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Determining the Optimal Evolution Time for Markov Processes with Final Sequence of States

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Abstract. This paper describes a class of dynamical stochastic systems that represents an extension of classical Markov decision processes. The Markov stochastic systems with given final sequence of states and unitary transition time, over a finite or infinite state space, are studied. Such dynamical system stops its evolution as soon as given sequence of states in given order is reached. The evolution time of the stochastic system with fixed final sequence of states depends on initial distribution of the states and probability transition matrix. The considered class of processes represents a generalization of zero-order Markov processes, studied in [3]. We are seeking for the optimal initial distribution and optimal probability transition matrix that provide the minimal evolution time for the dynamical system. We show that this problem can be solved using the signomial and geometric programming approaches.

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1 Introduction and Problem Formulation

Let L be a stochastic discrete system with finite set of states V, $|V| = \omega$. At every discrete moment of time $t \in \mathbb{N}$ the state of the system is $v(t) \in V$. The system L starts its evolution from the state v with the probability $p^*(v)$, for all $v \in V$, where $\sum\limits_{v\in V} p^*(v) = 1$. Also, the transition from one state u to another state v is performed according to given probability $p(u, v)$ for every $u \in V$ and $v \in V$, where $\sum\limits_{v\in V} p(u, v) = 1$, $\forall u \in V$ and $p(u, v) \geq 0$, $\forall u, v \in V$. Additionally we assume that a sequence of states $x_1, x_2, \ldots, x_m \in V$ is given and the stochastic system stops transitions as soon as the sequence of states $x_1, x_2, \ldots, x_m$ is reached in given order. The time T when the system stops is called evolution time of the stochastic system L with given final sequence of states $x_1, x_2, \ldots, x_m$.

Various classes of such systems have been studied in [1] and [5], where polynomial algorithms for determining the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time of the given stochastic systems were proposed. Another interpretations of these Markov processes were analyzed in 1981 by Leo J. Guibas and Andrew M. Odlyzko in [9] and by G. Zbaganu in 1992 in [8]. First article considers the evolution of these stochastic systems as a string, composed from the states of the systems, and studies the periods in this string. In the second paper the author considers that the evolution of Markov process is similar with a poem written by an ape. The evolution time of the system is associated with the time that needs for the ape to write that poem (the final sequence of states of the system).

Next, we consider that the distributions p and p^* are not fixed. So, we have the Markov process $L(p^*, p)$ with final sequence of states X, distribution of the states p^* and transition matrix p, for every parameters p and p^*. The problem is to determine the optimal distribution $p^* = \overline{p}^*$ and optimal transition matrix $p = \overline{p}$ that minimize the expectation of the evolution time $T(p^*, p)$ of the stochastic system $L(p^*, p)$.

Based on the results mentioned above, efficient methods for minimizing the expectation of the evolution time of zero-order Markov processes with final sequence of states and unitary transition time were obtained in [3]. The main idea was that the expectation of the evolution time can be written as a posynomial minus one unit. The geometric programming approach was applied and the problem was reduced to the case of convex optimization and solved using the interior-point methods.

In this paper we consider a generalization of this problem where the evolution time is minimized for Markov processes of order 1.

2 Preliminary Results

In order to determine the minimal evolution time for Markov processes with final sequence of states we will use the geometric and signomial programming approaches [6].

2.1 Geometric Programming

The geometric programming was introduced in 1967 by Duffin, Peterson, and Zener. Wilde and Beightler in 1967 and Zener in 1971 contributed with several results referred to many extensions and sensitivity analysis. A geometric program represents a type of optimization problem, described by objective and constraint functions that have a special form. A good tutorial on geometric programming was presented in [6].

First numerical methods, based on solving a sequence of linear programs, were elaborated by Avriel et al., Duffin, Rajpogal and Bricker. Nesterov and Nemirovsky in 1994 described the first interior-point method for geometric programs and proved the polynomial time complexity. Recent numerical approaches were presented by Andersen and Ye, Boyd and Vandenberghe, Kortanek.

