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Limits of solutions to the semilinear wave equation with small parameter

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Abstract. We study the existence of the limits of solution to singularly perturbed initial boundary value problem of hyperbolic - parabolic type with boundary Dirichlet condition for the semilinear wave equation. We prove the convergence of solutions and also the convergence of gradients of solutions to perturbed problem to the corresponding solutions to the unperturbed problem as the small parameter tends to zero. We show that the derivatives of solution relative to time-variable possess the boundary layer function of the exponential type in the neighborhood of $t = 0$.

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1 Introduction

Let $\Omega \in \mathbb{R}^3$ be an open and bounded set with the smooth boundary $\partial\Omega$. Consider the following initial boundary value problem for the wave equation, which in what follows will be called (P_ε):

$$\begin{cases} \varepsilon u_{tt}(x,t) + u_t(x,t) - \Delta u(x,t) + u^3(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \\ u(x,t)\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases}$$

where ε is a small positive parameter.

We will study the behaviour of the solutions to the problem (P_ε) as $\varepsilon \to 0$. It is natural to expect that the solutions to the problem (P_ε) tend to the corresponding solutions to the unperturbed problem (P_0):

$$\begin{cases} v_t(x,t) - \Delta v(x,t) + v^3(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ v(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ v(x,t)\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases}$$

as $\varepsilon \to 0$. The main results are contained in Theorem 5. Under some conditions on u_0, u_1 and f we will prove that

$$u \to v \quad \text{in} \quad C([0,T]; L^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0, \tag{1}$$

$$u \to v \quad \text{in} \quad L^\infty(0,T;H_0^1(\Omega)), \quad \text{as} \quad \varepsilon \to 0, \tag{2}$$

$$u' - v' - \alpha e^{-t/\varepsilon} \to 0 \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0, \tag{3}$$

where $\alpha = f(0) - u_1 + \Delta u_0 - u_0^3$. The relationship (3) shows that the derivative u' has the singular behaviour relative to the small values of the parameter ε in neighborhood of the set $\{(x,t)|x \in \Omega, t = 0\}$. It means that the set $\{(x,t)|x \in \Omega, t = 0\}$ is *the boundary layer* for u' and the function α is *the boundary layer function* for u'. The proofs of the relations (1), (2) and (3) are based on two key points. The first one is the relationship between the solutions to the problem (P_0) and (P_ε) in the linear case (see Lemma 3 and Theorem 3). The second key point represents *apriori* estimates of solutions to the problem (P_ε), which are uniform relative to small parameter ε (see Lemma 2).

The singularly perturbed nonlinear problems of hyperbolic-parabolic type were studied by many authors. Without pretending to the complete list of the works in this area, we mention here only the works [1] - [6] in which the larger references can be found.

In that follows we need to use some notations. Let X be a Banach space. For $k \in \mathbb{N}$, $p \in [1,\infty)$ and $(a,b) \subset (-\infty,+\infty)$ we denote by $W^{k,p}(a,b;X)$ the usual Sobolev spaces of the vectorial distributions $W^{k,p}(a,b;X) = \{f \in D'(a,b,X); f^{(l)} \in L^p(a,b;X), l = 0,1,\dots,k\}$ equipped with the norm

$$||f||_{W^{k,p}(a,b;X)} = \Big(\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;X)}\Big)^{1/p}.$$

For each $k \in \mathbb{N}, W^{k,\infty}(a,b;X)$ is the Banach space equipped with the norm

$$||f||_{W^{k,\infty}(a,b;X)} = \max_{0 \le l \le k} ||f^{(l)}||_{L^\infty(a,b;X)}.$$

In the following for $k \in \mathbb{N}$ we denote by $H^k(\Omega)$ $(L^2(\Omega) = H^0(\Omega))$ the usual real Hilbert spaces equipped with the following scalar products and norms:

$$\Big(u,v\Big)_{H^k(\Omega)} = \int_\Omega \sum_{|\alpha| \le k} \partial^\alpha u(x) \partial^\alpha v(x) dx, \quad [u,v] = (u,v)_{H_0^1(\Omega)},$$

$$(u,v) = \int_\Omega u(x)v(x)dx, \quad |u| = ||u||_{L^2(\Omega)}, \quad ||u|| = ||u||_{H_0^1(\Omega)}.$$

By $H^{-k}(\Omega)$ we denote the dual space to $H^k(\Omega)$, i.e. $H^{-k}(\Omega) = (H_0^k(\Omega))'$. We will write $\langle \cdot, \cdot \rangle$ to denote the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

2 Solvability of the problems $(\mathbf{P}_\varepsilon)$ and $(\mathbf{P_0})$

First of all we shall remind the definitions of solutions to the problems (P_ε) and (P_0) and also the existence theorems for solutions to these problems.

Definition 1. *We say a function* $u \in L^2(0,T;H_0^1(\Omega))$ *with* $u' \in L^2(0,T;L^2(\Omega))$, $u'' \in L^2(0,T:H^{-1}(\Omega))$ *is a solution to the problem* (P_ε) *provided*

$$\varepsilon\langle u''(t),\eta\rangle + (u'(t),\eta) + [u(t),\eta] + (u^3(t),\eta) = (f(t),\eta), \quad \forall \eta \in H_0^1(\Omega), \tag{4}$$

a.e. $t \in [0,T]$ *and*

$$u(0) = u_0, \quad u'(0) = u_1. \tag{5}$$

Definition 2. *We say a function* $v \in L^2(0,T;H_0^1(\Omega))$ *with* $v' \in L^2(0,T;H^{-1}(\Omega))$ *is a solution to the problem* (P_0) *provided*

$$\langle v'(t),\eta\rangle + [v(t),\eta] + (v^3(t),\eta) = (f(t),\eta), \quad \forall \eta \in H_0^1(\Omega), \tag{6}$$

a.e. $t \in [0,T]$ *and*

$$v(0) = u_0. \tag{7}$$

Remark 1. *In view of the conditions* $u \in L^2(0,T;H_0^1(\Omega))$, $u' \in L^2(0,T;L^2(\Omega))$, *and* $u'' \in L^2(0,T;H^{-1}(\Omega))$ *we have* $u \in C([0,T];L^2(\Omega))$ *and* $u' \in C([0,T];H^{-1}(\Omega))$. *Consequently, we will understand the equalities (5) in the following sense:* $|u(t)-u_0| \to 0, ||u'(t)-u_1||_{H^{-1}(\Omega)} \to 0$ *as* $t \to 0$. *Similarly, in view of the conditions* $v \in L^2(0,T;H_0^1(\Omega))$ *with* $v' \in L^2(0,T:H^{-1}(\Omega))$, *we have* $v \in C([0,T];L^2(\Omega))$, *consequently, we will understand the equality (7) in the following sense:* $|v(t)-u_0| \to 0$ *as* $t \to 0$.

Theorem 1 [7]. *Let* $T > 0$. *If* $f \in W^{1,1}(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, *then there exists a unique solution to the problem* (P_ε) *such that* $u \in W^{1,\infty}(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$, $u'' \in L^\infty(0,T;L^2(\Omega))$, $u''' \in L^\infty(0,T;H^{-1}(\Omega))$.

