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Linear singular perturbations of hyperbolic-parabolic type

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Abstract. We study the behavior of solutions of the problem $\varepsilon u''(t)+u'(t)+Au(t) = f(t), u(0) = u_0$, $u'(0) = u_1$ in the Hilbert space H as $\varepsilon \to 0$, where A is a linear, symmetric, strong positive operator.

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1 Introduction

Let V and H be the real Hilbert spaces endowed with the norm $||\cdot||$ and $|\cdot|$, respectively, such that $V \subset H$, where the embedding is defined densely and continuously. By $(,)$ denote the scalar prodact in H. Let $A : V \to H$ be a linear, closed, symmetric operator and

$$(Au, u) \geq \omega ||u||^2, \quad \forall u \in V, \quad \omega > 0. \tag{1}$$

In this paper we shall study the behavior of the solutions of the problem

$$\begin{cases} \varepsilon u''(t) + u'(t) + Au(t) = f(t), \quad t > 0, \\ u(0) = u_0,\, u'(0) = u_1 \end{cases} \tag{P_ε}$$

as $\varepsilon \to 0$, where ε is a small positive parameter. Our aim is to show that $u \to v$ as $\varepsilon \to 0$, where v is the solution of the problem

$$\begin{cases} v'(t) + Av(t) = f(t), \quad t > 0 \\ v(0) = u_0. \end{cases} \tag{P_0}$$

The main tool of our approach is the relation between the solutions of the problems (P_ε) and (P_0).

For $k \in \mathbb{N}, p \in [1,\infty)$ and $(a,b) \subset (-\infty,+\infty)$ we denote by $W^{k,p}(a,b;H)$ the usual Sobolev spaces of vectorial distributions $W^{k,p}(a,b;H) = \{f \in D'(a,b;H);$ $f^{(l)} \in L^p(a,b;H), l = 0,1,\ldots,k\}$ with the norm

$$||f||_{W^{k,p}(a,b;H)} = (\sum_{l=0}^{k} ||f^{(l)}||^p_{L^p(a,b;H)})^{1/p}.$$

For each $k \in \mathbb{N}$, $W^{k,\infty}(a,b;H)$ is the Banach space equipped with the norm

$$||f||_{W^{k,\infty}(a,b;H)} = \max_{0\leq l\leq k} ||f^{(l)}||_{L^\infty(a,b;H)}$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1,\infty]$ we denote the following Banach space $W_s^{k,p}(a,b;H) = \{f : (a,b) \to H; f^{(l)}(t)e^{-st} \in L^p(a,b;H)\}$ with the norm

$$\|f\|_{W_s^{k,p}(a,b;H)} = \max_{0\leq l\leq k} ||f^{(l)}(\cdot)e^{-st}||_{L^p(a,b;H)}.$$

2 A *priori* estimates for solutions of the problem (P_ε)

In this section we shall prove the *a priori* estimates for the solutions of the problem (P_ε) which are uniform relative to the small values of parameter ε. First of all we shall remind the existence theorems for the solutions of the problems (P_ε) and (P_0).

Theorem A. *[1] For any $T > 0$ suppose that $f \in W^{1,1}(0,T;H), u_0, u_1 \in V$ and the operator A satisfies the condition (1). Then there exists a unique function $u \in C(0,T;H) \cap L^\infty(0,T;V)$ satisfying the problem (P_ε) and the conditions: $Au \in L^\infty(0,T;H)$, $u' \in L^\infty(0,T;V)$, $u'' \in L^\infty(0,T;H)$.*

Theorem B. *[1] If $f \in W^{1,1}(0,T;H), u_0 \in V$ and A satisfies the condition (1), then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ of the problem (P_0) and estimates*

$$|v(t)| \leq e^{-\omega t}\Big(|u_0| + \int_0^t e^{\omega\tau}|f(\tau)|d\tau\Big),$$

$$|v'(t)| \leq e^{-\omega t}\Big(|Au_0 - f(0)| + \int_0^t e^{\omega\tau}|f'(\tau)|d\tau\Big)$$

are true for $0 \leq t \leq T$.

Before to prove the estimates for solutions of problem (P_ε) we recall the following well-known lemma.

Lemma A. *[2] Let $\psi \in L^1(a,b)(-\infty < a < b < \infty)$ with $\psi \geq 0$ a. e. on (a,b) and let c be a fixed real constant. If $h \in C([a,b])$ verifies*

$$\frac{1}{2}h^2(t) \leq \frac{1}{2}c^2 + \int_a^t \psi(s)h(s)ds, \ \forall t \in [a,b],$$

then

$$|h(t)| \leq |c| + \int_a^t \psi(s)ds, \ \forall t \in [a,b]$$

also holds.