In the context of geometric programming, a monomial represents a function $f : \mathbb{R}^s \to \mathbb{R}$ of the form $f(x_1, x_2, \ldots, x_s) = cx_1^{\alpha_1} x_2^{\alpha_2} \ldots x_s^{\alpha_s}$, where $c > 0$ and $\alpha_i \in \mathbb{R}$, $i = \overline{1, s}$. An arbitrary sum of monomials, $f(x_1, x_2, \ldots, x_s) = \sum\limits_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \ldots x_s^{\alpha_{sk}}$, where $c_k > 0$, $k = \overline{1, K}$ and $\alpha_{ik} \in \mathbb{R}$, $i = \overline{1, s}$, $k = \overline{1, K}$, represents a posynomial. Posynomials are closed under addition, multiplication, and nonnegative scaling. A geometric program is an optimization problem of the form

$$f_0(x_1, x_2, \dots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \dots, x_s) \le 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \dots, x_s) = 1, & j = \overline{1, m} \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

where $f_i(x_1, x_2, \dots, x_s)$, $i = \overline{0, r}$, are posynomials and $g_j(x_1, x_2, \dots, x_s)$, $j = \overline{1, m}$, are monomials.

In order to efficiently solve a geometric program we need to convert it to a convex optimization problem. The conversion is based on a logarithmic change of variables $y_l = \ln x_l$, $l = \overline{1, s}$ and a logarithmic transformation of the objective and constraint functions. The obtained convex optimization problem has the form

$$\ln f_0(e^{y_1}, e^{y_2}, \dots, e^{y_s}) \to \min,$$

$$\begin{cases} \ln f_i(e^{y_1}, e^{y_2}, \dots, e^{y_s}) \le 0, & i = \overline{1, r} \\ \ln g_j(e^{y_1}, e^{y_2}, \dots, e^{y_s}) = 0, & j = \overline{1, m} \end{cases}$$

and can be efficiently solved using standard interior-point methods (see [6] and [7]).

2.2 Signomial Programming

In the context of signomial programming, a signomial monomial represents a function $f : \mathbb{R}^s \to \mathbb{R}$ of the form $f(x_1, x_2, \dots, x_s) = c x_1^{\alpha_1} x_2^{\alpha_2} \dots x_s^{\alpha_s}$, where $c \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}$, $i = \overline{1, s}$. An arbitrary sum of signomial monomials of the form $f(x_1, x_2, \dots, x_s) = \sum\limits_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \dots x_s^{\alpha_{sk}}$, where $c_k \in \mathbb{R}$, $k = \overline{1, K}$ and $\alpha_{ik} \in \mathbb{R}$, $i = \overline{1, s}$, $k = \overline{1, K}$, represents a signomial. Signomials are closed under addition, substraction, multiplication, and scaling. A signomial program is an optimization problem of the form:

$$f_0(x_1, x_2, \dots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \dots, x_s) \le 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \dots, x_s) = 1, & j = \overline{1, m} \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

where $f_i(x_1, x_2, \dots, x_s)$, $i = \overline{0, r}$ and $g_j(x_1, x_2, \dots, x_s)$, $j = \overline{1, m}$, are signomials.

So, a signomial has the same form as a posynomial, but the coefficients are allowed to be also negative. There is a huge difference between a geometric program and a signomial program. The global optimal solution of a geometric program can always be determined, but only a local solution of a signomial program can be calculated efficiently.

2.3 Geometric Programs with Posynomial Equality Constraints

In several particular cases the signomial programs can be handled as geometric programs. In [6] it was shown that the geometric programs with posynomial equality constraints represent such particular case, i.e. can be solved using geometric programming method. A geometric program with posynomial equality constraints is a signomial program of the form:

$$f_0(x_1, x_2, \ldots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \le 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1, m} \\ h_k(x_1, x_2, \ldots, x_s) = 1, & k = \overline{1, n} \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

where $f_i(x_1, x_2, \ldots, x_s)$, $i = \overline{0, r}$ and $h_k(x_1, x_2, \ldots, x_s)$, $k = \overline{1, n}$, are posynomials and $g_j(x_1, x_2, \ldots, x_s)$, $j = \overline{1, m}$, are monomials.

Suppose that for each posynomial equality constraint $h_k(x_1, x_2, \ldots, x_s)$, $k = \overline{1, n}$, we can find a different variable $x_{l(k)}$ with the following properties:

- The variable $x_{l(k)}$ does not appear in any of the monomial equality constraint functions;
- The posynomial $h_k(x_1, x_2, \ldots, x_s)$ is monotone strictly
  - increasing in $x_{l(k)}$, case in which we denote $\lambda(x_{l(k)}) = -1$ or
  - decreasing in $x_{l(k)}$, case in which we denote $\lambda(x_{l(k)}) = 1$;
- The functions $f_i(x_1, x_2, \ldots, x_s)$, $i = \overline{0, r}$, are all
  - monotone decreasing in $x_{l(k)}$ if $\lambda(x_{l(k)}) = -1$;
  - monotone increasing in $x_{l(k)}$ if $\lambda(x_{l(k)}) = 1$.

