}^{2\beta} & b_{13}^{2\beta} \\ b_{21}^{2\beta} & b_{22}^{2\beta} & b_{23}^{2\beta} \end{array} \right)$. We denote by $E_1(a_{i_1 j_\alpha}^{\alpha 1}, a_{i_2 j_\alpha}^{\alpha 2}) = p(\beta = 1|\alpha) a_{i_1 j_\alpha}^{\alpha 1} + p(\beta = 2|\alpha) a_{i_2 j_\alpha}^{\alpha 2}$

$E_2(b^{1\beta}_{i_\beta j_1}, b^{2\beta}_{i_\beta j_2}) = q(\alpha = 1|\beta) b^{1\beta}_{i_\beta j_1} + q(\alpha = 2|\beta) b^{2\beta}_{i_\beta j_2}$ for any $i \in I$, $j \in J$, $\alpha \in \Delta_1$, $\beta \in \Delta_2$, the average value if the player 1, respectively the player 2, knows the belief probabilities (or the probabilities setted by the Nature). We will construct the utility matrices when the player 1 is of type α and, at the same time, the player 2 is of type β. Based on the facts mentioned above we will obtain the next bimatrix game in which the utility of the players is described by the following matrices with 4 lines and 9 columns:

$A(\alpha) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$
$1_1 2_2$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$
$2_1 1_2$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$
$2_1 2_2$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$
$1_1 2_2$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{1 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$
$2_1 1_2$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{1 j_\alpha}\right)$
$2_1 2_2$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$	$E_1\left(a^{\alpha 1}_{2 j_\alpha}, a^{\alpha 2}_{2 j_\alpha}\right)$

(4.5)

$B(\beta) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E\left({}_2 b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$
$1_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 1}, b^{2\beta}_{i_\beta 3}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_1$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E\left({}_2 b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$1_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 1_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$
$2_1 2_2$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 1}\right)$	$E_2\left(b^{1\beta}_{i_\beta 2}, b^{2\beta}_{i_\beta 3}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 2}\right)$	$E_2\left(b^{1\beta}_{i_\beta 3}, b^{2\beta}_{i_\beta 3}\right)$

(4.6)

What is the meaning, for example, of elements at the intersection of line $1_1 2_2$ and column $1_1 3_2$? Using the belief probabilities for types of the players, we get that player 1, being of type α, will chose the line $i = 1$ from the matrix $\begin{pmatrix} a^{\alpha 1}_{11} & a^{\alpha 1}_{12} & a^{\alpha 1}_{13} \\ a^{\alpha 1}_{21} & a^{\alpha 1}_{22} & a^{\alpha 1}_{23} \end{pmatrix}$ (when the player 2 is of type $\beta = 1$) and line $i = 2$ from the matrix $\begin{pmatrix} a^{\alpha 2}_{11} & a^{\alpha 2}_{12} & a^{\alpha 2}_{13} \\ a^{\alpha 2}_{21} & a^{\alpha 2}_{22} & a^{\alpha 2}_{23} \end{pmatrix}$ (when player 2 is of type $\beta = 2$), and correspondingly, the player 2 being of type β, will chose the column $j = 1$ from the matrix $\begin{pmatrix} b^{1\beta}_{11} & b^{1\beta}_{12} & b^{1\beta}_{13} \\ b^{1\beta}_{21} & b^{1\beta}_{22} & b^{1\beta}_{23} \end{pmatrix}$ (when the type of the player 1 is $\alpha = 1$) and the column $j = 3$ in the matrix $\begin{pmatrix} b^{2\beta}_{11} & b^{2\beta}_{12} & b^{2\beta}_{13} \\ b^{2\beta}_{21} & b^{2\beta}_{22} & b^{2\beta}_{23} \end{pmatrix}$ (when the type of the player 1 is $\alpha = 2$), then the average value of the payoff for the player 1 is $E_1\left(a_{1j_\alpha}^{\alpha 1}, a_{2j_\alpha}^{\alpha 2}\right) = p(\beta = 1|\alpha)a_{1j_\alpha}^{\alpha 1} + p(\beta = 2|\alpha)a_{2j_\alpha}^{\alpha 2}$ and respectively, for the player 2 is $E_2\left(b_{i_\beta 1}^{1\beta}, b_{i_\beta 3}^{2\beta}\right) = q(\alpha = 1|\beta)b_{i_\beta 1}^{1\beta} + q(\alpha = 2|\beta)b_{i_\beta 3}^{2\beta}$.