Denote by

$$E_1(u,t)=\varepsilon|u'(t)|+|u(t)|+\Big(\varepsilon\Big(Au(t),u(t)\Big)\Big)^{1/2}+\Big(\varepsilon\int_0^t|u'(\tau)|^2d\tau\Big)^{1/2}+$$

$$+\Big(\int_0^t\Big(Au(\tau),u(\tau)\Big)d\tau\Big)^{1/2}.$$

Lemma 1. *Suppose that for any $T>0$ $f\in W^{1,1}(0,T;H),u_0,u_1\in V$ and the operator A satisfies the condition (1). Then there exist positive constants γ and C depending on ω such that for the solutions of the problem (P_ε) the following estimates*

$$E_1(u,t)\le C\Big(E_1(u,0)+\int_0^t\Big|f(\tau)\Big|d\tau\Big),\quad 0\le t\le T,\tag{2}$$

$$E_1(u',t)\le C\Big(E_1(u',0)+\int_0^t|f'(\tau)|d\tau\Big),\quad 0\le t\le T\tag{3}$$

are true.

Proof. Denote by

$$E(u,t)=\varepsilon^2|u'(t)|^2+\frac{1}{2}|u(t)|^2+\varepsilon\Big(Au(t),u(t)\Big)+\varepsilon\int_0^t|u'(\tau)|^2d\tau+$$

$$+\varepsilon\Big(u(t),u'(t)\Big)+\int_0^t\Big(Au(\tau),u(\tau)\Big)d\tau.$$

The direct computations show that for every solution of the problem (P_ε) the following equality

$$\frac{d}{dt}E(u,t)=\Big(f(t),u(t)+2\varepsilon u'(t)\Big)\tag{4}$$

is fulfilled. From (4) it follows that

$$\frac{d}{dt}E(u,t)\le|f(t)|\Big(|u(t)|+2\varepsilon|u'(t)|\Big).\tag{5}$$

As $E(u,t)\ge 0$ and $|u(t)|+2\varepsilon|u'(t)|\le C(E(u,t))^{1/2}$, then from (5) we have

$$\frac{d}{dt}\Big(E(u,t)\Big)\le C\Big|f(t)\Big|\Big(E(u,t)\Big)^{1/2}.$$

Integrating the last inequality we obtain

$$\frac{1}{2}E(u,t)\le\frac{1}{2}E(u,0)+C\int\Big(E(u,\tau)\Big)^{1/2}\Big|f(\tau)\Big|d\tau.$$

From the last inequality using Lemma A we get the estimate

$$\Big(E(u,t)\Big)^{1/2}\le C\Big[\Big(E(u,0)\Big)^{1/2}+\int_0^t\Big|f(\tau)\Big|d\tau\Big].\tag{6}$$

It is easy to see that there exist positive constants C_0, C_1 such that

$$C_0\Big(E(u,t)\Big)^{1/2} \leq E_1(u,t) \leq C_1\Big(E(u,t)\Big)^{1/2}. \tag{7}$$

Using the inequality (7) from (6) we obtain the estimate (2).

To prove the estimate (3) let us denote by

$$E_h(u,t) = \varepsilon^2|u'(t+h)-u'(t)|^2 + \varepsilon\Big(A(u(t+h)-u(t)), u(t+h)-u(t)\Big)+$$

$$+\frac{1}{2}|u(t+h)-u(t)|^2 + \varepsilon\Big(u'(t+h)-u'(t), u(t+h)-u(t)\Big)+$$

$$\varepsilon\int_0^t |u'(\tau+h)-u'(\tau)|^2 d\tau +$$

$$+\int_0^t \Big(A(u(\tau+h)-u(\tau)), u(\tau+h)-u(\tau)\Big)d\tau,\ h>0, t\geq 0.$$

For any solution of the problem (P_ε) we have

$$\frac{d}{dt}E_h(u,t) = \Big(2\varepsilon(u'(t+h)-u'(t)) + u(t+h)-u(t), f(t+h)-f(t)\Big),\ t\geq 0.$$

Dividing the last equality by h^2 and then passing to the limit as $h\to 0$ we get

$$\frac{d}{dt}E(u',t) = \Big(f'(t), 2\varepsilon u''(t) + u'(t)\Big). \tag{8}$$

Since $u'(0)=u_1, \varepsilon u''(0) = f(0) - u_1 - Au_0$, then the estimate (3) follows from (8) in the same way as the estimate (2) follows from (4). Lemma 1 is proved.

3 Relation between the solutions of the problems (P_ε) and (P_0)

In this section we shall give the relation between the solutions of the problems (P_ε) and (P_0). This relation was inspired by the work [3]. At first we shall prove some properties of the kernel $K(t,\tau)$ of transformation which realizes this connection.

