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Ergodic sets and mixing extensions of topological transformation semigroups

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Abstract. We extend the concept of the ergodic set [1] – [2] from topological transformation groups to topological transformation semigroups. We investigate, in particular, connections between ergodicity, weak ergodicity, topological transitivity and minimality of the Whitney's sum of extensions of topological transformation semigroups.

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1 Basic definitions and notations

In this paper we use terminology and notation generally accepted at present in the theory of topological transformation groups (semigroups). We give only definitions of concepts which are necessary in our opinion; for more detailed discussions the reader is referred to [2] – [5].

A topological transformation semigroup (for short transformation semigroup) is a triple (X, S, π), where X is a nonempty compact Hausdorff topological space with unique uniformity $\mathcal{U}[X]$ (phase space), S is a topological semigroup with the unit element e (phase semigroup) and $\pi : X \times S \to X$ is a continuous mapping satisfying the following conditions:

1) $\forall x \in X,\ \pi(x, e) = x$;

2) $\forall x \in X,\ \forall s, t \in S\ \pi(\pi(x, s), t) = \pi(x, st)$.

We shall refer to (X, S) rather than (X, S, π).

Let (X, S, π) be a transformation semigroup, $s \in S,\ A \subset X$. Usually we shall write π^s for the map $X \to X$ defined by $\pi^s(x) = \pi(x, s)$ $(x \in X)$; $xs = \pi^s(x)$ and $xS = \{\ xs \mid s \in S\ \}$ $(x \in X)$. For $x \in X$ we denote the set $xs^{-1} = \{y \mid y \in X \wedge ys = x\}$ and

$$AS^{-1} = \bigcup_{a \in A, s \in S} as^{-1}.$$

A is called invariant if $AS \subset A$. A is called minimal if $A \neq \emptyset$ and $\overline{xS} = A$ for every $x \in A$. A (X, S) is minimal if the set X is minimal. If for $x \in X$, $\overline{xS}$ is minimal x is called an almost periodic point. We denote by AJ the set of all almost periodic points from A.

An extension (a homomorphism) $\varphi : (X, S, \pi) \to (Y, S, \rho)$ of transformation semigroups is a continuous surjection $\varphi : X \to Y$ such that for $\forall x \in X$ $\forall s \in S$, $\varphi(\pi^s(x)) = \rho^s(\varphi(x))$. A homomorphism $\varphi : (X, S, \pi) \to (Y, S, \rho)$ is called an isomorphism if φ is a homeomorphic map. Let $\varphi : (X, S) \to (Y, S)$, $\psi : (Z, S) \to (Y, S)$ be two extensions. We denote:

$R_{\varphi\psi} = \{(x, z) \mid (x, z) \in X \times Z \wedge \varphi(x) = \psi(z)\}$; $R_\varphi = R_{\varphi\varphi}$; $\Delta(X) = \{(x, x) \mid x \in X\}$;

$$P(R_\varphi) = \bigcap_{\alpha \in \mathcal{U}[X]} \bigcup_{s \in S} \{(x, y) \mid (x, y) \in R_\varphi \wedge (xs, ys) \in \alpha\};$$

$$Q(R_\varphi) = \bigcap_{\alpha \in \mathcal{U}[X]} \overline{\bigcup_{s \in S} \{(x, y) \mid (x, y) \in R_\varphi \wedge (xs, ys) \in \alpha\}}.$$

The set $R_{\varphi\psi}$ is an invariant set in the direct product of (X, S) and (Z, S). Hence are defined the transformation semigroup $(R_{\varphi\psi}, S)$ and the Whitney's sum of the extensions φ and ψ $\eta : (R_{\varphi\psi}, S) \to (Y, S)$, where $\eta(x, y) = \varphi(x) = \psi(y)$ $((x, y) \in R_{\varphi\psi})$.

An extension $\varphi : (X, S) \to (Y, S)$ is called minimal if (X, S) is minimal.

An extension φ is called distal (proximal, regionally distal) if $P(R_\varphi) = \Delta(X)$ $(P(R_\varphi) = R_\varphi,\ Q(R_\varphi) = \Delta(X))$. If Y is a singleton, then the distal (proximal, regionally distal) extension $\varphi : (X, S) \to (Y, S)$ is called transformation semigroup (X, S) distal (proximal, regionally distal).