We first form the geometric program relaxation:

$$f_0(x_1, x_2, \ldots, x_s) \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \le 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1, m} \\ h_k(x_1, x_2, \ldots, x_s) \le 1, & k = \overline{1, n} \\ x_l > 0, & l = \overline{1, s} \end{cases}.$$

If f^* is the optimal value of the relaxed problem, then any optimal solution of the auxiliary problem

$$\prod_{k=1}^{n} (x_{l(k)})^{\lambda(x_{l(k)})} \to \min,$$

$$\begin{cases} f_i(x_1, x_2, \ldots, x_s) \leq 1, & i = \overline{1, r} \\ g_j(x_1, x_2, \ldots, x_s) = 1, & j = \overline{1, m} \\ h_k(x_1, x_2, \ldots, x_s) \leq 1, & k = \overline{1, n} \\ f_0(x_1, x_2, \ldots, x_s) \leq f^* & \\ x_l > 0, & l = \overline{1, s} \end{cases},$$

is an optimal solution of the original problem.

3 The Main Results

3.1 Stochastic Systems with Final Sequence of States Independent States

In this subsection we briefly describe the main results referred to the problem of optimization of the evolution time of stochastic systems with final sequence of states and independent states. These systems are also called zero order Markov processes with final sequence of states or strong memoryless stochastic systems with final sequence of states and are analyzed and studied in [2] and [3]. This problem was reduced to a geometric program using the main properties of homogeneous recurrent linear sequences and generating function, presented in [3–5] and [1].

The zero order Markov processes with final sequence of states represent a particular case of stochastic systems with final sequence of states studied in this paper. In this case the states of the system are independent, so, the rows of the transition matrix p are equal to initial distribution p^*. The expectation of the evolution time can be determined using the following theorem.

Theorem 1. *The expectation of the evolution time $T(p^*)$ of zero-order Markov process $L(p^*)$ is $\mathbb{E}(T(p^*)) = -1 + (m + w_m^{-1}) + \frac{1}{w_m} \sum_{k=0}^{m-1} (k+1) z_{mk}$, where m is the length of final sequence of states $X = (x_1, x_2, \ldots, x_m)$, $\pi_s = p^*(x_s)$, $w_s = \prod_{j=1}^{s} \pi_j$, $t(s) = \min(\{t \in \{2, 3, \ldots, s+1\} \mid x_{t-1+j} = x_j,\ j = \overline{1, s+1-t}\})$, $s = \overline{1, m}$ and for each $s = \overline{1, m}$ and $k = \overline{0, s-1}$ the following relation holds:*

$$z_{sk} = \begin{cases} 0 & if\ 0 \leq k \leq t(s) - 3 \\ -w_{t(s)-1} & if\ k = t(s) - 2 \\ w_{t(s)-1}(1 - \pi_1) & if\ t(s - t(s) + 1) = 2\ and\ k = t(s) - 1 \\ w_{t(s)-1} & if\ t(s - t(s) + 1) \geq 3\ and\ k = t(s) - 1 \\ w_{t(s)-1} z_{s-t(s)+1,\ k-t(s)+1} & if\ t(s) \leq k \leq s - 1 \end{cases}.$$

The following theorem shows how the problem of optimization of the evolution time can be reduced to the geometric program

$$\mathbb{E}(T(p^*)) + 1 \to \min,$$

$$\begin{cases} \sum_{x \in Y} p^*(x) \leq 1 \\ p^*(x) > 0,\ \forall x \in Y \end{cases},$$

where $Y = \{x_1, x_2, \ldots, x_m\}$. If $\pi^* = (\pi^*(x))_{x\in Y}$ represents the optimal solution of this geometric program, then $\overline{p}^* = (\overline{p}^*(x))_{x\in V}$ represents the optimal solution of the initial problem, where

$$\begin{cases} \overline{p}^*(x) = \pi^*(x), & x \in Y \\ \overline{p}^*(x) = 0, & x \in V\backslash Y \ . \end{cases}$$

Theorem 2. *The expression $\mathbb{E}(T(p^*)) + 1$ represents a posynomial in the variables $\pi_1,\ \pi_2,\ \ldots,\ \pi_m$.*

Also, several particular cases were analyzed and presented in [3] and the explicit optimal solutions were obtained.