For $\varepsilon > 0$ denote

$$K(t,\tau) = \frac{1}{2\varepsilon\sqrt{\pi}}\Big(K_1(t,\tau) + 3K_2(t,\tau) - 2K_3(t,\tau)\Big),$$

where

$$K_1(t,\tau) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big\{\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big\}, \tag{9}$$

$$K_2(t,\tau) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big), \tag{10}$$

$$K_3(t,\tau) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \tag{11}$$

and $\lambda(s) = \int_s^\infty e^{-\eta^2}d\eta$.

Lemma 2. *The function $K(t,\tau)$ possesses the following properties:*

(i) $K \in C(\overline{R}_+ \times \overline{R}_+) \cap C^2(R_+ \times R_+)$;

(ii) $K_t(t,\tau) = \varepsilon K_{\tau\tau}(t,\tau) - K_\tau(t,\tau), \quad t > 0, \tau > 0$;

(iii) $\varepsilon K_\tau(t,0) - K(t,0) = 0, \quad t \geq 0$;

(iv) $K(0,\tau) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}, \quad \tau \geq 0$;

(v) *For each fixed $t > 0$, there exist constants $C_1(t,\varepsilon) > 0$ and $C_2(t) > 0$ such that*

$$|K(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\}, \quad |K_t(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\},$$

$$|K_\tau(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\}, \quad |K_{\tau\tau}(t,\tau)| \leq C_1(t,\varepsilon)\exp\{-C_2(t)\tau/\varepsilon\}$$

for $\tau > 0$;

(vi) $K(t,\tau) > 0, \quad t \geq 0, \quad \tau \geq 0$;

(vii) *For any $\varphi : [0,\infty) \to H$ continuous on $[0,\infty)$ such that $|\varphi(t)| \leq M\exp\{Ct\}$ for $t \geq 0$, the relation*

$$\lim_{t\to 0}\int_0^\infty K(t,\tau)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau$$

is valid in H for each fixed ε, $0 < \varepsilon \ll 1$;

(viii) $\int_0^\infty K(t,\tau)d\tau = 1, \quad t \geq 0$;

(ix) *Let $\rho : [0,\infty) \to \mathbb{R}, \rho \in C^1[0,\infty), \rho$ and ρ' be increasing functions and $|\rho(t)| \leq Me^{ct}, |\rho'(t)| \leq Me^{ct}$, for $t \in [0,\infty)$. Then there exist positive constants C_1 and C_2 such that*

$$\int_0^\infty K(t,\tau)|\rho(t) - \rho(\tau)|d\tau \leq C_1\sqrt{\varepsilon}e^{C_2 t}, \quad t > 0;$$

(x) *Let $f(t)e^{-Ct}, f'(t)e^{-Ct} \in L^\infty(0,\infty;H)$ with some $C \geq 0$. Then there exist positive constants C_1, C_2 such that*

$$\Big|f(t) - \int_0^\infty K(t,\tau)f(\tau)d\tau\Big|_H \leq C_1\sqrt{\varepsilon}e^{C_2 t}\|f'\|_{L^\infty_C(0,\infty;H)}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1;$$

(xi) *There exists $C > 0$ such that*

$$\int_0^t\int_0^\infty K(\tau,\theta)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Proof. The properties **(i)-(iv)** can be verified by direct calculation.

Proof **(v).** From (9), (10) and (11) we have

$$K_t(t,\tau)=\frac{1}{8\pi\varepsilon^2}\Big[3K_1(t,\tau)+9K_2(t,\tau)-6\sqrt{\frac{\varepsilon}{t}}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big], t>0,\tau>0, \quad (12)$$

$$K_\tau(t,\tau)=\frac{1}{4\pi\varepsilon^2}\Big[-K_1(t,\tau)+9K_2(t,\tau)-4K_3(t,\tau)\Big],\quad t>0,\tau>0, \quad (13)$$

$$K_{\tau\tau}(t,\tau)=\frac{1}{8\pi\varepsilon^3}\Big[K_1(t,\tau)+27K_2(t,\tau)-8K_3(t,\tau)-$$
$$-6\sqrt{\frac{\varepsilon}{t}}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big],\quad t>0,\tau>0. \quad (14)$$

As $|\lambda(s)|\le\sqrt{\pi}$ for $s\in\mathbb{R}$ and $|\exp\{s^2\}\lambda(s)|\le C$ for $s\in[0,\infty)$, then

$$\Big|K_1(t,\tau)\Big|\le\exp\Big\{\frac{t-2\tau}{4\varepsilon}\Big\},\quad \tau>0,t>0, \quad (15)$$

$$\Big|K_2(t,\tau)\Big|\le C\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\quad t>0,\tau>0, \quad (16)$$

$$\Big|K_3(t,\tau)\Big|\le C\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\quad t>0,\tau>0. \quad (17)$$

Using (15), (16) and (17) from (12), (13) and (14) we get the estimates from property **(v)**. The property **(v)** is proved.