2 Ergodic transformation semigroups

The transformation semigroup (X, S) is said to be ergodic (weakly ergodic) if $X = \overline{VS^{-1}}$ for any nonempty and open (nonempty, invariant and open) set $V \subset X$. (X, S) is said to be topological transitive if $\overline{xS} = X$ for some $x \in X$.

It is clear that for transformation groups the concepts of the ergodicity and weak ergodicity are the same and every ergodic transformation semigroup is weakly ergodic.

Theorem 1. *If for every nonempty open set $V \subset X$ there exists nonempty, open and invariant set $U \subset VS^{-1}$, then any weakly ergodic transformation semigroup (X, S) is ergodic.*

Proof. Let $V \subset X$ be a nonempty and open set and $U \subset VS^{-1}$ be a nonempty, open and invariant set. Since $X = \overline{US^{-1}} \subset \overline{(VS^{-1})S^{-1}} \subset \overline{VS^{-1}} \subset X$ then $X = \overline{VS^{-1}}$.

Theorem 2. *Let (X, S) be a transformation semigroup. The following assertions are equivalent.*

1) (X, S) *is ergodic.*

2) *X does not contain an invariant closed proper subset with the nonempty interior.*

3) *$X = \overline{US}$ for any nonempty open set $U \subset X$.*

4) *$U \cap Vs \neq \emptyset$ for any nonempty open sets U and V from X and some $s \in S$.*

5) *$Us^{-1} \cap V \neq \emptyset$ for any nonempty open sets U and V from X and some $s \in S$.*

Proof. Suppose 1) holds, $B \subset X$ is a closed and invariant set, $V = int\ B \neq \emptyset$ and $U \subset X$ is nonempty and open. Since $X = \overline{US^{-1}}$ then $V \cap Us^{-1} \neq \emptyset$ for some $s \in S$, hence $U \cap VS \neq \emptyset$. Then $X = \overline{VS}$. $B = X$ because $X = \overline{(int\ B)S} \subset \overline{BS} \subset B \subset X$. We proved 1) $\Longrightarrow$ 2). Suppose 2) holds and $U \subset X$ is nonempty and open. Then $X = \overline{US}$, because X contains the nonempty, closed and invariant subset $\overline{US}$ with the nonempty interior . We proved 2) $\Longrightarrow$ 3). Suppose 3) holds and U and V are nonempty open sets from X. Then $X = \overline{VS}$ and $U \cap VS \neq \emptyset$, hence $U \cap Vs \neq \emptyset$ for some $s \in S$. We proved 3) $\Longrightarrow$ 4). Suppose 4) holds and U and V are nonempty open sets from X. Then $U \cap Vs \neq \emptyset$ for some $s \in S$. Therefore there is an $x \in U$ such that $x \in Vs$. Then $x = ys$ for some $y \in V$, hence $y \in Us^{-1}$ and $Us^{-1} \cap V \neq \emptyset$. We proved 4) $\Longrightarrow$ 5). Suppose 5) holds and $V \subset X$ is a nonempty open set, $x \in X$ and U is an open neighborhood of x. Then $Vs^{-1} \cap U \neq \emptyset$ for some $s \in S$. Therefore $x \in \overline{VS^{-1}}$ and $X = \overline{VS^{-1}}$. We proved 5) $\Longrightarrow$ 1).

It is clear that the minimal transformation semigroup is ergodic and the topological transitive transformation group is ergodic. The following example shows that for the transformation semigroups the notions of ergodicity and weak ergodicity are not the same and topological transitive transformation semigroups are not obligatory ergodic.

Let $S = \{0, 1, 2, 3\}$, $S(\cdot)$ be a discrete semigroup with respect to modulo 4 multiplication, i.e. $s \cdot t = r$ where r is the remainder by the division of the product of the numbers s and t by 4. If $\pi(s,t) = s \cdot t$ $(s,t \in S)$, then (S, S, π) is a topological transitive but not ergodic and not weakly ergodic transformation semigroup. There is also the following general proposition.