Theorem 3. *If $t(m) = 2$, then the optimal solution is $\overline{p}^* = (\overline{p}^*(x))_{x\in V}$, where $\overline{p}^*(x_1) = 1$ and $\overline{p}^*(y) = 0$, for all $y \in V\backslash\{x_1\}$, and the minimal value of the expectation of evolution time is $\mathbb{E}(T(\overline{p}^*)) = m - 1$.*

Theorem 4. *If $t(m) = m + 1$, then the components $\overline{p}^*(y)$, $y \in V$, of the optimal solution $\overline{p}^*$ are direct by proportional to the multiplicities $m(y)$, $y \in V$, of the respective states in final sequence of states X and the minimal value of the expectation of evolution time is $\mathbb{E}(T(\overline{p}^*)) = -1 + \prod\limits_{y\in Y} \left(\frac{m}{m(y)}\right)^{m(y)}$.*

3.2 Stochastic Systems with Final Sequence of States and Interdependent States

In this subsection we study the problem of optimization of the evolution time of stochastic systems with final sequence of states and interdependent states. The optimal initial distribution and optimal transition matrix are obtained, using signomial and geometric programming approaches.

Theorem 5 offers us the way for determining the optimal initial distribution of the system.

Theorem 5. *The optimal initial distribution of the states is $\overline{p}^*$, where $\overline{p}^*(x_1) = 1$ and $\overline{p}^*(x) = 0$, $\forall x \in V\backslash\{x_1\}$.*

Proof. For finishing the evolution of the system it is necessary to pass consecutively through the final states $x_1,\ x_2,\ \ldots,\ x_m$. So, the evolution time will be minimal when the state x_1 will be reached as soon as possible. For this reason, it is optimal to start the evolution of the system from the state x_1, i.e. $\overline{p}^*(x_1) = 1$. Since $\sum\limits_{x\in V} \overline{p}^*(x) = 1$, we have $\overline{p}^*(x) = 0$, $\forall x \in V\backslash\{x_1\}$. $\square$

Theorem 6 describes several important properties of the optimal transit matrix.

Theorem 6. *We consider the set of active final states $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$, the set of final transitions $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$ and the set of branch states $\overline{Z} = \{y \in \overline{X}\backslash\{x_1\} \mid \exists x \in \overline{X}, \exists z \in \overline{X} \cup \{x_m\}, z \neq y :\ (x, y) \in \overline{Y}, (x, z) \in \overline{Y}\}$. The optimal transition matrix $\overline{p}$ has the following properties:*

1. $\overline{p}(x, x_1) = 1$ *if* $(x, x_1) \in \overline{Y}$ *and* $(x, z) \notin \overline{Y}$, $\forall z \neq x_1$;

2. $\overline{p}(x, x_1) = 1$, $\forall x \notin \overline{X}$;

3. $\overline{p}(x, x_1) > 0$, $\forall x \in \overline{Z}$ *and* $\overline{p}(x, x_1) = 0$ *if* $(x, x_1) \notin \overline{Y}$, $x \in \overline{X} \backslash \overline{Z}$;

4. $\overline{p}(x, y) = 0$ *if* $(x, y) \notin \overline{Y}$ *and* $y \neq x_1$;

5. $\overline{p}(x, y) > 0$, $\forall (x, y) \in \overline{Y}$;

6. $\sum\limits_{(x,y) \in \overline{Y} \cup \{(x,x_1)\}} \overline{p}(x, y) = 1, \ \forall x \in \overline{X}$.

Proof. Let $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$ be the set of the states from which it is possible to perform an optimal transition, $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$ – the set of the optimal transitions (that follow optimal realization of the final sequence of states), $\overline{Z} = \{y \in \overline{X} \backslash \{x_1\} \mid \exists x \in \overline{X}, \exists z \in \overline{X} \cup \{x_m\}, z \neq y : \ (x, y) \in \overline{Y}, (x, z) \in \overline{Y}\}$ – the set of branch states, in which the stochastic system, having as goal the realization of the final sequence of states, can make a mistake and need to have a chance to return in the state x_1.