Proof **(vi).** We shall prove property **(vi)** using the maximum principle for the solutions of equation **(ii)**. It is easy to see that

$$K(t,0)=\frac{1}{\varepsilon\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],\quad t\ge 0. \quad (18)$$

We intend to prove that

$$K(t,0)>0,\quad t\ge 0. \quad (19)$$

To this end we consider the function $f(s)=2q(s)-q(s/2)$, where $q(s)=\exp\{s^2\}\lambda(s), s\in[0,\infty)$. Because $K(t,0)=(\sqrt{\varepsilon\pi})^{-1}\exp\{-t/4\varepsilon\}f(\sqrt{t/\varepsilon})$, to prove (19) it is sufficient to show that $f(s)>0$ for $s\in[0,\infty)$. At first we shall prove that $q'(s)<0$ for $s\in[0,\infty)$. Since

$$q'(s)=2sq(s)-1,\ q''(s)=2(2s^2+1)q(s)-2s,\ q'''(s)=(8s^3+12s)q(s)-4(s^2+1)$$

and $\lim_{s\to+\infty}2sq(s)=1$, then $q'(0)=-1$ and $\lim_{s\to+\infty}q'(s)=0$. Suppose that there exists $s_1\in(0,\infty)$ such that $q''(s_1)=0$, i. e. $q(s_1)=s_1(2s_1^2+1)^{-1}$. As $q'''(s_1)=-4(2s_1^2+1)^{-1}$, then s_1 is the point of maximum for $q'(s)$, and $q'(s_1)<0, s_1\in[0,\infty)$ and consequently the function $q(s)$ is decreasing on $(0,\infty)$. Further, we note that

$$f(0)=q(0)=\frac{\sqrt{\pi}}{2},\quad \lim_{s\to+\infty}f(s)=0. \quad (20)$$

Suppose that $s_1 \in (0, \infty)$ is any critical point for function $f(s)$, i. e. $f'(s_1) = 0$, then we have: $4s_1 q(s_1) - 2^{-1} s_1 q(s_1/2) - 3/2 = 0$, from which follows

$$f(s_1) = 2q(s_1) - q\Big(\frac{s_1}{2}\Big) = \frac{3}{s_1} - 6q(s_1). \tag{21}$$

As $q'(s) < 0$ for $s \in (0, \infty)$, then $2s_1 q(s_1) < 1$. Hence from (21) it follows that $f(s_1) > 0$. The last condition and conditions (20) permit us to conclude that $f(s) > 0$ for $s \in [0, \infty)$, i. e. $K(t, 0) > 0$ for $t \geq 0$. Finally, from **(ii)**, **(iv)**, **(v)** and (18) it follows that the function $V(t, \tau) = \exp\{(t - 2\tau)/4\varepsilon\} K(t, \tau)$ is the bounded solution of the problem

$$\begin{cases} V_t(t,\tau) = \varepsilon V_{\tau\tau}(t,\tau), \quad t > 0, \tau > 0 \\ V(0,\tau) = \dfrac{1}{2\varepsilon} \exp\Big\{ -\dfrac{\tau}{\varepsilon}\Big\}, \quad \tau \geq 0, \\ V(t,0) = \dfrac{1}{\varepsilon\sqrt{\pi}} f\Big(\sqrt{\dfrac{t}{\varepsilon}}\Big), \quad t \geq 0, \end{cases} \tag{$P.V$}$$

in $Q_T = \{(t, \tau) : \tau \geq 0, 0 \leq t \leq T\}$, for any $T > 0$. Using the maximum principle for the solutions of problem $(P.V)$ we conclude that $V(t, \tau) > 0$ and consequently $K(t, \tau) > 0$. The property **(vi)** is proved.

Proof **(vii).** For any fixed $C > 0$ and for any fixed $\varepsilon > 0$, we get

$$\int_0^\infty K_2(t,\tau) e^{C\tau} d\tau = \frac{2\varepsilon}{3 + 2C\varepsilon}\Big[\exp\Big\{C(1 + C\varepsilon)t\Big\}\lambda\Big(-\frac{1 + 2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big) -$$

$$-\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)\Big] = \frac{2\varepsilon}{3 + 2C\varepsilon}\Big[\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)\Big(1 - \exp\Big\{\frac{3t}{4\varepsilon}\Big\}\Big) + \int\limits_{-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}}^{\sqrt{\frac{t}{\varepsilon}}} e^{-\eta^2} d\eta -$$

$$-\Big(1 - \exp\Big\{C(1 + C\varepsilon)t\Big\}\Big)\lambda\Big(-\frac{1 + 2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] = O(\sqrt{t}), \quad t \to 0. \tag{22}$$