Theorem 2 [8]. *Let* $T > 0$. *If* $f \in W^{1,1}(0,T;L^2(\Omega))$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, *then there exists a unique solution* $v \in W^{1,\infty}(0,T;L^2(\Omega))$ *to the problem* (P_0) *and the estimates*

$$|v(t)| + \Big(\int_0^t ||v(\tau)||^2 d\tau\Big)^{1/2} + \Big(\int_0^t |v^2(\tau)|^2 d\tau\Big)^{1/2} \leq$$

$$\leq |u_0| + \int_0^t |f(\tau)| d\tau, \quad \forall t \in [0,T], \tag{8}$$

$$||v'||_{L^\infty(0,t;L^2(\Omega)} + \Big(\int_0^t ||v'(\tau)||^2 d\tau\Big)^{1/2} + \Big(\int_0^t (v'^2(\tau), v^2(\tau)) d\tau\Big)^{1/2} \leq$$

$$\leq |\Delta u_0 + f(0) - u_0^3| + \int_0^t |f'(\tau)| d\tau, \quad \forall t \in [0,T], \tag{9}$$

are true.

Remark 2. *If* $f \in W^{1,1}(0,T;L^2(\Omega)), u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, *then according to the conclusion of Theorem 1 in fact* $u \in C^1([0,T];L^2(\Omega)) \cap$

$C([0,T];H_0^1(\Omega))$. Consequently, the term $\varepsilon\langle u''(t),\eta\rangle$ in (4) can be expressed in the form $\varepsilon(u''(t),\eta)$ and we will understand the equalities (5) in the following sense: $||u(t)-u_0||\to 0, |u'(t)-u_1|\to 0$ as $t\to 0$. Similarly, in view of the conclusion of Theorem 2, $v\in C([0,T];L^2(\Omega)), v'\in L^\infty(0,T;L^2(\Omega))$, the term $\langle v'(t),\eta\rangle$ in (6) can be expressed in the form $(v'(t),\eta)$.

3 Apriori estimates for solutions to the problem $(\mathbf{P}_\varepsilon)$

In this section we shall prove an *apriori* estimates for the solutions to the problem (P_ε) which are uniform relative to the small values of parameter ε. Before proving the estimates for the solutions to problem (P_ε) we recall the following well-known lemma.

Lemma 1 (see for example [9]). *Let $\psi\in L^1(a,b)(-\infty<a<b<\infty)$ with $\psi\geq 0$ a. e. on (a,b) and let c be a fixed real constant. If $h\in C([a,b])$ verifies*

$$\frac{1}{2}h^2(t)\leq\frac{1}{2}c^2+\int_a^t\psi(s)h(s)ds,\quad\forall t\in[a,b],$$

then

$$|h(t)|\leq|c|+\int_a^t\psi(s)ds,\quad\forall t\in[a,b],$$

also holds.

Denote by $u(t)=u(t,\cdot)$,

$$E_0(u,t)=\varepsilon|u'(t)|^2+|u(t)|^2+||u(t)||^2+2(1-\varepsilon)\int_0^t|u'(\tau)|^2d\tau+$$

$$+2\varepsilon(u(t),u'(t))+2\int_0^t||u(\tau)||^2d\tau+2\int_0^t|u^2(\tau)|^2d\tau+\frac{1}{2}|u^2(t)|^2$$

and

$$E_1(u,t)=\varepsilon^2|u'(t)|^2+\frac{1}{2}|u(t)|^2+\varepsilon||u(t))||^2+\varepsilon\Big(u'(t),u(t)\Big)+$$

$$+\varepsilon\int_0^t|u'(\tau)|^2d\tau+\int_0^t||u(\tau)||^2d\tau.$$

Lemma 2. *Let $f\in W^{1,1}(0,\infty;L^2(\Omega))$, $u_0\in H_0^1(\Omega)\cap H^2(\Omega)$, $u_1\in H_0^1(\Omega)$. Then there exists the positive constant $C=C(\Omega)$ such that for any solution u to the problem (P_ε) the following estimates*

$$E_0^{1/2}(u,t)\leq CM_0,\quad t\in[0,\infty),\quad 0<\varepsilon<1,\tag{10}$$

$$E_1^{1/2}(u',t)\leq CM_1,\quad a.e.\quad t\in[0,\infty),\quad 0<\varepsilon\leq 1/2,\tag{11}$$

hold, where

$$M_0=M_0(||u_0||,|u_1|,||f||_{W^{1,1}(0,\infty;L^2(\Omega))}),\quad M_0(0,0,0)=0,$$

$$M_1 = M_1(||u_0||_{H^2(\Omega)}, ||u_1||, ||f||_{W^{1,1}(0,\infty;L^2(\Omega))}), \quad M_1(0,0,0) = 0. \tag{12}$$

If in addition $f \in W^{2,1}(0,\infty;L^2(\Omega))$, $u_1, \alpha \in H_0^1(\Omega) \cap H^2(\Omega)$, *then there exists* $\varepsilon_0 = \varepsilon_0(\Omega, M_0) \in (0,1)$ *such that the function*

$$z(t) = u'(t) + \alpha e^{-t/\varepsilon} \tag{13}$$

satisfies the estimate

$$||z||_{L^\infty(0,\infty;H_0^1(\Omega))} + ||z'||_{L^\infty(0,\infty;L^2(\Omega))} + ||z||_{W^{1,2}(0,\infty;H_0^1(\Omega))} \le CM_2, \tag{14}$$

for $0 < \varepsilon \le \varepsilon_0$, *where*

$$\alpha = f(0) - u_1 + \Delta u_0 - u_0^3, \tag{15}$$

$$M_2 = M_2(||f||_{W^{2,1}(0,\infty;L^2(\Omega))}, ||u_1||_{H^2(\Omega)}, ||\alpha||_{H^2(\Omega)}), \quad M_2(0,0,0) = 0. \tag{16}$$

Proof. In what follows let us agree to denote all constants depending only on Ω by the same constant C. The direct computations show that for every solution to the problem (P_ε) the following equality

$$\frac{d}{dt}E_0(u,t) = 2\Big(f(t), u(t) + u'(t)\Big), \quad a.e. \quad t \in [0,\infty) \tag{17}$$

is fulfilled. For $\varepsilon \in (0,1)$ we have that $E_0(u,t) \ge 0$ and $|u(t)| \le (E_0(t,u))^{1/2}$. Then integrating the equality (17) on $(0,t)$ we get

$$E_0(u,t) = E_0(u,0) + 2\Big(f(t) - f(0), u(t)\Big) + 2\Big(f(0), u(t) - u(0)\Big) +$$

$$+2\int_0^t \Big(f(\tau) - f'(\tau), u(\tau)\Big)d\tau \le E_0(u,0) + \frac{1}{2}|u(t)|^2 + 8\Big(\int_0^t |f'(\tau)|d\tau\Big)^2 +$$

$$+|u_0|^2 + 9|f(0)|^2 + 2\int_0^t \Big(|f(\tau)| + |f'(\tau)|\Big)E_0^{1/2}(u,\tau)d\tau, \quad t \in [0,\infty).$$

From the last inequality we have that

$$E_0(u,t) \le 2(c_0+5)^2 M_0^2 +$$

$$+4\int_0^t \Big(|f(\tau)| + |f'(\tau)|\Big)E_0^{1/2}(u,\tau)d\tau\Big), \quad t \in [0,\infty), \tag{18}$$

where c_0 is the constant from the inequality $|u|^2 \le c_0||u||^2, u \in H_0^1(\Omega)$. Since $E_0(u,t) \in C([0,\infty))$ due to Lemma 1, from (18) the estimate (10) follows.

