Proceedings of the Third Conference of Mathematical Society of Moldova
IMCS-50, August 19-23, 2014, Chisinau, Republic of Moldova

On the holomorph of π-quasigroups of type T_1

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Abstract

Quasigroups satisfying the identity $x\cdot(x\cdot(x\cdot y)) = y$ are called π-quasigroups of type T_1. Necessary and sufficient conditions for the holomorph of a π-quasigroup of type T_1 to be a π-quasigroup of type T_1 are established. Also, it is proved that the left (right) multiplication group of a π-quasigroup of type T_1 is isomorphic to some normal subgroup of the left (right) multiplication group of its holomorph, respectively.

Keywords: π-quasigroup of type T_1, holomorph, multiplication group, normal subgroup.

Quasigroups satisfying identities of length five, with two variables, are called π-quasigroups. V. Belousov [1] and, independently F. Bennett [2], have given a classification of minimal identities, consisting of seven identities. The general form of an identity of length five with two variables in a quasigroup (Q, A) is:

$$^{\alpha}A(x,^{\beta} A(x,^{\gamma} A(x, y))) = y, \quad (1)$$

where $\alpha, \beta, \gamma \in S_3$. In this case, the tuple $[\alpha, \beta, \gamma]$ is called the type of the identity (1), and the quasigroup (Q, A) which satisfies (1), is called a π-quasigroup of type $T = [\alpha, \beta, \gamma]$. Using the notations from [1], a quasigroup $(Q, \cdot)$ which satisfies the identity

$$x \cdot (x \cdot (x \cdot y)) = y \quad (2)$$

is called a π-quasigroup of type $T_1 = [\varepsilon, \varepsilon, \varepsilon]$, where ε is the identity of S_Q. If $(Q, \cdot)$ is a quasigroup, then the groupoid $(Hol(Q, \cdot), \circ)$, where $Hol(Q, \cdot) = Aut(Q, \cdot) \times Q$ and

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, \beta(x) \cdot y),$$

for $\forall(\alpha, x), (\beta, y) \in Hol(Q, \cdot)$, is called the holomorph of the quasigroup $(Q, \cdot)$ [3]. From the definition it follows that the holomorph of a quasigroup is a quasigroup. Moreover, the mapping $Q \mapsto Hol(Q, \cdot), x \mapsto (\varepsilon, x)$, is an embedding of the quasigroup $(Q, \cdot)$ into its holomorph $(Hol(Q, \cdot), \circ)$. Thus, denoting $Q_1 = \{(\varepsilon, x)|x \in Q\}$, we obtain $(Q, \cdot) \cong (Q_1, \circ)$. If the quasigroup $(Q, \cdot)$ has a right or a left unit, then the mapping $Aut(Q, \cdot) \mapsto Hol(Q, \cdot), \alpha \mapsto (\alpha, e)$ is an embedding, too. In particular, if $(Q, \cdot)$ is a group with an abelian group of automorphisms, then $(Hol(Q, \cdot), \circ)$ is a group, $(Q_1, \circ) \triangleleft Hol(Q, \cdot)$, where $Q_1 = \{(\varepsilon, x)|x \in Q\}$, and $Hol(Q, \cdot)/(Q_1, \circ) \cong Aut(Q, \cdot)$. This isomorphism is given by the surjection:

$$\xi : Hol(Q, \cdot) \to Aut(Q, \cdot), \xi(\varphi, x) = \varphi,$$

for which $Ker\xi = (Q_1, \circ)$.

Proposition 1. *The holomorph of a $\pi-$quasigroup of type T_1 is a $\pi-$quasigroup of type T_1, if and only if the following conditions hold:*
$1)\alpha^3 = \varepsilon, \forall\alpha \in Aut(Q, \cdot)$;
$2)x \cdot (\alpha^2(x) \cdot (\alpha(x) \cdot y)) = y, \forall\alpha \in Aut(Q, \cdot), \forall x, y \in Q$.

Proof. If $(Q, \cdot)$ is a $\pi-$quasigroup of type T_1 and $(\alpha, x), (\beta, y) \in Hol(Q, \cdot)$, then: $(\alpha, x) \circ ((\alpha, x) \circ ((\alpha, x) \circ (\beta, y))) = (\alpha, x) \circ ((\alpha, x) \circ (\alpha\beta, \beta(x) \cdot y)) = (\alpha, x) \circ (\alpha^2\beta, \alpha\beta(x) \cdot (\beta(x) \cdot y)) = (\alpha^3\beta, \alpha^2\beta(x) \cdot (\alpha\beta(x) \cdot (\beta(x) \cdot y))) = (\beta, y)$, hence:

$$\begin{cases} \alpha^3 = \varepsilon \\ \alpha^2\beta(x) \cdot (\alpha\beta(x) \cdot (\beta(x) \cdot y)) = y. \end{cases}$$

Making the replacement $x \mapsto \beta^{-1}\alpha(x)$, and using the equality $\alpha^3 = \varepsilon$, the second relation implies:

$$x \cdot (\alpha^2(x) \cdot (\alpha(x) \cdot y)) = y,$$

for $\forall x, y \in Q, \forall\alpha \in Aut(Q, \cdot)$.

$\square$

Remark 1. The holomorph of a $\pi-$quasigroup of type T_1 with a trivial group of automorphisms is a $\pi-$quasigroup of type T_1. This fact is used below to obtain $\pi-$quasigroups of type T_1, the holomorphs of which are from the same class.

Example. The quasigroup $(Q,\cdot)$, where $Q=\{1,2,3\}$, given by the left translations $L_1=(132)$, $L_2=(123)$, $L_3=\varepsilon$, is a $\pi-$quasigroup of type T_1 with $Aut(Q,\cdot)=\{\varepsilon\}$, so its holomorph $Hol(Q,\cdot)$ is a $\pi-$quasigroup of type T_1 as well.