If $\varphi : [0, \infty) \to H$, and $|\varphi(t)|_H \leq M e^{Ct}, t \geq 0$, then from (22) we have

$$\Big|\int_0^\infty K_2(t,\tau)\varphi(\tau))d\tau\Big|_H \leq M \int_0^\infty K_2(t,\tau) e^{C\tau} d\tau \leq MC(\varepsilon)\sqrt{t}, \quad 0 < t \ll 1, \tag{23}$$

for any fixed $\varepsilon > 0$. Similarly as was obtained (22) we get

$$\int_0^\infty K_3(t,\tau) e^{C\tau} d\tau = \frac{\varepsilon}{1 + C\varepsilon}\Big[\exp\{C(1 + C\varepsilon)t\Big\}\lambda\Big(-\frac{1 + 2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] =$$

$$= \frac{\varepsilon}{1 + C\varepsilon}\Big[\Big(\exp\Big\{C(1 + C\varepsilon)t\Big\} - 1\Big)\lambda\Big(-\frac{1 + 2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big) +$$

$$+\int\limits_{-\frac{1+2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}}^{\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}} e^{-\eta^2}d\eta\Big] = O(\sqrt{t}), \quad t \to 0, \tag{24}$$

for any fixed $\varepsilon > 0$. If $\varphi : [0, \infty) \to H$, and $|\varphi(t)|_H \le M \exp\{Ct\}, t \ge 0$, then from (24) it follows that

$$\Big|\int_0^\infty K_3(t,\tau)\varphi(\tau)d\tau\Big|_H \le M\int_0^\infty K_3(t,\tau)\exp\{C\tau\}d\tau \le C(\varepsilon)M\sqrt{t} \tag{25}$$

for $0 < t \ll 1$. For $\varphi : [0, \infty) \to H$, $\varphi \in C(0, \infty; H)$ and $|\varphi(t)|_H \le M \exp\{Ct\}$, $t \ge 0$, we have

$$\int_0^\infty K_1(t,\tau)\varphi(\tau)d\tau = \exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\Big[\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big) - \lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)\Big]\varphi(\tau)d\tau +$$

$$+\Big(\exp\Big\{\frac{3t}{4\varepsilon}\Big\} - 1\Big)\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)\varphi(\tau)d\tau +$$

$$+\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big)\varphi(\tau)d\tau = I_1 + I_2 + I_3. \tag{26}$$

Let us evaluate the integrals I_i, $i = 1, 2, 3$, from (26). For any fixed $0 < \varepsilon < (2C)^{-1}$ we have

$$\begin{aligned}|I_1|_H &\le M\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{-\frac{\tau}{4\varepsilon} + C\tau\Big\}\int_{-\frac{\tau}{2\sqrt{\varepsilon t}}}^{\frac{2t-\tau}{2\sqrt{\varepsilon t}}} \exp\Big\{-\eta^2\Big\}d\eta d\tau \le \\ &\le \frac{2M}{1-2C\varepsilon}\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\sqrt{\varepsilon t} \le C(\varepsilon)\sqrt{t}, \quad 0 < t \ll 1,\end{aligned} \tag{27}$$

and

$$\begin{aligned}|I_2|_H &\le M\Big|\exp\Big\{\frac{3t}{4\varepsilon}\Big\} - 1\Big|\sqrt{\pi}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon} + C\tau\Big\}d\tau \le \\ &\le C(\varepsilon)t, \quad 0 < t \ll 1.\end{aligned} \tag{28}$$

At last, let us investigate the behaviour of integral I_3 as $t \to 0$. I_3 can be represented in the form

$$I_3 = \int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\Big[\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big) - \sqrt{\pi}\Big]\varphi(\tau)d\tau + \sqrt{\pi}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\varphi(\tau)d\tau. \tag{29}$$

The first term of the right side of (29) can be evaluated as follows

$$\Big|\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\Big[\lambda\Big(-\frac{\tau}{2\sqrt{\varepsilon t}}\Big) - \sqrt{\pi}\Big]\varphi(\tau)d\tau\Big|_H \le$$

$$\leq M\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}+C\tau\Big\}\lambda\Big(\frac{\tau}{2\sqrt{\varepsilon t}}\Big)d\tau =$$

$$=\frac{2M\varepsilon}{1-2C\varepsilon}\Big[\lambda(0)-\exp\Big\{\frac{(1-2C\varepsilon)^2 t}{4\varepsilon}\Big\}\lambda\Big(\frac{1-2C\varepsilon}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] =$$

$$=\frac{2M\varepsilon}{1-2C\varepsilon}\Big[\Big(1-\exp\Big\{\frac{(1-2C\varepsilon)^2 t}{4\varepsilon}\Big\}\Big)\lambda(0)+$$

$$+\exp\Big\{\frac{(1-2C\varepsilon)^2 t}{4\varepsilon}\Big\}\int\limits_0^{\frac{(1-2C\varepsilon)^2}{2}\sqrt{\frac{t}{\varepsilon}}}\exp\Big\{-\eta^2\Big\}d\eta\Big] \leq C(\varepsilon)\sqrt{t},\quad 0<t\ll 1. \tag{30}$$