To prove the estimate (11) let us denote by $u_h(t) = h^{-1}(u(t+h) - u(t))$, $h > 0$. For any solution of the problem (P_ε) the equality

$$\frac{d}{dt}E_1(u_h,t) = \Big(F_h(t), 2\varepsilon u_h'(t) + u_h(t)\Big), \quad a.e. \quad t \in [0,\infty),$$

is true, where

$$F_h(t) = f_h(t) - u_h(t)\Big(u^2(t+h) + u(t+h)u(t) + u^2(t)\Big).$$

Integrating the last equality on $(0,t)$, we obtain

$$E_1(u_h, t) = E_1(u_h, 0) + \int_0^t \Big(F_h(\tau), 2\varepsilon u_h'(\tau) + u_h(\tau)\Big) d\tau, \quad t \in [0, \infty).$$

As $|u_h(\tau) + 2\varepsilon u_h'(\tau)| \le 2E_1^{1/2}(u_h, \tau)$, then from the last equality we get

$$E_1(u_h, t) \le E_1(u_h, 0) + 2\int_0^t |F_h(\tau)| E_1^{1/2}(u_h, \tau) d\tau, \quad t \in [0, \infty).$$

Using Lemma 1, from the last inequality we obtain the estimate

$$E_1^{1/2}(u_h, t) \le E_1^{1/2}(u_h, 0) + \int_0^t |F_h(\tau)| d\tau, \quad t \in [0, \infty). \tag{19}$$

Since for $1 \le p < \infty, k \in \mathbb{N}$ and $u \in W^{1,p}(0, T; H^k(\Omega))$ the inequality

$$\int_0^t ||u_h(\tau)||^p_{H^k(\Omega)} d\tau \le \int_0^t ||u'(\tau)||^p_{H^k(\Omega)} d\tau, \quad t \in [0, \infty), \tag{20}$$

is true, then we obtain

$$\begin{aligned}\int_0^t |F_h(\tau)| d\tau \le \int_0^t |f'(\tau)| d\tau + 2\Big(\int_0^t |u'(\tau)|^2 d\tau\Big)^{1/2} \Big[\Big(\int_0^t |u^2(\tau+h)|^2 d\tau\Big)^{1/2} + \\ + \Big(\int_0^t |u^2(\tau)|^2 d\tau\Big)^{1/2}\Big] \le M_0 + (1-\varepsilon)^{-1} E_0(u,t) \le \\ \le CM_0\Big(1 + M_0\Big), \quad t \in [0, \infty), \quad 0 < \varepsilon \le 1/2.\end{aligned} \tag{21}$$

As $u'(0) = u_1, \varepsilon u''(0) = f(0) - u_1 + \Delta u_0 - u_0^3$, and $|u_0^3| \le 4\sqrt{3}||u_0||^3$, then

$$E_1^{1/2}(u', 0) \le C\Big(M_0 + M_0^3 + ||u_0||_{H^2(\Omega)} + ||u_1||\Big). \tag{22}$$

Using the estimates (21), (22) and passing to the limit in the inequality (19) as $h \to 0$ we obtain the estimate (11).

Now let us prove the estimate (14). Under the conditions on f, u_0 and u_1 we have that $z \in W^{1,\infty}(0, \infty; H_0^1(\Omega)) \cap L^\infty(0, \infty; H^2(\Omega))$, $z'' \in L^\infty(0, \infty; L^2(\Omega))$ and z is the solution to the problem

$$\begin{cases} \varepsilon\Big(z''(t), \eta\Big) + \Big(z'(t), \eta\Big) + [z(t), \eta] + 3\Big(u^2(t)z(t), \eta\Big) = \\ = \Big(f_1(t, \varepsilon), \eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0, \infty), \\ z(0) = z_0, \quad z'(0) = 0, \end{cases} \tag{23}$$

where

$$f_1(t,\varepsilon) = f'(t) + \Big(3u^2(t)\alpha - \Delta\alpha\Big)e^{-t/\varepsilon}, \quad z_0 = f(0) - u_0^3 + \Delta u_0.$$

Denote by

$$E_2(z,t) = \varepsilon^2|z'(t)|^2 + \frac{1}{2}|z(t)|^2 + \varepsilon||z(t))||^2 + \varepsilon\Big(z'(t), z(t)\Big) +$$

$$+\varepsilon\int_0^t |z'(\tau)|^2 d\tau + \int_0^t ||z(\tau)||^2 d\tau + 3\int_0^t \Big(u^2(\tau)z(\tau), z(\tau)\Big)d\tau.$$

For the solution z to the problem (23) we have

$$\frac{d}{dt}E_2(z,t) = \Big(f_1(t,\varepsilon), z(t) + 2\varepsilon z'(t)\Big) - 6\varepsilon\Big(z'(t), u^2(t)z(t)\Big), \quad a.e. \quad t \in (0,\infty).$$

Integrating the last equality on $(0,t)$ we obtain

$$E_2(z,t) = E_2(z,0) + \int_0^t \Big(f_1(\tau,\varepsilon), z(\tau) + 2\varepsilon z'(\tau)\Big)d\tau -$$

$$-6\varepsilon\int_0^t \Big(z'(\tau), u^2(\tau)z(\tau)\Big)d\tau, \quad t \in [0,\infty). \tag{24}$$

Using Holder's inequality, the estimate (10) and the inequality

$$||z||_{L^6(\Omega)} \le \gamma||z||, \quad \forall z \in H_0^1(\Omega), \quad \gamma = (48)^{1/6}, \tag{25}$$

we get the estimate

$$\Big|\Big(z'(\tau), u^2(\tau)z(\tau)\Big)\Big| \le |z'(\tau)|||z(\tau)||_{L^6(\Omega)}||u(\tau)||^3_{L^6(\Omega)} \le CM_0^3|z'(\tau)|||z(\tau)||,$$

from which it follows that

$$6\varepsilon\Big|\int_0^t \Big(z'(\tau), u^2(\tau)z(\tau)\Big)d\tau\Big| \le \frac{\varepsilon}{2}\int_0^t |z'(\tau)|^2 d\tau +$$

$$+CM_0^6\varepsilon\int_0^t ||z(\tau||^2 d\tau \le \frac{1}{2}E_2(z,t), \quad t \in [0,\infty), \quad 0 < \varepsilon \le \varepsilon_0, \tag{26}$$

where $\varepsilon_0 = min\{1/2, (2C)^{-1}M_0^{-6}\}$. As $|z(\tau) + 2\varepsilon z'(\tau)| \le 2E_2^{1/2}(z,\tau)$, then due to Lemma 1 from (24) and (26) follows the estimate

$$E_2^{1/2}(z,t) \le 2E_2^{1/2}(z,0) + 2\int_0^t |f_1(\tau,\varepsilon)|d\tau, \quad t \in [0,\infty), \quad 0 < \varepsilon \le \varepsilon_0. \tag{27}$$

The inequality $|u^2(\tau)\alpha| \le \gamma^4||\alpha||||u(\tau)||^2$ permits to get the estimate

$$\int_0^t |f_1(\tau,\varepsilon)|d\tau \le M_0 + CM_0^2||\alpha|| + ||\alpha||_{H^2(\Omega)}, \; t \in [0,\infty), \; 0 < \varepsilon < 1. \tag{28}$$

