Convergence estimates for some abstract second order differential equations in Hilbert spaces

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Abstract

In a real Hilbert space H we consider the following perturbed Cauchy problem

$$\begin{cases} \varepsilon\, u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B(u_{\varepsilon\delta}(t)) = f(t), t \in (0,T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases} \tag{$P_{\varepsilon\delta}$}$$

where $u_0, u_1 \in H$, $f : [0,T] \mapsto H$ and ε, δ are two small parameters, A is a linear self-adjoint operator, B is a locally Lipschitz and monotone operator.

We study the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(P_{\varepsilon\delta})$ in two different cases:

(i) when $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$;

(ii) when $\varepsilon \to 0$ and $\delta \to 0$.

We establish that the solution to the unperturbed problem has a singular behavior, relative to the parameters, in the neighborhood of $t = 0$. We show the boundary layer and boundary layer function in both cases.

Keywords: Singular perturbation; abstract second order Cauchy problem; boundary layer function; a priori estimate.

Let H and V be two real Hilbert spaces endowed with norms $|\cdot|$ and $||\cdot||$, respectively. Denote by $(\cdot,\cdot)$ the scalar product in H.

The framework of our studying is determined by the following conditions:

(H) *$V \subset H$ densely and continuously, i.e.*

$$||u|| \geq \omega_0\, |u|, \quad \forall u \in V, \quad \omega_0 > 0.$$

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(HA) *$A : D(A) = V \mapsto H$ is a linear, self-adjoint and positive definite operator, i.e.*

$$(Au, u) \geq \omega\, |u|^2, \quad \forall u \in V, \quad \omega > 0.$$

(HB1) *Operator $B : D(B) \subseteq H \to H$ is $A^{1/2}$ locally Lipschitz, i. e. $D(A^{1/2}) \subset D(B)$ and for every $R > 0$ there exists $L(R) \geq 0$ such that*

$$|B(u_1) - B(u_2)| \leq L(R)|A^{1/2}(u_1 - u_2)|, \forall u_i \in D(A^{1/2}), |A^{1/2}u_i| \leq R;$$

(HB2) *Operator B is the Fréchet derivative of some convex and positive functional $\mathcal{B}$ with $D(A^{1/2}) \subset D(\mathcal{B})$.*
(HB3) *Operator B possesses the Fréchet derivative B' in $D(A^{1/2})$ and there exists constant $L_1(R) \geq 0$ such that*

$$\big|\big(B'(u_1) - B'(u_2)\big)v\big| \leq L_1(R)\,\big|A^{1/2}(u_1 - u_2)\big|\,|A^{1/2}v|, \ \forall u_i,\, v \in D(A^{1/2}),$$

$$|A^{1/2}u_i| \leq R, \quad i = 1, 2.$$

The hypothesis **(HB2)** implies, in particular, that operator B is monotone and verifies condition

$$\frac{d}{dt}\mathcal{B}(u(t)) = \big(B(u(t)), u'(t)\big), \quad \forall t \in [a, b] \subset \mathbb{R},$$

in the case when $u \in C([a, b], D(A^{1/2})) \cap C^1([a, b], H)$.

Consider the following perturbed Cauchy problem

$$\begin{cases} \varepsilon\, u''_{\varepsilon\delta}(t) + \delta\, u'_{\varepsilon\delta}(t) + Au_{\varepsilon\delta}(t) + B(u_{\varepsilon\delta}(t)) = f(t), t \in (0, T), \\ u_{\varepsilon\delta}(0) = u_0, \quad u'_{\varepsilon\delta}(0) = u_1, \end{cases} \qquad (P_{\varepsilon\delta})$$

where $u_0, u_1 \in H$, $f : [0, T] \mapsto H$ and ε, δ are two small parameters.