Finally, for $\alpha = 1, \alpha = 2, \beta = 1$ and $\beta = 2$ we obtain:

$A(1) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1\left(a_{11}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{11}^{12}\right)$
$1_1 2_2$	$E_1\left(a_{11}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{11}^{11}, a_{21}^{12}\right)$
$2_1 1_2$	$E_1\left(a_{21}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{11}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{11}^{12}\right)$
$2_1 2_2$	$E_1\left(a_{21}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{21}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{21}^{11}, a_{21}^{12}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1\left(a_{13}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{13}^{12}\right)$
$1_1 2_2$	$E_1\left(a_{13}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{12}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{13}^{11}, a_{23}^{12}\right)$
$2_1 1_2$	$E_1\left(a_{23}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{12}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{13}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{13}^{12}\right)$
$2_1 2_2$	$E_1\left(a_{23}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{22}^{11}, a_{22}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{23}^{12}\right)$	$E_1\left(a_{23}^{11}, a_{23}^{12}\right)$

$A(2) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_1\left(a_{11}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{11}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{13}^{22}\right)$
$1_1 2_2$	$E_1\left(a_{11}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{11}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{23}^{22}\right)$
$2_1 1_2$	$E_1\left(a_{21}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{21}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{13}^{22}\right)$
$2_1 2_2$	$E_1\left(a_{21}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{21}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{23}^{22}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_1\left(a_{11}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{13}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{13}^{22}\right)$
$1_1 2_2$	$E_1\left(a_{11}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{23}^{22}\right)$	$E_1\left(a_{12}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{13}^{21}, a_{23}^{22}\right)$
$2_1 1_2$	$E_1\left(a_{21}^{21}, a_{11}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{13}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{12}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{13}^{22}\right)$
$2_1 2_2$	$E_1\left(a_{21}^{21}, a_{21}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{23}^{22}\right)$	$E_1\left(a_{22}^{21}, a_{22}^{22}\right)$	$E_1\left(a_{23}^{21}, a_{23}^{22}\right)$

$B(1) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2\left(b_{11}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{11}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{12}^{21}\right)$	$E\left({}_2b_{11}^{11}, b_{13}^{21}\right)$
$1_1 2_2$	$E_2\left(b_{11}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{11}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{11}^{11}, b_{13}^{21}\right)$
$2_1 1_2$	$E_2\left(b_{21}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{23}^{21}\right)$
$2_1 2_2$	$E_2\left(b_{21}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{21}^{11}, b_{23}^{21}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_1$	$E_2\left(b_{13}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{13}^{21}\right)$	$E\left({}_2b_{13}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{13}^{11}, b_{13}^{21}\right)$
$1_1 2_2$	$E_2\left(b_{13}^{11}, b_{11}^{21}\right)$	$E_2\left(b_{12}^{11}, b_{13}^{21}\right)$	$E_2\left(b_{13}^{11}, b_{12}^{21}\right)$	$E_2\left(b_{13}^{11}, b_{13}^{21}\right)$
$2_1 1_2$	$E_2\left(b_{23}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{23}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{23}^{21}\right)$
$2_1 2_2$	$E_2\left(b_{23}^{11}, b_{21}^{21}\right)$	$E_2\left(b_{22}^{11}, b_{23}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{22}^{21}\right)$	$E_2\left(b_{23}^{11}, b_{23}^{21}\right)$

$B(2) =$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$1_1 1_2$	$1_1 2_2$	$2_1 1_2$	$2_1 2_2$	$1_1 3_2$
$1_1 1_2$	$E_2\left(b_{11}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{11}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{12}^{22}\right)$	$E\left({}_2 b_{11}^{12}, b_{13}^{22}\right)$
$1_1 2_2$	$E_2\left(b_{21}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{i_\beta 1}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{i_\beta 2}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{23}^{22}\right)$
$2_1 1_2$	$E_2\left(b_{11}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{11}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{11}^{12}, b_{13}^{22}\right)$
$2_1 2_2$	$E_2\left(b_{21}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{21}^{12}, b_{23}^{22}\right)$