As

$$E_2^{1/2}(z,0) \leq C||f(0) - u_0^3 + \Delta u_0|| \leq CM_1, \quad 0 < \varepsilon < 1,$$

then from (27) and (28) follows the estimate

$$E_2^{1/2}(z,t) \leq CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \leq \varepsilon_0. \tag{29}$$

Further, if z is a solution to the problem (23), then the function $z_h(t) = h^{-1}(z(t+h) - z(t)), h > 0$ is the solution to the problem

$$\begin{cases} \varepsilon\Big(z_h''(t), \eta\Big) + \Big(z_h'(t), \eta\Big) + [z_h(t), \eta] + 3\Big(u^2(t)z_h(t), \eta\Big) = \\ = \Big(G_h(t,\varepsilon), \eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0,\infty), \\ z_h(0) = z_{0h}, \quad z_h'(0) = z_{1h}, \end{cases} \tag{30}$$

where

$$G_h(t,\varepsilon) = f_{1h}(t,\varepsilon) - 3u_h(t)z(t+h)\big(u(t+h) + u(t)\big),$$
$$z_{0h} = h^{-1}(z(h) - z_0), \quad z_{1h} = h^{-1}z'(h).$$

In exactly the same way as the inequality (27) was obtained we get the inequality

$$E_2^{1/2}(z_h,t) \leq 2E_2^{1/2}(z_h,0) + 2\int_0^t |G_h(\tau,\varepsilon)|d\tau, \; t \in [0,\infty), \; 0 < \varepsilon \leq \varepsilon_0. \tag{31}$$

As $u'(t) = z(t) - \alpha e^{-t/\varepsilon}$, then using Holder's inequality, the inequalities (25), (20) and the estimates (10), (29) we obtain

$$\int_0^t |u_h(\tau)z(\tau+h)\big(u(\tau+h) + u(\tau)\big)|d\tau \leq$$

$$\leq \int_0^t ||u_h(\tau)||_{L^6(\Omega)}||z(\tau+h)||_{L^6(\Omega)}\Big(||u(\tau+h)||_{L^6(\Omega)} + ||u(\tau)||_{L^6(\Omega)}\Big)d\tau \leq$$

$$\leq CM_0\int_0^t ||u_h(\tau)||||z(\tau+h)||d\tau \leq$$

$$\leq CM_0E_1^{1/2}(u',t)E_2^{1/2}(z,t+h) \leq CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \leq \varepsilon_0. \tag{32}$$

Using Holder's inequality, the inequalitis (20), (25) and the estimates (10), (11) we will estimate f_{1h} as follows

$$\int_0^t |f_{1h}(\tau)|d\tau \leq \int_0^t |f_1'(\tau,\varepsilon)d\tau \leq \int_0^t |f''(\tau)|d\tau +$$

$$+\frac{1}{\varepsilon}\int_0^t e^{-\tau/\varepsilon}(|\Delta\alpha| + 3|\alpha u^2(\tau)|)d\tau + 6\int_0^t e^{-\tau/\varepsilon}|\alpha u(\tau)u'(\tau)|d\tau \leq$$

$$\leq M_2 + C||\alpha||_{L^6(\Omega)}\int_0^t e^{-\tau/\varepsilon}||u(\tau)||_{L^6(\Omega)}\Big(\frac{1}{\varepsilon}||u(\tau)||_{L^6(\Omega)} +$$

$$+||u'(\tau)||_{L^6(\Omega)}\Big)d\tau \le CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \le 1/2. \tag{33}$$

The estimates (32) and (33) imply the following estimate for G_h

$$\int_0^t |G_h(\tau)|d\tau \le CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \le \varepsilon_0. \tag{34}$$

As

$$E_2^{1/2}(z',0) = |f'(0) + \Delta u_1 - 3u_0^2 u_1| \le CM_2, \tag{35}$$

then, using the estimates (34), (35) and passing to the limit in the inequality (31) as $h \to 0$, we obtain the estimate

$$E_2^{1/2}(z',t) \le CM_2, \quad t \in (0,\infty), \quad 0 < \varepsilon \le \varepsilon_0. \tag{36}$$

From (29) and (36) follows the estimate

$$||z||_{W^{1,\infty}(0,\infty;L^2(\Omega))} + ||z||_{W^{1,2}(0,\infty;H_0^1(\Omega))} \le CM_2, \quad 0 < \varepsilon \le \varepsilon_0. \tag{37}$$

Finally, let us estimate $||z||_{L^\infty(0,\infty;H_0^1(\Omega))}$. To this end we denote by

$$E_3(z,t) = \varepsilon|z'(t)|^2 + |z(t)|^2 + ||z(t)||^2 + 2(1-\varepsilon)\int_0^t |z'(\tau)|^2 d\tau +$$

$$+2\varepsilon(z(t),z'(t)) + 2\int_0^t ||z(\tau)||^2 d\tau + 6\int_0^t \Big(u^2(\tau), z^2(\tau)\Big)d\tau + 3\Big(u^2(t), z^2(t)\Big).$$

If z is a solution to the problem (23), then

$$\frac{d}{dt}E_3(z,t) = 2\Big(f_1(t,\varepsilon), z(t) + z'(t)\Big) + 6\Big(u(t)u'(t), z^2(t)\Big), \quad a.e. \quad t \in (0,\infty).$$

Integrating the last equality on $(0,t)$, similarly as the inequality (18) was obtained, we get

$$E_3(z,t) \le C\Big(E_3(z,0) + ||f_1'||^2_{L^1(0,\infty;L^2(\Omega))} + \int_0^t \Big|\Big(u(\tau)u'(\tau), z^2(\tau)\Big)\Big|d\tau +$$

$$+\int_0^t \Big(|f_1(\tau,\varepsilon)| + |f_1'(\tau,\varepsilon)|\Big)E_3^{1/2}(z,\tau)d\tau\Big), \quad t \in [0,\infty), \quad 0 < \varepsilon < 1. \tag{38}$$

In the obvious way we obtain the estimate

$$E_3(z,0) + ||f_1'||^2_{L^1(0,\infty;L^2(\Omega))} \le CM_2. \tag{39}$$

Using Holder's inequality, the inequality (25) and estimates (10), (11), (29), we get the estimate

$$\int_0^t \Big|\Big(u(\tau)u'(\tau), z^2(\tau)\Big)\Big|d\tau \le \int_0^t |u'(\tau)| ||z(\tau)||^2_{L^6(\Omega)} ||u(\tau)||_{L^6(\Omega)} d\tau \le$$

$$\leq \gamma^3 \int_0^t |u'(\tau)|||z(\tau)||^2||u(\tau)||d\tau \leq$$

$$\leq CM_1M_0E_2(z,t) \leq CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \leq \varepsilon_0. \tag{40}$$

Due to Lemma 1 from (38), (39) and (40) follows the estimate

$$||z||_{L^\infty(0,t;H_0^1(\Omega))} \leq E_3^{1/2}(z,t) \leq C\Big(M_2 + \int_0^t (|f_1(\tau,\varepsilon)| + |f_1'(\tau,\varepsilon)|)d\tau\Big) \leq$$

$$\leq CM_2, \quad t \in [0,\infty), \quad 0 < \varepsilon \leq \varepsilon_0. \tag{41}$$

The estimates (37), (41) imply (14). Lemma 2 is proved.