Theorem 3. *If $Ss \subset sS$ for $\forall s \in S$, then every topological transitive transformation semigroup (X,S) is weakly ergodic.*

Proof. Let (X,S) be a topological transitive transformation semigroup and U be a nonempty, invariant and open subset of X. And let $X = \overline{xS}$ for some $x \in X$. Then there is $s \in S$ with $xs \in U$. Let $t \in S$. Then $st = tp$ for some $p \in S$. Since $xtp = xst \in U$ then $xt \in US^{-1}$, hence $\overline{xS} \subset \overline{US^{-1}}$ and $X = \overline{US^{-1}}$.

Theorem 4. *Let (X,S) be an ergodic transformation semigroup, X be a metric space. Then (X,S) is topological transitive. Furthermore, there is $M \subset X$ such that $\overline{M} = X$ and for $\forall x \in M$, $\overline{xS} = X$.*

Proof. Let $B = \{V_i \mid i = 1, 2, \ldots\}$ be a countable base of the topology of X and U be any nonempty open subset of X. By ergodicity of (X,S) for every natural number i we have $X = \overline{V_i S^{-1}}$. Let $M = \cap_{i=1}^{\infty} V_i S^{-1}$. By Baire's theorem the set M is nonempty and $\overline{M} = X$. Let $\forall x \in M$. Then $x \in V_i S^{-1}$ for every natural number i. Since $V_k \subset U$ for some natural number k then $x \in US^{-1}$. Whence it follows that $xS \cap U \neq \emptyset$ and $\overline{xS} = X$.

The next example will demonstrate the existence of a weakly ergodic but not topological transitive transformation semigroup with metric phase space.

Let X be a compact metric space, $f: X \to X$ be a constant mapping, S be a semigroup of nonnegative integer numbers by addition, $\pi: X \times S \to X$ be a map by the definition: $\pi(x,s) = f^s(x)$ where f^s is the constant mapping $X \to X$ if $s = 0$ and $f^s = f$ if $s \neq 0$. Then (X, S, π) is weakly ergodic with metric phase space but not topological transitive.

Theorem 5. *A distal and ergodic transformation semigroup* (X, S) *is minimal.*

Proof. By Theorem 4 and Corollary 3 from [4] the transformation semigroup (X, S) is inclosed into some transformation group (X, T) and $E(X, S) = E(X, T)$ where $E(X, S)$ and $E(X, T)$ are Ellis groups of (X, S) and (X, T) accordingly. By the definition of the inclosure of a transformation semigroup into a transformation group and by Theorem 2 it follows that the transformation group (X, T) is ergodic. In this case by Ellis theorem [2] (X, T) is minimal. Since $E(X, S) = E(X, T)$ then (X, S) is minimal, too.

Theorem 6. *Let* $\varphi : (X, S) \to (Y, S)$ *be an extension. If* (X, S) *is ergodic (weakly ergodic), then* (Y, S) *is ergodic (weakly ergodic), too.*

Proof. Let $U \subset Y$ be nonempty and open (invariant, nonempty and open). Since $A_1 = \varphi^{-1}(U) \subset X$ is nonempty and open (invariant, nonempty and open) then $X = \overline{A_1 S^{-1}}$. Because $Y = \varphi(\overline{A_1 S^{-1}}) \subset \overline{\varphi(A_1 S^{-1})} = \overline{\varphi(\varphi^{-1}(U) S^{-1})} \subset \overline{US^{-1}} \subset Y$ then $Y = \overline{US^{-1}}$ and (Y, S) is ergodic (weakly ergodic).

Theorem 7. *Let* $\varphi : (X, S) \to (Y, S)$ *be a proximal extension. If* (Y, S) *is ergodic and* $\overline{XJ} = X$, *then* (X, S) *is ergodic, too.*