$\widetilde{\mathbf{i}} \diagdown \widetilde{\mathbf{j}}$	$3_1 1_2$	$2_1 3_2$	$3_1 2_2$	$3_1 3_2$
$1_1 1_2$	$E_2\left(b_{13}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{13}^{22}\right)$	$E\left({}_2 b_{13}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{13}^{12}, b_{13}^{22}\right)$
$1_1 2_2$	$E_2\left(b_{23}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{23}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{23}^{22}\right)$
$2_1 1_2$	$E_2\left(b_{13}^{12}, b_{11}^{22}\right)$	$E_2\left(b_{12}^{12}, b_{13}^{22}\right)$	$E_2\left(b_{13}^{12}, b_{12}^{22}\right)$	$E_2\left(b_{13}^{12}, b_{13}^{22}\right)$
$2_1 2_2$	$E_2\left(b_{23}^{12}, b_{21}^{22}\right)$	$E_2\left(b_{22}^{12}, b_{23}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{22}^{22}\right)$	$E_2\left(b_{23}^{12}, b_{23}^{22}\right)$

So,the normal form of the Bayesian game from (4.4) is

$$\begin{aligned}\Gamma_{Bayes} &= \Big\langle \{1,2\}, \widetilde{\mathbf{I}} = \widetilde{\mathbf{I}}(\alpha=1) \cup \widetilde{\mathbf{I}}(\alpha=2), \widetilde{\mathbf{J}} = \widetilde{\mathbf{J}}(\beta=1) \cup \widetilde{\mathbf{J}}(\beta=1)\ , \\ &\quad \mathcal{A} = \|\mathbf{A}(\alpha=1), \mathbf{A}(\alpha=1)\|\ , \mathcal{B} = \|\mathbf{B}(\beta=1), \mathbf{B}(\beta=2)\| \Big\rangle\ .\end{aligned}$$

Bimatrix games $\langle A(1), B(1)\rangle$, $\langle A(1), B(2)\rangle$, $\langle A(2), B(1)\rangle$ and $\langle A(2), B(2)\rangle$ are subgames of the constructed above Bayesian game.

As a particular case we will examine the next example. We consider the following bimatrix game $H_1 = \begin{pmatrix} 3 & 5 & 4 \\ 6 & 7 & 2 \end{pmatrix}$, $H_2 = \begin{pmatrix} 0 & 5 & 1 \\ 4 & 3 & 2 \end{pmatrix}$ for which we construct the normal form of the Bayesian game associated to the informational extended game.

For example, suppose that the informational extended strategies of the player 1 are $\theta_1^1(j) = \begin{cases} 1 \text{ if } j = 1,2 \\ 2 \text{ if } j = 3 \end{cases}$, $\theta_1^2(j) = \begin{cases} 1 \text{ if } j = 1,3 \\ 2 \text{ if } j = 2 \end{cases}$ and respectively, for the player 2 are $\theta_2^1(i) = \begin{cases} 1 \text{ if } i = 1 \\ 2 \text{ if } i = 2 \end{cases}$, $\theta_2^2(i) = \begin{cases} 1 \text{ if } i = 2 \\ 2 \text{ if } i = 1 \end{cases}$. Using the notations from Example 2.1 or 3.1, we have $\theta_1^1(j) \equiv \theta_1^3(j)$, $\theta_1^2(j) \equiv \theta_1^4(j)$, $\theta_2^1(i) \equiv \theta_2^4(i)$, $\theta_2^2(i) \equiv \theta_2^5(i)$.