Corollary 1. *If $f \in W^{2,1}(0,\infty;L^2(\Omega)), u_1, f(0) - u_0^3 + \Delta u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, then, in fact, the function $u(t)$ satisfies the estimates*

$$||u||_{W^{1,p}(0,\infty;H_0^1(\Omega))} \leq CM_2, \quad 0 < \varepsilon \leq \varepsilon_0, \quad p \in [2,\infty], \tag{42}$$

$$\varepsilon||u''||_{L^\infty(0,\infty;L^2(\Omega))} + ||\Delta u||_{L^\infty(0,\infty;L^2(\Omega))} \leq CM_2, \quad 0 < \varepsilon \leq \varepsilon_0. \tag{43}$$

4 Relationship between solutions to the problems ($\mathbf{P}_\varepsilon$) and ($\mathbf{P_0}$) in the linear case

In this section we shall give the relationship between solutions to the problem (P_ε) and (P_0) in the linear case, i. e. in the case when the term u^3 in the problems (P_ε) and (P_0) is missing. This relation was inspired by the work [10]. At first we shall give some properties of the kernel $K(t,\tau,\varepsilon)$ of transformation which realizes this connection.

For $\varepsilon > 0$ denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}d\eta.$$

Lemma 3 [11]. *The function $K(t,\tau,\varepsilon)$ possesses the following properties*:

(i) *For any fixed $\varepsilon > 0$ $K \in C(\{t \geq 0\} \times \{\tau \geq 0\}) \cap C^\infty(R_+ \times R_+)$;*

(ii) $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad t > 0, \tau > 0;$

(iii) $\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad t \geq 0;$

(iv) $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}, \quad \tau \geq 0;$

(v) *For each fixed* $t > 0$, $s, q \in \mathbb{N}$ *there exist constants* $C_1(s,q,t,\varepsilon) > 0$ *and* $C_2(s,q,t) > 0$ *such that*

$$|\partial_t^s \partial_\tau^q K(t,\tau,\varepsilon)| \leq C_1(s,q,t,\varepsilon)\exp\{-C_2(s,q,t)\tau/\varepsilon\}, \quad \tau > 0;$$

(vi) $K(t,\tau,\varepsilon) > 0, \quad t \geq 0, \quad \tau \geq 0;$

(vii) *Let* ε *be fixed,* $0 < \varepsilon \ll 1$ *and* H *be a Hilbert space. For any* $\varphi : [0,\infty) \to H$ *continuous on* $[0,\infty)$ *such that* $|\varphi(t)| \leq M\exp\{Ct\}$, $t \geq 0$, *the relationship*

$$\lim_{t\to 0}\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau,$$

is valid in H;

(viii) $\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad t \geq 0;$

(ix)

$$\int_0^\infty K(t,\tau,\varepsilon)|t-\tau|^q d\tau \leq C\varepsilon^{q/2}\big(1+t^{q/2}\big), \quad q \in [0,1].$$

(x) *Let* $f \in W^{1,\infty}(0,\infty;H)$. *Then there exists positive constant* C *such that*

$$\Big|\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big|\Big|_H \leq C\sqrt{\varepsilon}(1+\sqrt{t})\|f'\|_{L^\infty(0,\infty;H)}, \quad t \geq 0.$$

(xi) *There exists* $C > 0$ *such that*

$$\int_0^t\int_0^\infty K(\tau,\theta,\varepsilon)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Theorem 3. *Suppose that* $f \in L^\infty(0,\infty;L^2(\Omega))$ and $u \in W^{2,\infty}(0,\infty;L^2(\Omega)) \cap L^\infty(0,\infty;H_0^1(\Omega))$ *is the solution to the problem*:

$$\varepsilon(u''(t),\eta) + (u'(t),\eta) + [u(t),\eta] = (f(t),\eta), \quad \forall \eta \in H_0^1(\Omega), \tag{44}$$

a.e. $t \in [0,\infty)$,

$$u(0) = u_0, \quad u'(0) = u_1. \tag{45}$$

Then the function v_0 *which is defined by*

$$v_0(t) = \int_0^\infty K(t,\tau,\varepsilon)u(\tau)d\tau \tag{46}$$

is the solution to the problem

$$(v_0'(t),\eta)+[v_0(t),\eta]=(f_0(t,\varepsilon),\eta),\quad \forall\eta\in H_0^1(\Omega),\quad \forall t>0, \tag{47}$$

$$v_0=\varphi_\varepsilon, \tag{48}$$

where

$$f_0(t,\varepsilon)=F_0(t,\varepsilon)+\int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau,$$

$$F_0(t,\varepsilon)=\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1,\quad \varphi_\varepsilon=\int_0^\infty e^{-\tau}u(2\varepsilon\tau)d\tau.$$

Moreover, $v_0\in W^{2,\infty}(0,\infty;L^2(\Omega))\cap L^\infty(0,\infty;H_0^1(\Omega))$.

Proof. As u is the solution to the problem (44), (45) and $u,u',u''\in L^\infty(0,\infty;L^2(\Omega))$, then $v_0\in W^{2,\infty}(0,\infty;L^2(\Omega))\cap L^\infty(0,\infty;H_0^1(\Omega))$, $u\in C^1([0,\infty);L^2(\Omega))$ and $|u(t)-u_0|\to 0,|u'(t)-u_1|\to 0$ as $t\to 0$. Therefore, integrating by parts and using the properties (**i**) - (**iii**) and (**v**) of Lemma 3, we get

$$(v_0'(t),\eta)=\Big(\int_0^\infty K_t(t,\tau,\varepsilon)u(\tau)d\tau,\eta\Big)=$$

$$\Big(\int_0^\infty\Big(\varepsilon K_{\tau\tau}(t,\tau,\varepsilon)-K_\tau(t,\tau,\varepsilon)\Big)u(\tau)d\tau,\eta\Big)=$$

$$=\Big(\int_0^\infty K(t,\tau,\varepsilon)\Big(\varepsilon u''(\tau)+u'(\tau)\Big)d\tau+\varepsilon K(t,0,\varepsilon)u_1,\eta\Big)=$$

$$=\Big(f_0(t,\varepsilon)u_1,\eta\Big)-[v(t),\eta],\quad \forall\eta\in H_0^1(\Omega),\quad \forall t>0.$$

Thus $v_0(t)$, which is defined by (46) satisfies the equation (47). From property (**vii**) of Lemma 3 the validity of the initial condition (48) follows. Thus Theorem 3 is proved.

5 Limits of solutions to the problem $(\mathbf{P}_\varepsilon)$ as $\varepsilon\to\mathbf{0}$

In this section we shall study the behavior of solutions to the problem (P_ε) as $\varepsilon\to 0$.

Theorem 4. *Suppose that* $f\in W^{2,1}(0,\infty;L^2(\Omega)),u_0,u_1,f(0)-u_0^3+\Delta u_0\in H_0^1(\Omega)\cap H^2(\Omega)$*, then there exist constants* $C=C(\Omega)$ *and* $\varepsilon_0=\varepsilon_0(\Omega,M_0)$ *such that the following estimates*

$$||u-v||_{C([0,t];L^2(\Omega))}\le CM_2\big(t^{3/2}+1\big)\sqrt{\varepsilon},\quad t\ge 0,\quad 0<\varepsilon\le\varepsilon_0, \tag{49}$$

$$||u-v||_{L^\infty(0,t;H_0^1(\Omega))}\le CM_2\big(1+t^{3/2}\big)\varepsilon^{1/4},\quad t\ge 0,\quad 0<\varepsilon\le\varepsilon_0, \tag{50}$$

are fulfilled. If in addition $f\in W^{2,\infty}(0,\infty;L^2(\Omega))$*, then*

$$||u'-v'-\alpha he^{-t/\varepsilon}||_{L^\infty(0,t;L^2(\Omega))}\le CM_3\varepsilon^{1/4}(1+t^{5/2}),\quad t\ge 0,\ 0<\varepsilon\le\varepsilon_0, \tag{51}$$

is fulfilled, where u is a solution to the problem (P_ε), v is the solution to the problem (P_0) and $M_3 = M_2 + ||f''||_{L^\infty(0,\infty;L^2(\Omega))}$.