Proof. We suppose that A is an invariant and closed subset of X with $V = int\, A \neq \emptyset$ and will prove that $A = X$. First we will prove that the set $B = X \setminus V$ is invariant. It is sufficient to show that $VS^{-1} = V$. For the latter is sufficient to show that $VS^{-1} \subset A$. Let $y \in VS^{-1}$ and $y \notin A$. At this point $Ut_0 \subset V$ and $U \cap A = \emptyset$ for some point $t_0 \in S$ and some neighborhood U of y. There is an almost periodic point x belonging to U. Then $\overline{xS} = \overline{xt_0 S} \subset A$ and consequently $x \in A$. But this contradicts $U \cap A = \emptyset$. The contradiction proved that $VS^{-1} \subset A$. Thus B is an invariant and closed subset of X and $B \neq X$, too. If $B = \emptyset$, then $A = X$. Suppose $B \neq \emptyset$. It is clear the set $Y \setminus \varphi(B)$ is open and $Y \setminus \varphi(B) \subset \varphi(A)$. Suppose that $Y \setminus \varphi(B) = \emptyset$ then $Y = \varphi(B)$. Let $x \in XJ$, then $\varphi(x) = \varphi(b)$ for some $b \in B$ and $\overline{xS} \cap \overline{bS} \neq \emptyset$ by proximality of φ. From the latter we have $x \in \overline{bS}$ by minimality of the set $\overline{xS}$. Hence $x \in B$, $XJ \subset B$ and $X = \overline{XJ} \subset B$. At this point $X = B$. But this contradicts $B \neq X$. Therefore $Y \setminus \varphi(B)$ is nonempty. Thus Y contains an invariant and closed subset $\varphi(A)$ with the nonempty interior. At this point $Y = \varphi(A)$. By the same argument as in the proof of the equality $X = B$ we have that $A = X$.

Corollary 1. *Every proximal transformation semigroup (X, S) with $\overline{XJ} = X$ is ergodic.*

Theorem 8. *Let X, Y be metric spaces and $\varphi : (X, S) \to (Y, S)$ be a distal extension with $YJ = Y$. If (X, S) is ergodic, then it is minimal.*

Proof. By Teorem 4 (X, S) is topological transitive. Since $YJ = Y$ and φ is distal, then $XJ = X$. At this point (X, S) is minimal.

Theorem 9. *Let $\varphi : (X, S) \to (Y, S)$ be a regionally distal extension with (Y, S) minimal. If (X, S) is ergodic, then it is minimal.*

Proof. Let X' be a minimal subset of X, $x \in X$, $x' \in X'$ and $\varphi(x) = \varphi(x')$. Since (X, S) is ergodic, then by Theorem 2 $x\alpha \cap x'\alpha s_\alpha^{-1} \neq \emptyset$ for any open index $\alpha \in \mathcal{U}[X]$ and some $s_\alpha \in S$. Therefore $x_\alpha s_\alpha \in x'\alpha$ for some $x_\alpha \in x\alpha$. Without loss of generality we may suppose that $\lim_\alpha x_\alpha = x$ and $\lim_\alpha x_\alpha s_\alpha = x'$. Since the restriction of φ to X' is an open map and $\lim_\alpha \varphi(x_\alpha) s_\alpha = \varphi(x')$, then for α there is some point $x'_\alpha \in X'$ with $\varphi(x'_\alpha) = \varphi(x_\alpha)$ and $\lim_\alpha x'_\alpha s_\alpha = x'$. Suppose that $\lim_\alpha x'_\alpha = z \in X'$. Then $(x, z) \in Q(R_\varphi)$ and $x = z$ because φ is regionally distal. Therefore $x \in X'$ and $X' = X$.

3 Mixing extensions

We shall say that the pair (φ, ψ) of the extensions $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ is disjoint (weakly disjoint, mixing, weakly mixing) if $(R_{\varphi\psi}, S)$ is minimal (topological transitive, ergodic, weakly ergodic). The extension $\varphi : (X, S) \to (Y, S)$ is called mixing (weakly mixing) if the pair (φ, φ) is mixing (weakly mixing). We denote the disjointness (weak disjointness) of pair (φ, ψ) by $\varphi \perp \psi$ ($\varphi \mathrel{\widetilde{\perp}} \psi$).

Theorem 10. *Let X, Y, Z be metric spaces, $\varphi : (X, S) \to (Y, S)$ be a distal extension and $\psi : (Z, S) \to (Y, S)$ be an extension with $ZJ = Z$. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since the projection map $R_{\varphi\psi} \to Z$ is a distal extension $(R_{\varphi\psi}, S) \to (Z, S)$, then $\varphi \perp \psi$ by Theorem 8.

Corollary 2. *Let X, Y, Z be metric spaces, $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ be distal extensions with $YJ = Y$. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since ψ is distal and $YJ = Y$, then $ZJ = Z$ and by Theorem 10 $\varphi \perp \psi$.