As mentioned above, the informational extended strategies $\left\{\theta_1^1, \theta_1^2, \theta_2^1, \theta_2^2\right\}$ generate an incomplete information game in which the payoff matrix may be one of the following matrices (one in which the utility of the players is determined by one of the matrix bellow):

$$\begin{aligned} AB\left(\theta_1^1, \theta_2^1\right) &= \begin{pmatrix} (3,0) & (3,0) & (6,4) \\ (5,5) & (5,5) & (7,3) \end{pmatrix},\ AB\left(\theta_1^2, \theta_2^1\right) = \begin{pmatrix} (3,0) & (6,4) & (3,0) \\ (5,5) & (7,3) & (5,5) \end{pmatrix}, \\ AB\left(\theta_1^1, \theta_2^2\right) &= \begin{pmatrix} (5,5) & (5,5) & (7,3) \\ (3,0) & (3,0) & (6,0) \end{pmatrix},\ AB\left(\theta_1^2, \theta_2^2\right) = \begin{pmatrix} (5,5) & (7,3) & (5,5) \\ (3,0) & (6,4) & (3,0) \end{pmatrix}. \end{aligned} \tag{4.7}$$

We will construct the Bayesian game for the game in incomplete and imperfect information over the set of informational non extended strategies I, J from (4.7). The set of types of the player 1 is $\alpha \in \Delta_1 = \{1,2\}$ and of the player 2 is $\beta \in \Delta_2 = \{1,2\}$. Let's consider that the belief probabilities of the types are: for the player 1 : $p(\beta|\alpha) = \begin{cases} p \text{ for } \beta = 1 \\ 1-p \text{ for } \beta = 2 \end{cases}$ and for the player 2 : $q(\alpha|\beta) = \begin{cases} q \text{ for } \alpha = 1 \\ 1-q \text{ for } \alpha = 2 \end{cases}$, $0 \le p \le 1$, $0 \le q \le 1$. Thus we get a Bayesian game in which the utility functions of the players, depending of their types, will be:

$$\mathbf{A}(\alpha=1) = \begin{pmatrix} 5-2p & 5-2p & 5-2p & 5-2p & 5-2p & 6 & 3 & 6 & 6 \\ 3 & 3 & 3 & 3 & 3 & 6p & 6-3p & 3+3p & 6 \\ 5 & 5 & 5 & 5 & 5 & 7 & 5 & 7 & 7 \\ 3+2p & 3+2p & 3+2p & 3+2p & 3+2p & 6+p & 3+2p & 6+p & 6+p \end{pmatrix},$$

$$\mathbf{B}(\beta=1)=\begin{pmatrix} 0 & 4 & 0 & 4 & 0 & 4q & 0 & 4 & 4q \\ 0 & 4 & 0 & 4 & 0 & 4q & 0 & 4 & 4q \\ 5 & 3+2q & 5 & 3+2q & 5 & 5-q & 5 & 3 & 5-2q \\ 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q \end{pmatrix},$$

$$\mathbf{A}(\alpha=2)=\begin{pmatrix} 5-2p & 7-p & 5-2p & 7-p & 5-2p & 5-2p & 5-2p & 7-p & 5-2p \\ 3 & 6 & 3 & 6 & 3 & 3 & 3 & 6 & 3 \\ 3+2p & 7 & 5 & 7-4p & 5 & 5 & 5 & 7 & 5 \\ 3+2p & 6+p & 3+2p & 3+2p & 6+p & 3+2p & 3+2p & 6+p & 3+2p \end{pmatrix},$$

$$\mathbf{B}(\beta=2)=\begin{pmatrix} 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q \\ 0 & 4-4q & 0 & 4-4q & 0 & 0 & 0 & 4-4q & 0 \\ 5 & 3+2q & 5 & 3+2q & 5 & 5-2q & 5 & 3 & 5-2q \\ 0 & 4-4q & 0 & 4-4q & 0 & 0 & 0 & 4-4q & 0 \end{pmatrix}.$$