Proof. *Proof of estimate (49).* If u is the solution to the problem (P_ε), then according to Theorem 3 the function

$$w(t) = \int_0^\infty K(t,\tau,\varepsilon)u(\tau)d\tau$$

is the solution to the problem

$$\begin{cases} \Big(w'(t),\eta\Big) + [w(t),\eta] = \Big(F(t,\varepsilon),\eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad t>0, \\ w(0) = \varphi_\varepsilon, \end{cases} \tag{52}$$

where

$$F(t,\varepsilon) = F_0(t,\varepsilon) + \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau.$$

Using the estimates (11), (41) and properties (**viii**) and (**x**) from Lemma 3 we obtain the following estimates

$$|u(t)-w(t)| \le C\sqrt{\varepsilon}(1+\sqrt{t})||u'||_{L^\infty(0,\infty;L^2(\Omega))} \le$$

$$\le CM_1\sqrt{\varepsilon}(1+\sqrt{t}), \quad t\ge 0, \quad 0<\varepsilon\le 1/2, \tag{53}$$

and

$$||u(t)-w(t)|| \le CM_2\varepsilon^{1/2}(1+\sqrt{t}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{54}$$

Therefore, as $||w(t)|| \le M_0$, then

$$|u^3(t)-w^3(t)| \le C||u(t)-w(t)||_{L^6(\Omega)}\Big(||u(t)||^2_{L^6(\Omega)} + ||w(t)||^2_{L^6(\Omega)}\Big) \le$$

$$\le C\gamma^3||u(t)-w(t)||||u(t)||^2 \le CM_2\sqrt{\varepsilon}(1+\sqrt{t}), \quad t\ge 0,\ 0<\varepsilon\le\varepsilon_0. \tag{55}$$

Denote by $y(t) = v(t) - w(t)$, where v is the solution to the problem (P_0) and w is the solution to the problem (52). Then the function y is the solution to the following problem:

$$\begin{cases} \Big(y'(t),\eta\Big) + [y(t),\eta] + \Big((v^2(t)+v(t)w(t)+w^2(t))y(t),\eta\Big) = \\ = \Big(F_1(t,\varepsilon),\eta\Big), \quad \forall\eta\in H_0^1(\Omega), \quad t>0, \\ y(0) = u_0 - \varphi_\varepsilon, \end{cases} \tag{56}$$

where

$$F_1(t,\varepsilon) = f(t) - F_0(t,\varepsilon) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau + \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau - w^3(t),$$

Due to the estimate (8) (in the linear case) for the function y we get the estimate

$$|y(t)| \leq |y(0)| + \int_0^t |F_1(\tau,\varepsilon)|d\tau, \quad t \geq 0. \tag{57}$$

From the estimate (11) it follows that

$$|y(0)| \leq \int_0^\infty e^{-\tau}|u_0 - u(2\varepsilon\tau)|d\tau \leq$$

$$\leq \int_0^\infty e^{-\tau} \int_0^{2\varepsilon\tau} |u'(s)|ds d\tau \leq C\varepsilon ||u'||_{L^\infty(0,\infty;L^2(\Omega))} \leq C\varepsilon M_1. \tag{58}$$

As $q(s) = e^{s^2}\lambda(s) \leq C$, for $s \in [0,\infty]$ then

$$\int_0^t |F_0(\tau,\varepsilon)|d\tau \leq C|u_1| \int_0^t \Big[\exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big) + \lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\Big]d\tau =$$

$$= C|u_1| \int_0^t \exp\Big\{-\frac{\tau}{4\varepsilon}\Big\}\Big[q\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big) + q\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\Big]d\tau \leq C\varepsilon|u_1|, \quad t \geq 0. \tag{59}$$

Using the properties (**viii**) and (**x**) from Lemma 3, we have

$$\int_0^t \Big|f(\tau) - \int_0^\infty K(\tau,s,\varepsilon)f(s)ds\Big|d\tau \leq$$

$$\leq C\sqrt{\varepsilon}||f'||_{L^\infty(0,\infty:L^2(\Omega))}(1+t^{3/2}) \leq C\sqrt{\varepsilon}M_2(1+t^{3/2}), \quad t \geq 0. \tag{60}$$

Let us evaluate the difference

$$I(t) = w^3(t) - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau.$$

Due to inequality (25) and the estimates (10), (41), we get

$$|(u^3(s))'| = 3|u'(s)u^2(s)| \leq 3||u'(s)||_{L^6(\Omega)}||u(s)||^2_{L^6(\Omega)} \leq$$

$$\leq 3\gamma^3||u'(s)||||u(s)||^2 \leq CM_2, \quad s \in [0,\infty), \quad 0 < \varepsilon \leq \varepsilon_0,$$

and, consequently,

$$\Big|u^3(t) - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau\Big| \leq CM_2\sqrt{\varepsilon}(1+\sqrt{t}), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Hence, from the last estimate and (55), we get

$$|I(t)| \leq |w^3(t) - u^3(t)| + \Big|u^3(t) - \int_0^\infty K(t,\tau,\varepsilon)u^3(\tau)d\tau\Big| \leq$$

$$\leq CM_2\sqrt{\varepsilon}\big(1+\sqrt{t}\big), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0.$$

Therefore

$$\int_0^t |I(\tau)|d\tau \le CM_2\sqrt{\varepsilon}\big(1+t^{3/2}\big), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{61}$$

Gathering the estimates (59), (60) and (61), we have

$$\int_0^t |F_1(\tau,\varepsilon)|d\tau \le CM_2\sqrt{\varepsilon}(1+t^{3/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0.$$

Using the last estimate and (58), from (57) follows the estimate

$$|y(t)| \le CM_2\sqrt{\varepsilon}(1+t^{3/2}), \quad t\ge 0, \quad 0<\varepsilon\le\varepsilon_0. \tag{62}$$

Finally, the estimates (53) and (62) involve (49).