1. If $(x, x_1) \in \overline{Y}$ and $(x, z) \notin \overline{Y}$, $\forall z \neq x_1$, then $x \in \overline{X}$ and the transition (x, x_1) is the unique possible transition from the state x that belongs to the set $\overline{Y}$. For ensuring the realization of this transition when the system is in the state $x \in \overline{X}$, it is necessary to have $\overline{p}(x, x_1) = 1$.

2. For finishing the evolution of the system it is necessary to pass consecutively through the final states x_1, x_2, $\ldots$, x_m. So, for minimizing the evolution time of the system it is necessary that the state x_1 to be reached as soon as possible. So, if the system is in the state $x \notin \overline{X}$, we need to have $\overline{p}(x, x_1) = 1$.

3. Since $\overline{Z}$ represents the set of branch states, in which the stochastic system, having as goal the realization of the final sequence of states, can make a mistake, we need to give as soon as possible a chance to return in the state x_1 for retrying from the beginning the realization of the final sequence of states. So, we can assume that $\overline{p}(x, x_1) > 0$, $\forall x \in \overline{Z}$ and $\overline{p}(x, x_1) = 0$ if $(x, x_1) \notin \overline{Y}$ and $x \in \overline{X} \backslash \overline{Z}$;

4. If the state $x \in \overline{X}$, then $\exists y_1, y_2, \ldots, y_k \in \overline{X} \cup \{x_m\}$ such that $(x, y_j) \in \overline{Y}$, $j = \overline{1, k}$, where $k \geq 1$. For ensuring the realization of one of these transitions when the system is in the state $x \in \overline{X}$ or return to the initial state x_1 when it is necessary, we need the nonexistence of another transition $(x, y) \notin \overline{Y}$ with $y \neq x_1$, i.e. need to have $\overline{p}(x, y) = 0$. If $x \notin \overline{X}$, from Property 2 of this Theorem, since $\sum\limits_{y \in V} \overline{p}(x, y) = 1$ and $\overline{p}(x, y) \geq 0$, $\forall x, y \in V$, we have $\overline{p}(x, y) = 0$, $\forall y \neq x_1$.

5. We have $\overline{p}(x,y) > 0$, $\forall (x,y) \in \overline{Y}$, because, otherwise we have $\overline{p}(x,z) = 0$ for at least one transition $(x,z) \in \overline{Y}$, i.e. this transition is not realizable, which implies that the evolution time is infinite (non optimal), contradiction with our minimization goal.

6. The relation $\sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} \overline{p}(x,y) = 1,\ \forall x \in \overline{X}$ is obtained from the formula $\sum\limits_{y\in V} \overline{p}(x,y) = 1$, $\forall x \in \overline{X}$ and the Property 4 from this Theorem.

Such we proved these six properties of the optimal transition matrix $\overline{p}$. $\square$

Theorem 7 offers us the way for determining the optimal transition matrix of the system.

Theorem 7. *If $\delta_{i,j}(p) \not\equiv 0$, $i,j = \overline{1,2}$, then the optimal transition matrix can be determined by solving the following geometric programs with posynomial equality constraints:*

$$\mathbb{E}(T(p)) = d_1 d_2^{-1} \to \min, \tag{1}$$

$$\begin{cases} (2a): & \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1,\ \forall x \in \overline{X} \\ (2b): & d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ (2c): & d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ (2d): & d_{1,1}^{-1} \delta_{1,1}(p) = 1 \\ (2e): & d_{1,2}^{-1} \delta_{1,2}(p) = 1 \\ (2f): & d_{2,1}^{-1} \delta_{2,1}(p) = 1 \\ (2g): & d_{2,2}^{-1} \delta_{2,2}(p) = 1 \\ (2h): & d_i > 0,\ i = \overline{1,2} \\ (2i): & d_{i,j} > 0,\ i,j = \overline{1,2} \\ (2j): & p(x,y) > 0,\ \forall (x,y) \in \overline{Y} \\ (2k): & \overline{p}(x,x_1) > 0,\ \forall x \in \overline{Z} \end{cases} \tag{2}$$