From (29) and (30) follows the estimate

$$\Big|I_3-\sqrt{\pi}\int_0^\infty \exp\Big\{-\frac{\tau}{2\epsilon}\Big\}\varphi(\tau)d\tau\Big|_H \leq C(\varepsilon)\sqrt{t},\quad 0<t\ll 1. \tag{31}$$

Hence due to (26), (27), (28) and (31) we have

$$\Big|\int_0^\infty K_1(t,\tau)\varphi(\tau)d\tau-2\varepsilon\sqrt{\pi}\int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau\Big|_H \leq C\sqrt{t},\quad 0<t\ll 1, \tag{32}$$

for any fixed ε, $0<\varepsilon\ll 1$. Finally, from (23), (25) and (32) we get the proof of the property **(vii)**.

Proof **(viii).** Integrating by parts we have

$$\int_0^\infty K_1(t,\tau)d\tau = 2\varepsilon\Big[\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)+\lambda\Big(-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],$$

$$\int_0^\infty K_2(t,\tau)d\tau = \frac{2\varepsilon}{3}\Big[\lambda\Big(-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)-\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)\Big],$$

$$\int_0^\infty K_3(t,\tau)d\tau = \varepsilon\Big[\lambda\Big(-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big],$$

from which follows the proof of the property **(viii)**.

Proof **(ix).** As ρ is increasing and $|\rho(t)|\leq M\exp(Ct)$, then integrating by parts and using the property **(v)** we get

$$\int_0^\infty K_1(t,\tau)|\rho(t)-\rho(\tau)|d\tau = \exp\Big\{\frac{3t}{4\varepsilon}\Big\}\Big[\int_0^t \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)\Big(\rho(t)-\rho(\tau)\Big)d\tau+$$

$$+\int_t^{\infty}\exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)\Big(\rho(\tau)-\rho(t)\Big)d\tau\Big] = 2\varepsilon\Big(\rho(t)-\rho(0)\Big)\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)+$$

$$+\sqrt{\frac{\varepsilon}{t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau - 2\varepsilon\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\times$$

$$\times\int_0^{\infty}\exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\rho'(\tau)\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)\,\mathrm{sign}(t-\tau)d\tau. \qquad (33)$$

Similarly can be obtained the equalities

$$\int_0^{\infty}K_2(t,\tau)|\rho(t)-\rho(\tau)|d\tau = -\frac{2\varepsilon}{3}\Big(\rho(t)-\rho(0)\Big)\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)+$$

$$+\frac{1}{3}\sqrt{\frac{\varepsilon}{t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau+$$

$$+\frac{2\varepsilon}{3}\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^{\infty}\exp\Big\{\frac{3\tau}{2\varepsilon}\Big\}\rho'(\tau)\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)\,\mathrm{sign}(t-\tau)d\tau, \qquad (34)$$

and

$$\int_0^{\infty}K_3(t,\tau)|\rho(t)-\rho(\tau)|d\tau = -\varepsilon\Big(\rho(t)-\rho(0)\Big)\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)+$$

$$+\frac{1}{2}\sqrt{\frac{\varepsilon}{t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau+$$

$$+\varepsilon\int_0^{\infty}\exp\Big\{\frac{\tau}{\varepsilon}\Big\}\rho'(\tau)\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big)\,\mathrm{sign}(t-\tau)d\tau, \qquad (35)$$

As a consequence from (33), (34) and (35) we get

$$\int_0^{\infty}K(t,\tau)|\rho(t)-\rho(\tau)|d\tau = \frac{1}{\sqrt{\pi}}\Big[\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big(\rho(t)-\rho(0)\Big)+$$

$$+\frac{1}{2\sqrt{\varepsilon t}}\int_0^{\infty}\exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}\Big|\rho(t)-\rho(\tau)\Big|d\tau+$$

$$+\int_0^{\infty}\rho'(\tau)\Big[\exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)-\exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)-$$

$$-\exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big)\Big]\,\mathrm{sign}(t-\tau)d\tau\Big], \qquad (36)$$