We study the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(P_{\varepsilon\delta})$ in two different cases:
(i) $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$, relative to the following unperturbed system:

$$\begin{cases} \delta l'_\delta(t) + Al_\delta(t) + B(l_\delta(t)) = f(t), \quad t \in (0, T), \\ l_\delta(0) = u_0, \end{cases} \qquad (P_\delta)$$

(ii) $\varepsilon \to 0$ and $\delta \to 0$, relative to the following unperturbed problem:

$$Av(t) + B(v(t)) = f(t), \quad t \in [0,T). \qquad (P_0)$$

The problem $(P_{\varepsilon\delta})$ is the abstract model of singularly perturbed problems of hyperbolic-parabolic type in the case (i) and of hyperbolic-parabolic-elliptic type in the case (ii). Such kind of problems arises in the mathematical modeling of elasto-plasticity phenomena. These abstract results can be applied to singularly perturbed problems of hyperbolic-parabolic-elliptic type with stationary part defined by strongly elliptic operators.

For the case $\delta \geq \delta_0 > 0$, in conditions **(H)**, **(HA)**, **(HB1)**, **(HB2)** and **(HB3)**, using results obtained in [1], we obtain some *a priori* estimates of solutions to the perturbed problem, which are uniform with respect to parameters, and get the relationship between the solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ) emphasised in the following inequalities

$$||u_{\varepsilon\delta} - l_\delta||_{C([0,T];H)} + ||A^{1/2}u_{\varepsilon\delta} - A^{1/2}l_\delta||_{L^2(0,T;H)} \leq C\,\varepsilon^\beta,$$

$$||u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta}e^{-\delta^2 t/\varepsilon}||_{C([0,T];H)} + \big|\big|A^{1/2}\big(u'_{\varepsilon\delta} - l'_\delta + H_{\varepsilon\delta}e^{-\delta^2 t/\varepsilon}\big)\big|\big|_{L^2(0,T;H)} \leq$$

$$\leq C\,\varepsilon^\beta, \forall \varepsilon \in (0,\varepsilon_0], T > 0, p > 1,$$

where $u_{\varepsilon\delta}$ and l_δ are strong solutions to problems $(P_{\varepsilon\delta})$ and (P_δ) respectively, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega, T, \delta_0, u_0, u_1, f) \in (0,1)$, $\beta = \min\{1/4,\ (p-1)/2p\}$, $C = C(T, p, \delta_0, \omega_0, \omega, u_0, u_1, H_{\varepsilon\delta}, f) > 0$,

$$H_{\varepsilon\delta} = \delta^{-1}f(0) - u_1 - \delta^{-1}Au_0 - \delta^{-1}B(u_0).$$

For the case $\varepsilon \to 0, \delta \to 0$, in conditions **(H)**, **(HA)**, **(HB1)** and **(HB2)**, in [2] we established the relationship between the solutions to the problems $(P_{\varepsilon\delta})$ and (P_δ) emphasised in the following inequality

$$||u_{\varepsilon\delta} - v - h_\delta||_{C([0,T];H)} \leq C\,\Theta(\varepsilon,\delta),\ \forall \varepsilon \in (0,\varepsilon_0],\ \forall \delta \in (0,1],$$

where $u_{\varepsilon\delta}$ and v are strong solutions to problems $(P_{\varepsilon\delta})$ and (P_0) respectively, $\varepsilon_0 = \varepsilon_0(\omega_0, \omega, T, \delta_0, u_0, u_1, f) \in (0,1)$, $\beta = \min\{1/4,\ (p-1)/2p\}$,

$C = C(T, p, \omega_0, \omega, u_0, u_1, f) > 0$, the function h_δ is the solution to the problem

$$\begin{cases} \delta h_\delta'(t) + Ah_\delta(t) + B(l_\delta(t)) - B\big(v(t)\big) = 0, \quad t \in (0,T), \\ h_\delta(0) = u_0 - (A+B)^{-1} f(0), \end{cases}$$

$$\Theta(\varepsilon, \delta) = \frac{\varepsilon^{1/4}}{\delta^{7/4+1/p}} + \sqrt{\delta}.$$

From the last three inequalities we can state that the solution to the unperturbed problem has a singular behavior, relative to the parameters, in the neighborhood of $t = 0$ in both cases.

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References

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