According to the Definition 4.2, we have $\left(\widetilde{\mathbf{i}^*},\widetilde{\mathbf{j}^*}\right) \equiv (i_1^* i_2^*, j_1^* j_2^*) \in NE\left(\Gamma_{Bayes}\right)$ in the game (4.5)-(4.6), if for all $i \in I, j \in J$ the following conditions hold

$$\begin{cases} E_1\left(a^{11}_{i_1^* j_1^*}, a^{12}_{i_2^* j_1^*}\right) \geq E_1\left(a^{11}_{i_1 j_1^*}, a^{12}_{i_2 j_1^*}\right), \\ E_1\left(a^{21}_{i_1^* j_2^*}, a^{22}_{i_2^* j_2^*}\right) \geq E_1\left(a^{21}_{i_1 j_2^*}, a^{22}_{i_2 j_2^*}\right), \\ E_2\left(b^{11}_{i_1^* j_1^*}, b^{21}_{i_1^* j_2^*}\right) \geq E_2\left(b^{11}_{i_1^* j_1}, b^{21}_{i_1^* j_2}\right), \\ E_2\left(b^{12}_{i_2^* j_1^*}, b^{22}_{i_2^* j_2^*}\right) \geq E_2\left(b^{12}_{i_2^* j_1}, b^{22}_{i_2^* j_2}\right). \end{cases}$$

Let $p = q = \dfrac{1}{2}$, then

$$\mathbf{A}(\alpha = 1)=\begin{pmatrix} 4 & 4 & 4 & 4 & 4 & 6 & 3 & 6 & 6 \\ 3 & 3 & 3 & 3 & 3 & 3 & 9/2 & 9/2 & 6 \\ 5 & 5 & 5 & 5 & 5 & 7 & 5 & 7 & 7 \\ 3 & 3 & 3 & 3 & 3 & 13/2 & 3 & 13/2 & 13/2 \end{pmatrix},$$

$$\mathbf{A}(\alpha = 2)=\begin{pmatrix} 5 & 13/2 & 5 & 13/2 & 5 & 5 & 5 & 13/2 & 5 \\ 3 & 6 & 3 & 6 & 3 & 3 & 3 & 6 & 3 \\ 3 & 7 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 3 & 13/2 & 3 & 3 & 13/2 & 3 & 3 & 13/2 & 3 \end{pmatrix},$$

$$\mathbf{B}(\beta = 1)=\begin{pmatrix} 0 & 4 & 0 & 4 & 0 & 2 & 0 & 4 & 2 \\ 0 & 4 & 0 & 4 & 0 & 2 & 0 & 4 & 2 \\ 5 & 3 & 5 & 3 & 5 & 9/2 & 5 & 3 & 5 \\ 5 & 3 & 5 & 3 & 5 & 5 & 5 & 3 & 5 \end{pmatrix},$$

$$\mathbf{B}(\beta = 2)=\begin{pmatrix} 5 & 3 & 5 & 3 & 5 & 5 & 5 & 3 & 5 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 5 & 3 & 5 & 3 & 5 & 5 & 5 & 3 & 5 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

For all $\alpha = \overline{1,2}$ the set of best response strategies of the player 1 is $Br_1(1) = \{1,3\}$, $Br_1(2) = \{1,4\}$, $Br_1(3) = \{1,3\}$, $Br_1(4) = \{1\}$, $Br_1(5) = \{4\}$, $Br_1(6) = \{3\}$, $Br_1(7) = \{2\}$, $Br_1(8) = \{3\}$, $Br_1(9) = \{3\}$. Respectively, for all $\beta = \overline{1,2}$ the set of best response strategies of the player 2 is $Br_2(1) = \{1,3,5,6,7,9\}$, $Br_2(2) = \{2,4,8\}$, $Br_2(3) = \{1,3,5,6,7,9\}$, $Br_2(4) = \{1,3,5,6,7,9\}$. Thus, the set of Bayese-Nash equilibrium profile is

$$BE\left[\Gamma_{Bayes}\right] = \left\{(1_1 1_2, 1_1 1_2), (1_1 1_2, 2_1 1_2), (2_1 1_2, 1_1 1_2), (2_1 1_2, 2_1 1_2), (2_1 1_2, 3_1 3_2)\right\}.$$

Using given above constructions and the Harsanyi theorem [7], we get the following theorem.