Remark 2. 1. If $L^{(\circ)}_{(\beta,b)}$ and $R^{(\circ)}_{(\beta,b)}$ are the left and, respectively, the right translations with (β,b) in the holomorph $(Hol(Q,\cdot),\circ)$, then, for every $(\alpha,a)\in(Hol(Q,\cdot)$, we have:

$$L^{(\circ)-1}_{(\beta,b)}(\alpha,a)=(\beta^{-1}\alpha,\beta^{-1}\alpha(b)\backslash a)=L^{(\circ)}_{(\beta^{-1},b_1)}(\alpha,a),$$

where $\alpha(b_1)\cdot a=\beta^{-1}\alpha(b)\backslash a)$ and, respectively,

$$R^{(\circ)-1}_{(\beta,b)}(\alpha,a)=(\alpha\beta^{-1},\beta^{-1}(a/b))=R^{(\circ)}_{(\beta^{-1},b_2)}(\alpha,a),$$

where $\alpha(b_2)=\beta^{-1}(a/b)$.

2. Let $(Q,\cdot)$ be a finite $\pi-$quasigroup of type T_1. If $(Hol(Q,\cdot),\circ)$ is a $\pi-$quasigroup of type T_1, then there exists a positive integer k such that $|Aut(Q,\cdot)|=3^k$ and $|Hol(Q,\cdot)|\equiv 0(mod\ 3)$.

Proposition 2. *Let $(Q,\cdot)$ be a $\pi-$quasigroup of type T_1 and let $Q_1=\{(\varepsilon,x)|x\in Q\}$. Then $(Q,\cdot)\cong(Q_1,\circ)$ and the following statements hold:* 1)$LM(Q_1,\circ)\lhd LM(Hol(Q,\cdot),\circ)$; 2)$RM(Q_1,\circ)\lhd RM(Hol(Q,\cdot),\circ)$.

Proof. 1) $(Q,\cdot)\cong(Q_1,\circ)$ implies $LM(Q,\cdot)\cong LM(Q_1,\circ)$. Moreover, $LM(Q_1,\circ)$ is a subgroup of $LM(Hol(Q,\cdot),\circ)$. Now, let $L^{(\circ)}_{(\varepsilon,x)}\in LM(Q_1,\circ)$ and $L^{(\circ)}_{(\beta,b)}\in LM(Hol(Q,\cdot),\circ)$, then:

$$L^{(\circ)}_{(\beta,b)}L^{(\circ)}_{(\varepsilon,x)}L^{(\circ)-1}_{(\beta,b)}(\alpha,a)=L^{(\circ)}_{(\beta,b)}L^{(\circ)}_{(\varepsilon,x)}(\beta^{-1}\alpha,\beta^{-1}\alpha(b)\cdot a)=$$

$$L^{(\circ)}_{(\beta,b)}(\beta^{-1}\alpha,\beta^{-1}\alpha(x)\cdot(\beta^{-1}\alpha(b)\backslash a))=$$

$$(\alpha, \beta^{-1}\alpha(b) \cdot (\beta^{-1}\alpha(x) \cdot (\beta^{-1}\alpha(b) \backslash a))) = L^{(\circ)}_{(\varepsilon,c)}(\alpha, a),$$

for $\forall(\alpha, a) \in Hol(Q, \cdot)$, where $\alpha(c) \cdot a = \beta^{-1}\alpha(b) \cdot (\beta^{-1}\alpha(x) \cdot (\beta^{-1}\alpha(b) \backslash a))$. Thus,

$$L^{(\circ)}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)} L^{(\circ)-1}_{(\beta,b)} \in LM(Q_1, \circ). \tag{3}$$

Analogously,

$$L^{(\circ)-1}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)} L^{(\circ)}_{(\beta,b)}(\alpha, a) = L^{(\circ)-1}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)}(\beta\alpha, \alpha(b) \cdot a) =$$

$$L^{(\circ)-1}_{(\beta,b)}(\beta\alpha, \beta\alpha(x) \cdot (\alpha(b) \cdot a)) =$$

$$(\alpha, \alpha(b) \backslash (\beta\alpha(x) \cdot (\alpha(b) \cdot a))) = L^{(\circ)}_{(\varepsilon,g)}(\alpha, a),$$

for $\forall(\alpha, a) \in Hol(Q, \cdot)$, where $\alpha(g) \cdot a = \alpha(b) \backslash (\beta\alpha(x) \cdot (\alpha(b) \cdot a))$. So,

$$L^{(\circ)-1}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)} L^{(\circ)}_{(\beta,b)} \in LM(Q_1, \circ). \tag{4}$$

(3) and (4) imply that $LM(Q_1, \circ) \lhd LM(Hol(Q, \cdot), \circ)$. The proof of the second relation is similar. $\square$

Remark 3. The function $\xi : LM(Hol(Q, \cdot)) \mapsto Aut(Q, \cdot)$, $\xi(L^{\delta_1}_{(\alpha_1,x_1)} L^{\delta_2}_{(\alpha_2,x_2)}, ..., L^{\delta_n}_{(\alpha_n,x_n)}) = \alpha_1^{\delta_1}\alpha_2^{\delta_2}...\alpha_n^{\delta_n}$, where $\delta_i = 1$ or -1, for every $i = 1, 2, ..., n$, is a surjective homomorphism with $Ker\xi = LM(Q_1, \circ)$, so $LM(Hol(Q, \cdot))/LM(Q_1, \circ) \cong Aut(Q, \cdot)$.

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