Since $\rho'(t)$ is increasing and $|\rho'(t)| \leq M\exp(Ct)$, then it follows that

$$\begin{aligned}\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big(\rho(t)-\rho(0)\Big) &\leq \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)Mt\exp\{Ct\} \leq \\ &\leq C_1 t\exp\Big\{-\frac{t}{4\varepsilon}+Ct\Big\} \leq C_1\varepsilon\exp\{C_2 t\}, \quad t\geq 0, \quad \varepsilon\leq\frac{1}{8C}.\end{aligned} \tag{37}$$

Further we have

$$\int_0^\infty \exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}\Big\}|\rho(t)-\rho(\tau)|d\tau \leq$$

$$\leq M\int_0^\infty \exp\Big\{-\frac{(t-\tau)^2}{4\varepsilon t}+C\max\{t,\tau\}\Big\}|t-\tau|d\tau =$$

$$= 4M\varepsilon t\int\limits_{-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}}^{\infty}|\eta|\exp\Big\{-\eta^2+C\max\{t,t+2\eta\sqrt{\varepsilon t}\}\Big\}d\eta =$$

$$= 4M\varepsilon t\exp\{Ct\}\Big(\int\limits_{-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}}^{0}|\eta|\exp\{-\eta^2\}d\eta+\int_0^\infty \eta\exp\Big\{-\eta^2+2C\sqrt{\varepsilon t}\eta\Big\}d\eta\Big) \leq$$

$$\leq C_1\varepsilon t\exp\{C_2 t\}, \quad t\geq 0. \tag{38}$$

As $|\lambda(s)\exp\{s^2\}| \leq C$, for $s\geq 0$, then we have

$$\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty |\rho'(\tau)|\exp\Big\{\frac{3\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)d\tau \leq$$

$$\leq M\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{C\tau+\frac{3\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big)d\tau \leq C_1\int_0^\infty \exp\Big\{C\tau-\frac{(t-\tau)^2}{4\varepsilon\tau}\Big\} =$$

$$= C_1\sqrt{\varepsilon t}\exp\{Ct\}\int\limits_{-\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}}^{\infty}\exp\Big\{2C\sqrt{\varepsilon t}\eta-\eta^2\Big\}d\eta \leq C_1\sqrt{\varepsilon t}\exp\Big\{C_2 t\Big\}, \quad t\geq 0. \tag{39}$$

Similarly we get the estimates

$$\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\int_0^\infty \exp\Big\{-\frac{\tau}{2\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big)|\rho'(\tau)|d\tau \leq C_1\sqrt{\varepsilon t}\exp\Big\{C_2 t\Big\}, \quad t\geq 0, \tag{40}$$

and

$$\int_0^\infty \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{\tau+t}{2\sqrt{\varepsilon t}}\Big)|\rho'(\tau)|d\tau \leq C_1\sqrt{\varepsilon t}\exp\Big\{C_2 t\Big\}, \quad t\geq 0. \tag{41}$$

Finally from (36) and the estimates (37)-(41) follows the estimate from property (**ix**).

Proof **(x).** From the properties (**viii**) and (**ix**) it follows that

$$\Big|f(t)-\int_0^\infty K(t,\tau)f(\tau)d\tau\Big|_H \le \int_0^\infty K(t,\tau)|f(t)-f(\tau)|_H\, d\tau \le$$
$$\le \int_0^\infty K(t,\tau)\Big|\int_\tau^t |f'(\theta)|_H\, d\theta\Big| \le M\int_0^\infty K(t,\tau)\,|e^{C\tau}-e^{Ct}|d\tau \le$$
$$\le C_1\sqrt{\varepsilon}\, e^{C_2 t}\|f'\|_{L^\infty_C(0,\infty;H)},$$

for $t\ge 0, 0\le\varepsilon\ll 1$. Property (**x**) is proved.

Proof **(xi).** Denote by $\mathcal{K}(t,\tau)=K(t,\tau)|_{\varepsilon=1}, \mathcal{K}_i(t,\tau)=K_i(t,\tau)|_{\varepsilon=1}, i=1,2,3$. Then

$$I=\int_0^t\int_0^\infty K(\tau,\theta)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau=\varepsilon\int_0^{\frac{t}{\varepsilon}}\int_0^\infty \mathcal{K}(\tau,\theta)\exp\{-\theta\}d\theta d\tau=$$
$$=\frac{\varepsilon}{2\sqrt{\pi}}\Big(I_1+3I_2-2I_3\Big). \tag{42}$$

As $0<\mathcal{K}_i(\tau,\theta)\le C\exp\Big\{-\frac{(\tau-\theta)^2}{4\tau}\Big\}, i=2,3,$ then

$$I_i\le\int_0^{\frac{t}{\varepsilon}}\int_0^\infty \exp\Big\{\frac{(\tau+\theta)^2}{4\tau}\Big\}d\theta d\tau\le C,\quad t\ge 0, i=2,3. \tag{43}$$