and (1) subject to

$$\begin{cases} (3a): & \sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1,\ \forall x \in \overline{X} \\ (3b): & d_{1,1}^{-1} d_1 + d_{1,1}^{-1} d_{1,2} = 1 \\ (3c): & d_{2,1}^{-1} d_2 + d_{2,1}^{-1} d_{2,2} = 1 \\ (3d): & d_{1,1}^{-1} \delta_{1,2}(p) = 1 \\ (3e): & d_{1,2}^{-1} \delta_{1,1}(p) = 1 \\ (3f): & d_{2,1}^{-1} \delta_{2,2}(p) = 1 \\ (3g): & d_{2,2}^{-1} \delta_{2,1}(p) = 1 \\ (3h): & d_i > 0,\ i = \overline{1,2} \\ (3i): & d_{i,j} > 0,\ i,j = \overline{1,2} \\ (3j): & p(x,y) > 0,\ \forall (x,y) \in \overline{Y} \\ (3k): & \overline{p}(x,x_1) > 0,\ \forall x \in \overline{Z} \end{cases} \tag{3}$$

according to the properties described by Theorems 5 and 6, where $\delta_{i,j}(p),\ \ i,j=\overline{1,2}$, *are the posynomials from the decomposition*

$$\mathbb{E}(T(p)) = (\delta_{1,1}(p) - \delta_{1,2}(p))(\delta_{2,1}(p) - \delta_{2,2}(p))^{-1} \tag{4}$$

which follows from the algorithm developed in [1]. The signomial programs $(1)-(2)$ *and* $(1)-(3)$ *can be handled as geometric programs using the way followed in [6] and described in Section 2.3. If* $\overline{p}^1$ *is the optimal solution of the problem* $(1)-(2)$ *and* $\overline{p}^2$ *is the optimal solution of the problem* $(1)-(3)$*, then the optimal transition matrix is* $\overline{p} \in \{\overline{p}^1, \overline{p}^2\}$ *for which* $\mathbb{E}(T(\overline{p}))$ *is minimal. If there exists at least one posynomial* $\delta_{i^*,j^*}(p) \equiv 0$*, then in* (2) *and* (3) *the corresponding posynomial equality constraints just disappear and the corresponding substitution* $d_{i^*,j^*}=0$ *is performed in* (2) *and substitution* $d_{i^*,3-j^*}=0$ *is performed in* (3).

Proof. From Theorem 5 and theoretical argumentation of the algorithm developed and presented in [1], which determines the generating vector of the distribution of the evolution time, we can observe that the components of generating vector $q(p)$ of the distribution $a = rep(T(p))$ of the evolution time $T(p)$ represent signomials in the variables $p(x,y)$, $x,y \in V$. Since $\mathbb{E}(T(p)) = G^{[a]\prime}(1)$, we obtain that $\mathbb{E}(T(p))$ represents a fraction with signomial numerator and denominator. Because every signomial can be written as a difference between two posynomials, we have the relation (4), where $\delta_{i,j}(p),\ \ i,j=\overline{1,2}$, are posynomials.

If we denote $d_{i,1} = \max\{\delta_{i,1}(p), \delta_{i,2}(p)\}$, $d_{i,2} = \min\{\delta_{i,1}(p), \delta_{i,2}(p)\}$, $i=\overline{1,2}$ and $d_i = d_{i,1} - d_{i,2} > 0$, $i=\overline{1,2}$, we obtain $0 < \mathbb{E}(T(p)) = d_1 d_2^{-1}$, $d_{i,1}^{-1} d_i + d_{i,1}^{-1} d_{i,2} = 1$, $i=\overline{1,2}$ and $d_{i,j}^{-1}\delta_{i,j}(p) = 1$ or $d_{i,j}^{-1}\delta_{i,3-j}(p) = 1$, $i,j=\overline{1,2}$. The relations $\sum\limits_{(x,y)\in\overline{Y}\cup\{(x,x_1)\}} p(x,y) = 1$, $\forall x \in \overline{X}$ and $p(x,y) > 0$, $\forall (x,y) \in \overline{Y}$, follow from Theorem 6.

Also, applying the same Theorem 6, we can eliminate the variables $p(x,y)$ for which $p(x,y)=0$ or $p(x,y)=1$ performing the corresponding substitutions. In such way, we obtain the signomial programs $(1)-(2)$ and $(1)-(3)$. Because all constraints $(2a)-(2g)$ and $(3a)-(3g)$ are posynomial equality constraints, these signomial programs are geometric programs with posynomial equality constraints.

