ANALYSIS OF SOLUTIONS FOR THE SYSTEMS OF NONLINEAR EQUATIONS DEPENDING ON PARAMETERS

ANALIZA SOLUȚIILOR SISTEMELOR DE ECUAȚII NELINIARE ÎN FUNCȚIE DE PARAMETRI

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Let us represent a system of n order nonlinear equations with respect to the variables $x_1,\ldots,x_n$, and depending on m parameters $y_1,\ldots,y_m$, that is, matrices A and B, and vectors $\vec{x}$, $\vec{y}$ and $\vec{F}$, in the block form [1]:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix} = -\left(\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} \vec{y}_1 \\ \vec{y}_2 \end{pmatrix} + \begin{pmatrix} \vec{F}_1(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2) \\ \vec{F}_2(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2) \end{pmatrix}\right) \tag{1}$$

or Eq. (1) is given in a more detailed notation as follows

$$A_{11}\vec{x}_1 + A_{12}\vec{x}_2 = -\left(B_{11}\vec{y}_1 + B_{12}\vec{y}_2 + \vec{F}_1(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2)\right),$$
$$A_{21}\vec{x}_1 + A_{22}\vec{x}_2 = -\left(B_{21}\vec{y}_1 + B_{22}\vec{y}_2 + \vec{F}_2(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2)\right). \tag{2}$$

By construction $rank(A_{11}) = r$, and thus there is an inverse matrix A_{11}^{-1}, taking which into account the first expression in Eq. (2) can be represented in an equivalent form

$$\vec{x}_1 = -\left(A_{11}^{-1}A_{12}\vec{x}_2 + A_{11}^{-1}B_{11}\vec{y}_1 + A_{11}^{-1}B_{12}\vec{y}_2 + A_{11}^{-1}\vec{F}_1(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2)\right), \tag{3}$$

and, taking into account Eq. (3), the second expression is given by

$$D\vec{x}_2 = -\left(\left(B_{21} - A_{21}A_{11}^{-1}B_{11}\right)\vec{y}_1 + \left(B_{22} - A_{21}A_{11}^{-1}B_{12}\right)\vec{y}_2 + \vec{F}_2(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2) - A_{21}A_{11}^{-1}\vec{F}_1(\vec{x}_1,\vec{x}_2;\vec{y}_1,\vec{y}_2)\right), \tag{4}$$

where $D = A_{22} - A_{21}A_{11}^{-1}A_{12}$. One can be proved that the rank of matrix D is zero, and thus all the elements of this matrix vanish. The system of equations (4) is essentially simplified in this case, so it takes the form

$$(B_{21} - A_1^{-1}B_1)\vec{y}_1 + (B_2 - A_1^{-1}B_{12})\vec{y}_2 + \vec{F}_2(\vec{x}_1, \vec{x}_2; \vec{y}_1, \vec{y}_2) - A_1^{-1}\vec{F}_1(\vec{x}_1, \vec{x}_2; \vec{y}_1, \vec{y}_2) = 0. \quad (5)$$

One should consider that, as a result of solving the system of nonlinear equations (3) using the converging iterative procedure, we come to the dependence $\vec{x}_1 = \vec{x}_1(\vec{x}_2; \vec{y}_1, \vec{y}_2)$, after substitution of which in Eq. (5), one can write the system of nonlinear equations regarding the variables $\vec{x}_2$ or $x_{r+1},\dots,x_n$ in the case of a complete set of variables. Solutions of nonlinear system (5) with respect to these variables are some functions of parameters $y_1,\dots,y_m$, i.e.

$$x_{r+1} = x_{r+1}(y_1,\dots,y_m),\ x_{r+2} = x_{r+2}(y_1,\dots,y_m),\ \dots,\ x_n = x_n(y_1,\dots,y_m).$$

Keeping the small values up to the second order of smallness with respect to variables $\vec{x}_2$ and parameters $\vec{y}$, the system of equations (5) can be represented in a form without explicit expressions for coefficients a_{ij} , b_j , c_j , d_j , that is

$$\sum_{j=1}^{n-r} a_{ij}\, x_{r+i}x_{r+j} + \sum_{j=1}^{m} b_j\, x_{r+i}y_j + \sum_{j=1}^{m} c_j\, y_i y_j + \sum_{j=1}^{m} d_j\, y_j = 0,\ i = 1,\dots,n-r. \quad (6)$$

Since $|x_i| < 1$ and $|y_i| < 1$, then the relations $|y_i y_j| < (|y_i|, |y_j|)$ and $|x_{r+i} y_j| < |y_i|$ are valid, and, taking into account only the main part in the asymptotic representation, the system of equations (6) can be rewritten accordingly as

$$\sum_{j=1}^{n-r} a_{ij}\, x_{r+i}x_{r+j} + \sum_{j=1}^{m} d_j\, y_j = 0,\ i = 1,\dots,n-r. \quad (7)$$