Corollary 3. *Let X, Y be metric spaces and $\varphi : (X, S) \to (Y, S)$ be a distal extension with $YJ = Y$. If φ is mixing, then it is minimal and it is an isomorphism.*

Corollary 4. *A distal transformation semigroup with metric phase space is trivial if it is mixing.*

Theorem 11. *Let $\varphi : (X, S) \to (Y, S)$ be regionally distal and $\psi : (Z, S) \to (Y, S)$ be minimal. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since the projection map $R_{\varphi\psi} \to Z$ is a regionally distal extension $(R_{\varphi\psi}, S) \to (Z, S)$, then $\varphi \perp \psi$ by Theorem 9.

Theorem 12. *Let $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ be regionally distal extensions with (Y, S) minimal. If the pair (φ, ψ) is mixing, then $\varphi \perp \psi$.*

Proof. Since φ and ψ are regionally distal then the Whitney's sum $(R_{\varphi\psi}, S) \to (Y, S)$ of φ and ψ is regionally distal, then $\varphi \perp \psi$ by Theorem 9.

Corollary 5. *Let $\varphi : (X, S) \to (Y, S)$ be regionally distal and (Y, S) be minimal. If φ is mixing, then it is minimal and it is an isomorphism.*

Corollary 6. *A regionally distal transformation semigroup is trivial if it is mixing.*

Let u be a fixed idempotent from a fixed minimal right ideal I of the Ellis enveloping semigroup of a universal minimal transformation semigrup of the class of all minimal transformation semigroups with the fixed phase semigroup S, $\mathcal{E} = Iu$ [5]. Henceforth it is assumed that $\varphi : (X, S) \to (Y, S)$ and $\psi : (Z, S) \to (Y, S)$ are minimal extensions; $x_0 \in Xu$, $y_0 \in Yu$ and $z_0 \in Zu$ such that $\varphi(x_0) = \psi(z_0) = y_0$; $\mathcal{A} = \{p \mid p \in \mathcal{E} \wedge x_0p = x_0\}$, $\mathcal{B} = \{p \mid p \in \mathcal{E} \wedge z_0p = z_0\}$, $\mathcal{F} = \{p \mid p \in \mathcal{E} \wedge y_0p = y_0\}$ are the Ellis groups of (X, S), (Z, S) and (Y, S), respectively [5].

The regionally distal extension $\psi : (Z, S) \to (Y, S)$ is called an RD-factor of the extension $\varphi : (X, S) \to (Y, S)$ if $\varphi = \eta \circ \psi$ for some extension $\eta : (X, S) \to (Z, S)$. The pair (φ, ψ) is called RD-prime if every common RD-factor η of φ and ψ, $\eta \neq \varphi$ and $\eta \neq \psi$, is an isomorphism. The extension φ is called RD-prime if the pair (φ, φ) is RD-prime.

The pair (φ, ψ) is called B-pair if $R_{\varphi\psi} = \overline{R_{\varphi\psi}J}$. The extension φ is called B-extension if $R_{\varphi} = \overline{R_{\varphi}J}$. The transformation semigroup (X, S) is called B-transformation semigroup if $X \times X = \overline{(X \times X)J}$.

Theorem 13. *Every mixing pair of the extensions is RD-prime.*

Proof. Let η be a maximal RD-factor of the mixing pair (φ, ψ), δ is an extension such that $\varphi = \delta \circ \eta$; $\theta = \delta \times id\, Z$; $q : R_{\eta\psi} \to Z$ is a projection map. Since the set $R_{\varphi\psi}$ is ergodic and $\theta(R_{\varphi\psi}) = R_{\eta\psi}$ then by Theorem 6 $R_{\eta\psi}$ is ergodic. Because η is regionally distal then q is regionally distal, too. At this point by Theorem 9 the set $R_{\eta\psi}$ is minimal, hence η is an isomorphism.

Corollary 7. *Every mixing transformation semigroup is RD-prime.*

From theorems 4.4.5 and 4.1.12 from [5] we obtain the following two theorems.