Proof of estimate (50). To prove the estimate (50) we have to evaluate $||y||_{L^\infty(0,\infty;H_0^1(\Omega))}$. To this end we observe that due to (9) for y' is true the estimate

$$|y'(t)| \le |Y_0| + \int_0^t |F_1'(\tau,\varepsilon) - a'(\tau)y(\tau)|d\tau, \quad t\in[0,\infty), \tag{63}$$

where $a(t) = v^2(t) + v(t)w(t) + w^2(t)$, $Y_0 = \Delta y(0) + F_1(0,\varepsilon) - a(0)y(0)$. Using the estimate (43), we get

$$|\Delta y(0)| \le CM_2. \tag{64}$$

Due to the inequalities (11) and (25) we obtain that

$$|F_1(0,\varepsilon)| \le C\Big(||f||_{W^{1,1}(0,\infty;L^2(\Omega))} + |u_1| + ||u_0||^3\Big) \le CM_2 \tag{65}$$

and

$$|a(0)y(0)| = |u_0^3 - \varphi_\varepsilon^3| \le CM_2. \tag{66}$$

The estimates (64), (65) and (66) imply

$$|Y_0| \le CM_2. \tag{67}$$

As

$$\varepsilon K_\tau(t,\tau,\varepsilon) - K(t,\tau,\varepsilon) = -\frac{3}{4\varepsilon\sqrt{\pi}}\Big(K_1(t,\tau,\varepsilon) - K_2(t,\tau,\varepsilon)\Big)$$

and

$$\int_0^\infty \Big(K_1(t,\tau,\varepsilon) + K_2(t,\tau,\varepsilon)\Big)d\tau \le C\varepsilon\Big(\lambda\big(-\frac{1}{2}\sqrt{t/\varepsilon}\big) + e^{3t/4\varepsilon}\lambda\big(\sqrt{t/\varepsilon}\big)\Big) \le C\varepsilon,$$

then for $f\in W^{1,\infty}(0,\infty;L^2(\Omega))$, due to the properties (**ii**), (**iii**) and (**v**) from Lemma 3, we obtain the estimate

$$\Big|\int_0^\infty K_t(t,\tau,\varepsilon)f(\tau)d\tau\Big| = \Big|\int_0^\infty \big(\varepsilon K_\tau(t,\tau,\varepsilon) - K(t,\tau,\varepsilon)\big)f'(\tau)d\tau\Big| \le$$

$$\leq C\varepsilon^{-1}\int_0^\infty \big(K(t,\tau,\varepsilon)+K_2(t,\tau,\varepsilon)\big)|f'(\tau)|d\tau \leq$$
$$\leq C||f||_{W^{1,\infty}(0,\infty;L^2(\Omega))} \leq CM_2. \tag{68}$$

Similarly, using the inequalities (25), (42) and the properties (**ii**), (**iii**), (**v**) and (**viii**) from Lemma 3, we obtain the estimates

$$\Big|\int_0^\infty K_t(t,\tau,\varepsilon)u^3(\tau)d\tau\Big| \leq C||u||_{W^{1,\infty}(0,\infty;H_0^1(\Omega))} \leq CM_2, \quad 0<\varepsilon\leq\varepsilon_0, \tag{69}$$

$$|(w^3)'(t)| \leq C||w'(t)||||w(t)||^2 \leq$$
$$\leq CM_0\Big|\Big|\int_0^\infty K_t(t,\tau,\varepsilon)u(\tau)d\tau\Big|\Big| \leq CM_2 \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{70}$$

By direct computation we can show that

$$\int_0^\infty |F_0'(\tau,\varepsilon)|d\tau \leq C|u_1| \leq CM_2, \quad t\in[0,\infty). \tag{71}$$

The estimates (68), (69), (70) and (71) involve the estimate

$$\int_0^t |F_1'(\tau,\varepsilon)|d\tau \leq CM_2, \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{72}$$

Thanks to the inequality (25) and Holder's inequality we have

$$|a'(t)y(t)| \leq C\Big(||v(t)||||v'(t)|| + ||v'(t)||||w(t)|| +$$
$$+||v(t)||||w'(t)|| + ||w(t)||||w'(t)||\Big)||y(t)||, \quad t\in[0,\infty).$$

Therefore, the estimates (8), (9), (62) and (33) imply

$$\int_0^t |a'(\tau)y(\tau)|d\tau \leq CM_2(1+t^{3/2}), \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{73}$$

Using the estimates (67), (72) and (73), from (63) we obtain

$$|y'(t)| \leq CM_2(1+t^{3/2}), \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty). \tag{74}$$

As

$$(y'(t),y(t)) + ||y(t)||^2 + (a(t),y(t)) = (F_1(t,\varepsilon),y(t))$$
$$|F_1(t,\varepsilon)| \leq CM_2, \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty),$$

then due to (74) we get

$$||y(t)||^2 \leq |y(t)|\big(|F_1(t)|+|y'(t)|\big) \leq CM_2\sqrt{\varepsilon}(1+t^{3/2})^2, \quad 0<\varepsilon\leq\varepsilon_0, \quad t\in[0,\infty).$$

From the last estimates and (54) the estimate (50) follows.

Proof of estimate (51). Denote by $z(t) = u'(t) + \alpha e^{-t/\varepsilon}$, where α is defined in (13). Then $z(t)$ is a solution to the problem (23). According to Theorem 3 the function $w_1(t)$, which is defined as

$$w_1(t) = \int_0^\infty K(t,\tau,\varepsilon) z(\tau) d\tau$$

is a solution to the problem

$$\begin{cases} (w_1'(t),\eta) + [w_1(t),\eta] = (\mathcal{F}(t,\varepsilon),\eta), \quad \forall \eta \in H_0^1(\Omega), \quad t > 0, \\ w_1(0) = w_{1\varepsilon}, \end{cases}$$

where

$$\mathcal{F}(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon) f_1(\tau,\varepsilon) d\tau - 3\int_0^\infty K(t,\tau,\varepsilon) u^2(\tau) z(\tau) d\tau.$$

$$w_{1\varepsilon} = \int_0^\infty e^{-\tau} z(2\varepsilon\tau) d\tau.$$

Using the estimates (11), (14) and properties (**viii**), (**ix**) and (**x**) from Lemma 3, similarly as the estimates (53) and (54) were obtained, we obtain the following estimates

$$\begin{aligned} |z(t) - w_1(t)| &\le C\sqrt{\varepsilon}(1+\sqrt{t})||z'||_{L^\infty(0,\infty;L^2(\Omega))} \le \\ &\le CM_2\sqrt{\varepsilon}(1+\sqrt{t}), \quad t \ge 0, \quad 0 < \varepsilon \le \varepsilon_0, \end{aligned} \tag{75}$$

and

$$\begin{aligned} ||z(t) - w_1(t)|| &\le \int_0^\infty K(t,\tau,\varepsilon) \Big| \int_\tau^t ||z'(s)|| ds \Big| d\tau \le \\ &\le ||z'(t)||_{L^2(0,\infty;H_0^1(\Omega))} \int_0^\infty K(t,\tau,\varepsilon) |t-\tau|^{1/2} d\tau \le \\ &\le CM_2 \varepsilon^{1/4}(1+t^{1/4}), \quad t \ge 0, \quad 0 < \varepsilon \le \varepsilon_0. \end{aligned} \tag{76}$$

Under the conditions on f, u_0 and u_1, if v is a solution to the problem (P_0), then the function $v_1(t) = v'(t)$ is a solution to the problem

$$\begin{cases} \Big(v_1'(t),\eta\Big) + [v_1(t),\eta] + 3\Big(v^2(t)v_1(t),\eta\Big) = \\ = \Big(f'(t),\eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0,\infty), \\ v_1(0) = z_0. \end{cases}$$

Denote by $R(t) = w_1(t) - v_1(t)$. Then the function R is a solution to the problem

$$\begin{cases} \Big(R'(t),\eta\Big) + [R(t),\eta] + 3\Big(v^2(t)R(t),\eta\Big) = \\ = \Big(\mathcal{F}_1(t,\varepsilon),\eta\Big), \quad \forall \eta \in H_0^1(\Omega), \quad a.e. \quad t \in (0,\infty), \\ R(0) = w_{1\varepsilon} - z_0, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \mathcal{F}(t,\varepsilon) + 3v^2(t)w_1(t) - f'(t).$$