Theorem 4.1 *The strategy profile* $(\mathbf{i}^*, \mathbf{j}^*)$ *is a Bayes-Nash equilibrium in the game* Γ_{Bayes} *from (4.4) if and only if, for all* $\alpha \in \Delta_1, \beta \in \Delta_2$, *the strategy profile* $(\mathbf{i}^*, \mathbf{j}^*)$ *is a Nash equilibrium for the subgame* $sub\Gamma_{Bayes} = \left\langle \{1,2\}, \widetilde{\mathbf{I}}(\alpha), \widetilde{\mathbf{J}}(\beta), \mathbf{A}(\alpha), \mathbf{B}(\beta) \right\rangle.$

Using the terms of the informational extended strategies, these theorem means the following.

Remark 4.2 *If the player* 1 *chooses the information extended strategy* $\theta_1^\alpha \in \Theta_1$ *(respectively, the player* 2 *choose the information extended strategy* $\theta_2^\beta \in \Theta_2$*) and assumes that the player* 2, *for all* $\beta \in \Delta_2$, *will choose the information extended strategies* θ_2^β *with the probability* $p\left(\theta_2^\beta | \theta_1^\alpha\right)$ *(respectively, the player* 2 *assumes that for all* $\alpha \in \Delta_1$, *the player* 1 *will choose the information extended strategies* θ_1^α *with the probability* $q\left(\theta_1^\alpha | \theta_2^\beta\right)$*), then the Nash equilibrium profiles of the bimatrix Bayesian game with matrices* $\mathbf{A}(\alpha), \mathbf{B}(\beta)$, *for all* $\alpha \in \Delta_1$, $\beta \in \Delta_2$, *from (4.2)-(4.3) is the Bayes-Nash equilibria of the bimatrix informational extended game* $\widetilde{\Gamma}$ *from (4.1).*

Finally, to determine Bayes-Nash equilibria profiles of the bimatrix incomplete information game $\widetilde{\mathbf{\Gamma}} = \left\langle \{1,2\}, I, J, \left\{ AB(\alpha,\beta) = \left\| \left(a_{ij}^{\alpha\beta}, b_{ij}^{\alpha\beta}\right) \right\|_{i\in I}^{j\in J} \right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}} \right\rangle$ from (4.1), we have to follow next steps:

- using the "combinatorial algorithm", we construct, for all α, β, the corteges $\mathcal{I}^\alpha$ and $\mathcal{J}^\beta$ that represent the informational extended strategies θ_1^α and θ_2^β, respectively;
- construct the game of incomplete information on the set of information non extended strategies, i.e. construct, for each player, the set of possible utility matrices $\left\{ A(\alpha) = ||a_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, B(\beta) = ||b_{i_j^\alpha j_i^\beta}||_{i\in I}^{j\in J}, i_j^\alpha \in I^\alpha, j_i^\beta \in J^\beta \right\}_{\alpha=\overline{1,\kappa_1}}^{\beta=\overline{1,\kappa_2}}$;
- for all $\alpha \in \Delta_1$, $\beta \in \Delta_2$, construct the "belief probabilities" $p\left(\theta_2^\beta | \theta_1^\alpha\right)$ and $q\left(\theta_1^\alpha | \theta_2^\beta\right)$;
- generate the sets $\left\{\widetilde{\mathbf{I}}(\alpha)\right\}_{\alpha\in\Delta_1}, \left\{\widetilde{\mathbf{J}}(\beta)\right\}_{\beta\in\Delta_2}$ of pure strategies for Bayesian game which correspond to the game $\widetilde{\mathbf{\Gamma}}$;
- for all fixed $\alpha \in \Delta_1$ and $\beta \in \Delta_2$, construct the payoff matrices $\mathbf{A}(\alpha)$ from (4.2) and $\mathbf{B}(\beta)$ *from* (4.3);
- using the existent algorithms, determine for all $\alpha \in \Delta_1$, $\beta \in \Delta_2$ the set of Nash equilibrium profiles in the bimatrix game $\left\langle \{1,2\}, \widetilde{\mathbf{I}}(\alpha), \widetilde{\mathbf{J}}(\beta), \mathbf{A}(\alpha), \mathbf{B}(\beta) \right\rangle$.
- using the theorem 4.1,construct the set of all Bayes-Nash equilibria in the game Γ_{Bayes} from (4.4).

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