Theorem 14. *For the RD-prime B-pair (φ, ψ) of the extensions the following assertions are valid:*

1) *If $Ss \subset sS$ for any $s \in S$ and $\mathcal{AB}$ is a group, then the pair (φ, ψ) is weakly mixing.*

2) *If X, Y, Z are metric spaces and $\mathcal{A} = \mathcal{B}$ or $\mathcal{A}$ $(\mathcal{B})$ is an invariant subgroup of $\mathcal{F}$, then $\varphi \,\widetilde{\perp}\, \psi$.*

Theorem 15. *For the RD-prime B-extension φ we have the following assertions:*

1) *If $Ss \subset sS$ for any $s \in S$, then φ is weakly mixing.*

2) *If X and Y are metric spaces, then $\varphi \,\widetilde{\perp}\, \varphi$.*

From Theorems 13 – 15 we obtain the following results.

Theorem 16. *Let (φ, ψ) be a B-pair with the conditions: $\mathcal{AB}$ is a group; for $\forall s \in S$ $Ss \subset sS$ and for any nonempty and open set $V \subset R_{\varphi\psi}$ there exists a nonempty, open and invariant set $U \subset VS^{-1}$ (in partucular S is a group). Then the following statements are equivalent.*

1) *(φ, ψ) is RD-prime.*

2) *(φ, ψ) is weakly mixing.*

3) *(φ, ψ) is mixing.*

Theorem 17. *Let φ be a B-extension such that $Ss \subset sS$ $(s \in S)$ and for any nonempty and open set $V \subset R_{\varphi}$ there exists a nonempty, open and invariant set $U \subset VS^{-1}$ (in partucular S is a group). Then the following statements are equivalent.*

1) *φ is RD-prime.*

2) *φ is weakly mixing.*

3) *φ is mixing.*

Theorem 18. *Let S be a group, X, Y, Z be metric spaces and (φ, ψ) be a B-pair such that $\mathcal{A}$ or $\mathcal{B}$ is an invariant subgroup of $\mathcal{F}$. Then the following statements are equivalent.*

1) *(φ, ψ) is RD-prime.*

2) *$\varphi \,\widetilde{\perp}\, \psi$.*

3) *(φ, ψ) is mixing.*

Theorem 19. *Let S be a group, X and Y be metric spaces and φ be a B-extension. Then the following statements are equivalent.*

1) *φ is RD-prime.*

2) $\varphi \,\widetilde{\perp}\, \varphi$.

3) *φ is mixing.*

Corollary 8. *Let (X,S) be a RD-prime B-transformation semigroup. Then:*

1) *If for $\forall s \in S$, $Ss \subset sS$, then (X,S) is weakly mixing.*

2) *If X is metric, then $(X \times X, S)$ is topological transitive.*

Corollary 9. *Let S be a group, X be a metric space and (X,S) be a B-transformation semigroup. Then the following statements are equivalent.*

1) *(X,S) is RD-prime.*

2) *(X,S) is mixing.*

3) *$(X \times X, S)$ is topological transitive.*

Remark 1. 1) *(φ,ψ) is a B-pair, in particular, if φ or ψ is a RIC-extension* [5].

2) *$\mathcal{A}$ or $\mathcal{B}$ is an invariant subgroup of $\mathcal{F}$, in particular, if φ or ψ is regular (an extension φ is called regular if for $\forall (x,y) \in R_\varphi J$ there exists a homomorphism $\alpha : (X,S) \to (X,S)$ such that $\alpha(x) = y$).*

References

[1] Furstenberg H.,*Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation.* Math. Systems Theory, 1967, **1**, 1–50.

[2] Woude, J.C.S.P., van der, *Topological dynamics.* Mathematisch Centrum.-Amsterdam, 1982.

[3] Bronstein I.U., *Rasshirenia minimalnyh grupp preobrazovanii.* Kishinev: RIO AN MSSR, 1975 *Extensions of Minimal Transformation Groups.* Sijthoff & Noordhoff International Publishers, 1979).

[4] Bronstein I.U., Gherco A.I., *O vlozenii nekotoryh topologiceskih polugrupp preobrazovanii v topologiceskie gruppy preobrazovanii.* Izv. AN MSSR, seria fiz.-tehn. i matem. nauk, 1970, **3**, 18–24.

[5] Gherco A.I., *Rasshirenia topologiceskih polugrupp preobrazovanii.* Kishinev: Izdatelskii tsentr Mold. gosuniversiteta, 2001.

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