Based on the structure of the system of equations (7), we arrive at an important conclusion on the nature of the dependence on parameters for the solutions of this system. First of all, we note that if the quantities $x_{r+i} = x_{r+i}(y_1,\dots,y_m)$ ($i = 1,\dots,n-r$) satisfy the system of equations (7), then the same system has the solutions $x_{r+i} = -x_{r+i}(y_1,\dots,y_m)$, and thus the bifurcation of solutions of the system of equations occurs at zero values of parameters $y_1,\dots,y_m$. Along with this, one can mention that the dependencies $x_{r+i} = x_{r+i}(y_1,\dots,y_m)$ are not analytical regarding the parameters, i.e. these solutions cannot be represented in the form of power series expansion in terms of the integer powers of parameters $y_1,\dots,y_m$. Indeed, Eq. (7) is transformed to the identities regarding the parameters $y_1,\dots,y_m$ by subsequent substitution of these expansions into the equation, and, under the assumption of the fulfillment of conditions with respect to these identities, we come to a conclusion that all coefficients d_{ij} in Eq. (7) must vanish.

The system of equations (7) can be represented for convenience in an equivalent form in some cases to analyze the solutions of this system

$$\sum_{j=1}^{n-r} a_{ij}\, x_{r+j} = -\frac{b_i(\vec{y})}{x_{r+i}},\ b_i(\vec{y}) = \sum_{j=1}^{m} d_j\, y_j,\ i = 1,\dots,n-r \quad (8)$$

or in the form of a system of homogeneous linear equations

$$\sum_{j=1}^{n-r} \tilde{a}_{ij} x_{r+j} = 0\,,\ \tilde{a}_{ii} = a_i - \frac{b_i(\vec{y})}{x_{r+i}^2}\,,\ \tilde{a}_{ij} = a_j\ \ (i \neq j),\ i = 1,\ldots,n-r\,. \tag{9}$$

For the existence of nonzero solutions of the system of equations (9), the determinant of this system must vanish, i.e.

$$\begin{vmatrix} a_{11} - \dfrac{b_1(\vec{y})}{x_{r+1}^2} & a_{12} & \cdots & a_{1,n-r} \\ a_{21} & a_{22} - \dfrac{b_2(\vec{y})}{x_{r+2}^2} & \cdots & a_{2,n-r} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-r,1} & a_{n-r,2} & \cdots & a_{n-r,n-r} - \dfrac{b_{n-r}(\vec{y})}{x_n^2} \end{vmatrix} = 0\,. \tag{10}$$

Eq. (10) represents a condition for determining the range of values for the parameters of the system under consideration, for which there are solutions of the system of equations (7) (or Eqs. (8)).

Let us turn further to a more detailed analysis of the characteristics of solutions of the nonlinear system of equations (7) based on parameters $y_1,\ldots,y_m$ for systems defined by Eq. (7) depending only on one parameter y. In this case, Eq. (7) is written as

$$\sum_{j=1}^{n-r} a_j\, x_{r+i} x_{r+j} + d_i y = 0\,,\ i = 1,\ldots,n-r\,. \tag{11}$$

In accordance with the non-analytical character of the dependencies $x_{r+i} = x_{r+i}(y)$ ($i = 1,\ldots,n-r$) on parameter y, we investigate these dependencies in the form of power series

$$x_{r+i} = x_{r+i}(y) = \alpha_{i1} y^{\varepsilon_{j1}} + \alpha_{i2} y^{\varepsilon_{j2}} + \ldots,\ \varepsilon_{i1} < \varepsilon_{i2} < \ldots,\ i = 1,\ldots,n-r\,. \tag{12}$$

After substitution of Eq. (12) into Eq. (11), the system of algebraic equations (11) turns into an identity regarding the parameter y, based on the fulfillment of which we come to the conditions for determining the coefficients $\varepsilon_{i1}, \varepsilon_{i2}, \ldots$ and $\alpha_{i1}, \alpha_{i2}, \ldots$ In particular, as a result of an analysis of the expressions obtained in this way, we find $\varepsilon_{i1} = \frac{1}{2}$, $\alpha_{ij} = 0$ $(j \geq 2)$, $i = 1,\ldots,n-r$, while the values of the coefficients α_{i1} are determined by solving the system of equations no longer depending on the parameter y, that is $\sum_{j=1}^{n-r} a_{ij} \alpha_{i1} \alpha_{j1} + d_i = 0$, $i = 1,\ldots,n-r$.

Literature:

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