Next we will illustrate how these geometric programs with posynomial equality constraints can be handled as geometric programs using the way described in Section 2.3. We will consider only the problem $(1)-(2)$, the argumentation for the problem $(1)-(3)$ can be performed in similar way.

So, if the posynomials $\delta_{i,j}(p)$, $i,j=\overline{1,2}$, are not monomials, we can fix the variable $d_{1,1}$ for constraint $(2b)$, d_2 for $(2c)$, $d_{1,2}$ for $(2e)$, $d_{2,1}$ for $(2f)$, $d_{2,2}$ for $(2g)$, an arbitrary variable $p(x^*,y^*)$ that appears in the posynomial $\delta_{1,1}(p)$ for constraint $(2d)$ and an arbitrary variable $p(x,y^*(x)) \neq p(x^*,y^*)$ that appears in the posynomial from $(2a)$ for every $x \in \overline{X}$ for the constraints $(2a)$. These selected variables verify the properties described in Section 2.3, i.e. the problem $(1)-(2)$ can be handled as geometric programs.

If the posynomial $\delta_{1,2}(p)$, $\delta_{2,1}(p)$ or $\delta_{2,2}(p)$ is a monomial, then the respective constraint $(2e)$, $(2f)$ or $(2g)$ just disappears and the respective substitution $d_{i^*,j^*} = \delta_{i^*,j^*}(p)$ is performed in the signomial program $(1)-(2)$. The selected variables for the rest of constraints are not changed. So, the problem $(1)-(2)$ can be handled as geometric programs.

If the posynomial $\delta_{1,1}(p)$ is a non-constant monomial, then the corresponding constraint $(2d)$ just disappears and the corresponding substitution $d_{1,1} = \delta_{1,1}(p)$ is performed in the signomial program $(1) - (2)$. The selected variables for the constraints $(2a)$, $(2c)$, $(2e)$, $(2f)$ and $(2g)$ are not changed. Additionally, the variable $p(x^*, y^*)$ that appears in the posynomial $\delta_{1,1}(p)$ is selected for constraint $(2b)$. These selected variables verify the properties described in Section 2.3, i.e. the problem $(1)-(2)$ can be handled as geometric programs.

If the posynomials $\delta_{1,1}(p)$ and $\delta_{1,2}(p)$ are two constants, then also d_1 is a constant. In this case the constraints $(2b)$, $(2d)$ and $(2e)$ just are eliminated. The selected variables for the rest of constraints are not changed. So, in this way, the problem $(1)-(2)$ can be handled as geometric programs.

If the posynomial $\delta_{1,1}(p)$ is a constant and $\delta_{1,2}(p)$ is not a constant, then the constraint $(2d)$ is eliminated and substitution $d_{1,1} = \delta_{1,1}(p)$ is performed in the signomial program $(1)-(2)$. We can fix the variable $d_{1,2}$ for constraint $(2b)$, an arbitrary variable $p(x^{**}, y^{**})$ that appears in the posynomial $\delta_{1,2}(p)$ for constraint $(2e)$ and an arbitrary variable $p(x, y^{**}(x)) \neq p(x^{**}, y^{**})$ that appears in the posynomial from $(2a)$ for every $x \in \overline{X}$ for the constraints $(2a)$. The selected variables for the rest of constraints are not changed. These selected variables verify the properties described in Section 2.3, i.e. the problem $(1)-(2)$ can be handled as geometric programs.

In this way, we analyzed all the possible cases. So, the problems $(1)-(2)$ and $(1)-(3)$ can be handled as geometric programs. □

4 Particular cases and generalizations

In the previous section a method for determining the optimal evolution time of stochastic systems with final sequence of states, based on geometric and signomial programming approaches, was theoretically grounded. Theorems 5 and 6 present the main properties of the optimal distribution and optimal transition matrix. From these theorems we can easy remark several particular cases and generalizations.

We consider the particular case $x_1 = x_2 = \ldots = x_m$. From Theorem 6 the optimal transition matrix is obtained. The following formula holds:

$$\overline{p}(x,y) = \begin{cases} 1 & if \;\; y = x_1 \\ 0 & if \;\; y \neq x_1 \end{cases}, \ \forall x, y \in V.$$

So, the expectation of the evolution time is minimal (equal to $m-1$) when the stochastic system starts the evolution from the state x_1 and remains with probability 1 at every moment of time in this state.