For the function R the estimate

$$|R(t)| \le |R(0)| + \int_0^t |\mathcal{F}_1(\tau,\varepsilon)|d\tau, \quad t \ge 0, \tag{77}$$

holds. Using the estimate (14) we get

$$|R(0)| \le \int_0^\infty e^{-\tau}|z_0 - z(2\varepsilon\tau)|d\tau \le C\varepsilon||z'||_{L^\infty(0,\infty;L^2(\Omega))} \le C\varepsilon M_2. \tag{78}$$

To estimate the second term of the right-hand side of (77) we will present $\mathcal{F}_1(t,\varepsilon)$ in the following form:

$$\mathcal{F}_1(t,\varepsilon) = I_1(t,\varepsilon) + I_2(t,\varepsilon) + 3I_3(t,\varepsilon),$$

where

$$I_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)f'(\tau)d\tau - f'(t),$$

$$I_2(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\Big(3u^2(\tau)\alpha - \Delta\alpha\Big)e^{-\tau/\varepsilon}d\tau,$$

$$I_3(t,\varepsilon) = v^2(t)w_1(t) - \int_0^\infty K(t,\tau,\varepsilon)u^2(\tau)z(\tau)d\tau.$$

Using the properties (**viii**) and (**x**) from Lemma 3, we obtain the estimate

$$|I_1(t,\varepsilon)| \le C||f''||_{L^\infty(0,\infty;L^2(\Omega))}\sqrt{\varepsilon}(1+\sqrt{t}), \quad t \ge 0, \quad \varepsilon > 0$$

and, consequently,

$$\int_0^t |I_1(\tau,\varepsilon)|d\tau \le C||f''||_{L^\infty(0,\infty;L^2(\Omega))}\sqrt{\varepsilon}(1+t^{3/2}), \quad t \ge 0, \quad \varepsilon > 0. \tag{79}$$

In view of the inequality (25) and the estimate (10) we have

$$|u^2(t)\alpha| \le ||u(t)||^2_{L^6(\Omega)}||\alpha||_{L^6(\Omega)} \le \gamma^3||u(t)||^2||\alpha|| \le CM_2, \quad t \ge 0.$$

Therefore the property (**xi**) from Lemma 3 permits to estimate $I_2(t,\varepsilon)$

$$\int_0^t |I_2(\tau,\varepsilon)|d\tau \le CM_2\int_0^t\int_0^\infty K(\tau,s,\varepsilon)e^{-s/\varepsilon}dsd\tau \le$$

$$\le CM_2\varepsilon, \quad t \ge 0, \quad 0 < \varepsilon \le \varepsilon_0. \tag{80}$$

Further, using the inequalities (25), (50) and (76), we get

$$|v^2(t)w_1(t) - u^2(t)z(t)| \le C\Big(||w_1(t)||||u(t) - v(t)||(||u(t)|| + ||v(t)||)+$$

$$+||w_1(t) - z(t)||||u(t)||^2\Big) \leq CM_2\Big(||u(t) - v(t)|| + ||w_1(t) - z(t)||\Big) \leq$$

$$\leq CM_2\varepsilon^{1/4}(1 + t^{3/2}), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0. \tag{81}$$

Using the estimates (10), (14), (42) and property (**ix**) from Lemma 3, we obtain the following estimate

$$\int_0^\infty K(t, \tau, \varepsilon)\big|u^2(\tau)z(\tau) - u^2(t)z(t)\big|d\tau \leq$$

$$\leq \int_0^\infty K(t, \tau, \varepsilon)\Big|\int_\tau^t |2u(s)u'(s)z(s) + u^2(s)z'(s)|ds\Big|d\tau \leq$$

$$\leq CM_2 \int_0^\infty K(t, \tau, \varepsilon)\Big|\int_\tau^t (1 + |z'(s)|)ds\Big|d\tau \leq$$

$$\leq CM_2\varepsilon^{1/4}(1 + t^{1/4}), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0. \tag{82}$$

The estimates (81), (82) imply

$$\int_0^t |I_3(\tau)|d\tau \leq CM_2\varepsilon^{1/4}(1 + t^{5/2}), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0.$$

From the last estimate and (79), (80) it follows that

$$\int_0^t |\mathcal{F}_1(\tau, \varepsilon)|d\tau \leq C\mathcal{M}_2\varepsilon^{1/4}(1 + t^{5/2}), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0. \tag{83}$$

From (77), due to (78) and (83) it follows that

$$|R(t)| \leq C\mathcal{M}_2\varepsilon^{1/4}(1 + t^{5/2}), \quad t \geq 0, \quad 0 < \varepsilon \leq \varepsilon_0.$$

The last estimate and (75) imply the estimate (51). Theorem 4 is proved.

Theorem 5. *Let $T > 0$. Suppose that $f \in W^{2,1}(0, T; L^2(\Omega)), u_0, u_1, f(0) - u_0^3 + \Delta u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, then there exist constants $C = C(\Omega, T)$ and $\varepsilon_0 = \varepsilon_0(\Omega, M_0)$ such that the following estimates*

$$||u - v||_{C([0,T];L^2(\Omega))} \leq C\mathcal{M}_2\sqrt{\varepsilon}, \quad 0 < \varepsilon \leq \varepsilon_0, \tag{84}$$

$$||u - v||_{L^\infty(0,T;H_0^1(\Omega))} \leq C\mathcal{M}_2\varepsilon^{1/4}, \quad 0 < \varepsilon \leq \varepsilon_0, \tag{85}$$

are fulfilled. If in addition $f \in W^{2,\infty}(0, T; L^2(\Omega))$, then

$$||u' - v' - \alpha h e^{-t/\varepsilon}||_{L^\infty(0,T;L^2(\Omega))} \leq C\mathcal{M}_3\varepsilon^{1/4}, \quad 0 < \varepsilon \leq \varepsilon_0, \tag{86}$$

is fulfilled, where u is the solution to the problem (P_ε), v is the solution to the problem (P_0). The constants $\mathcal{M}_i, i = 1, 2, 3$, depend on the same values as M_i, the difference being that the norms $||f||_{W^{k,l}(0,\infty;L^2(\Omega))}$ in M_i are replaced with the norms $||f||_{W^{k,l}(0,T;L^2(\Omega))}$.

Proof. For any function $f \in W^{2,l}(0,T;L^2(\Omega)), l \in [1,\infty]$, there exits the extension $\tilde{f} \in W^{2,l}(0,\infty;L^2(\Omega))$ such that

$$||\tilde{f}||_{W^{2,l}(0,\infty;L^2(\Omega))} \leq C(T)||\tilde{f}||_{W^{2,l}(0,T;L^2(\Omega))} \tag{87}$$

(see, for instance, [12]). If $\tilde{u}$ is the solution to the problem (4), (5) with the same initial conditions u_0, u_1 and the right-hand side $\tilde{f}$, then according to Theorem 1 we have that $u(t) = \tilde{u}(t)$ for $t \in [0,T]$. Similarly, if $\tilde{v}$ is a solution to the problem (6), (7) with the same initial condition u_0 and the right-hand side $\tilde{f}$, then according to Theorem 2 we have that $v(t) = \tilde{v}(t)$ for $t \in [0,T]$.

Consequently, using the estimates (49), (50), (51) and (87) we obtain the estimates (84), (85) and (86). Theorem 5 is proved.

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