Also, in the case $x_i \neq x_j$, $\forall i, j,\ 1 \leq i < j \leq m$, we have

$$\overline{p}(x,y) = \begin{cases} 1 & if \quad x \notin \{x_1, x_2, \ldots, x_{m-1}\} \ and \ y = x_1 \\ 1 & if \quad \exists\, i,\ 1 \leq i < m,\ such\ that\ x = x_i \ and \ y = x_{i+1} \\ 0 & otherwise \end{cases}, \ \forall x, y \in V.$$

The expectation of the evolution time is minimal (equal to $m-1$) when the stochastic system starts the evolution from the state x_1, passes with probability 1 in the state x_2, next, in similar way, passes in the state x_3, ..., until it reaches the state x_m.

Another particular case is when $\forall i, j,\ 1 \leq i < j \leq m$, if $x_i = x_j$ then $j = m$ or $x_{i+1} = x_{j+1}$. This case is an extension of the previous particular case. We consider the minimal values i^* and j^*, $i^* < j^*$, for which $x_{i^*} = x_{j^*}$. We have $x_{i^*+k} = x_{j^*+k}$, $k = \overline{0, m - j^*}$. So, $\{x_1, x_2, \ldots, x_m\} = \{x_1, x_2, \ldots, x_{j^*-1}\}$, which implies

$$\overline{p}(x,y) = \begin{cases} 1 & if \quad x \notin \{x_1, x_2, \ldots, x_{j^*-1}\} \ and \ y = x_1 \\ 1 & if \quad \exists\, i,\ 1 \leq i < j^* - 1,\ such\ that\ x = x_i \ and \ y = x_{i+1} \\ 1 & if \quad x = x_{j^*-1} \ and \ y = x_{i^*} \\ 0 & otherwise \end{cases}, \ \forall x, y \in V.$$

The expectation of the evolution time is minimal (equal to $m-1$) when the stochastic system starts the evolution from the state x_1, passes with probability 1 in the state x_2, next, in similar way, passes in the state x_3, ..., until it reaches the state $x_m \in \{x_1, x_2, \ldots, x_{j^*-1}\}$.

Next we present a generalization of the problem studied in this paper for the case in which the number of the states of the system is not finite, i.e. we have $\omega = |V| = \infty$. This case cannot be handled in the same way as finite case, because it is not known any formula and any algorithm for determining the expectation of the evolution time of stochastic system with final sequence of states and interdependent states when the number of the states is not finite and the transition matrix and initial distribution are fixed and given. Nevertheless, the optimal distribution and optimal transition matrix can be determined using the result obtained above.

Indeed, we observed above that the given stochastic system, with finite or infinite number of states, can be reduced to a new stochastic system with maximum m states, $x_1, x_2, \ldots, x_m$, preserving the optimal solution. This reduction is possible thanks to Theorems 5 and 6, from which, in optimal case, the excluded states cannot be reached by system at any moment of time. So, if $\overline{p}$ is the optimal transition matrix for the stochastic system with infinite number of states and $\overline{p_r}$ is the optimal transition matrix for the reduced stochastic system, then

$$\overline{p}(x,y) = \begin{cases} \overline{p_r}(x,y) & if \quad x, y \in \{x_1, x_2, \ldots, x_m\} \\ 1 & if \quad x \notin \{x_1, x_2, \ldots, x_m\} \ and \ y = x_1 \\ 0 & otherwise \end{cases}, \ \forall x, y \in V.$$

5 Conclusions

In this paper the following results related to stochastic systems with final sequence of states and unitary transition time were established:

- The given stochastic system, with finite or infinite number of states, can be reduced to a new stochastic system with maximum m states, $x_1, x_2, \ldots, x_m$, preserving the optimal solution;
- The evolution time of the stochastic system with fixed final sequence of states depends on initial distribution of the states and probability transition matrix;
- In the case when the states of the system are independent, the expectation of the evolution time represents a posynomial minus one unit, that offers the possibility to minimize it using geometric programming approach;
- In the case when the states of the system are interdependent, the expectation of the evolution time can be minimized by solving two geometric programs with posynomial equality constraints, that represents signomial programs which can be handled as geometric programs using the models developed in this paper;
- In several particular cases, which were described in Section 4, the optimal initial distribution and optimal probability transition matrix are trivial.

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