For I_1 we have the estimate

$$I_1=\int_0^{\frac{t}{\varepsilon}}\int_0^\infty \mathcal{K}_1(\tau,\theta)e^{-\theta}d\theta d\tau=\int_0^{\frac{t}{\varepsilon}}\exp\Big\{-\frac{9\tau}{4}\Big\}\int_{-\infty}^{\sqrt{\tau}}\exp\Big\{3\eta\sqrt{\tau}\Big\}\lambda(\eta)d\eta d\tau=$$
$$=\frac{1}{3}\int_0^{\frac{t}{\varepsilon}}\tau^{-1/2}\exp\Big\{\frac{3\tau}{4}\Big\}\lambda(\sqrt{\tau})d\tau-\frac{1}{3}\int_0^{\frac{t}{\varepsilon}}\tau^{-1/2}\lambda(\frac{\sqrt{\tau}}{2})d\tau\le C,\quad t\ge 0. \tag{44}$$

From (42), (43) and (44) follows the property (**xi**). Lemma 2 is proved.

Now we are ready to establish the relation between the solutions of the problem (P_ε) and the corresponding solutions of the problem (P_0).

Theorem 1. *Let $A:D(A)\subset H\to H$ be a linear and closed operator, $f\in W^{1,\infty}_C(0,\infty;H)$ for some $C\ge 0$. If u is a solution of the problem (P_ε) such that $u\in W^{2,\infty}_C(0,\infty;H)$ with some $C\ge 0$, then the function v_0 which is defined by*

$$v_0(t)=\int_0^\infty K(t,\tau)u(\tau)d\tau$$

satisfies the following conditions:

$$\begin{cases} v_0'(t)+Av_0(t)=F_0(t,\varepsilon), & t>0,\\ v_0(0)=\varphi_\varepsilon, \end{cases} \tag{P.v_0}$$

where

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1 + \int_0^\infty K(t,\tau)f(\tau)d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}u(2\varepsilon\tau)d\tau.$$

Proof. Integrating by parts and using the properties (**i**) – (**iii**) and (**v**) of Lemma 2 we get

$$v_0'(t) = \int_0^\infty K_t(t,\tau)u(\tau)d\tau = \int_0^\infty \Big(\varepsilon K_{\tau\tau}(t,\tau) - K_\tau(t,\tau)\Big)u(\tau)d\tau =$$

$$= \int_0^\infty K(t,\tau)\Big(\varepsilon u''(\tau) + u'(\tau)\Big)d\tau + \varepsilon K(t,0)u_1 - Av_0(t) + \int_0^\infty K(t,\tau)f(\tau)d\tau.$$

Thus $v_0(t)$ satisfies the equation from $(P.v_0)$. From property (**viii**) of Lemma 2 follows the validity of the initial condition of $(P.v_0)$. Theorem 1 is proved.

4 The limit of the solutions of the problem (P_ε) as $\varepsilon \to 0$

In this section we shall study the behavior of the solutions of the problem (P_ε) as $\varepsilon \to 0$.

Theorem 2. *Suppose $f \in W_C^{1,\infty}(0,\infty;H)$, with some $C \geq 0$, $u_0, u_1 \in H, Au_0, Au_1 \in H$ and the operator A satisfies the condition (1). Then*

$$|u(t) - v(t)| \leq C_1 M e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{45}$$

where u and v are the solutions of the problems (P_ε) and $(P.v)$, respectively,

$$M = |f(0)| + |u_0| + |Au_0| + |u_1| + ||f'||_{L_C^\infty(0,\infty;H)},$$

and C_1 and C_2 are independent of M and ε.

If

$$u_0, Au_0, u_1, f(0) \in V, f \in W_C^{2,\infty}(0,\infty;H), \quad \textit{with some} \quad C \geq 0, \tag{46}$$

then

$$\Big|u'(t) - v'(t) + h\exp\Big\{-\frac{t}{\varepsilon}\Big\}\Big| \leq C_1 M_1 e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{47}$$

where $h = f(0) - u_1 - Au_0$, $M_1 = |f'(0)| + |Ah| + ||f''||_{L_C^\infty(0,\infty;H)}$, and C_1 and C_2 are independent of M_1 and ε.

If

$$u_0, Au_0, Au_1 \in V, Af \in W_C^{1,\infty}(0,\infty;H), \quad \textit{with some} \quad C \geq 0, \tag{48}$$

then

$$||u(t) - v(t)|| \leq C_1 M_2 e^{C_2 t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 \leq \varepsilon \ll 1, \tag{49}$$

where $M_2 = |Af(0)| + |Au_0| + |Au_1| + |A^2u_0| + ||Af'||_{L_C^\infty(0,\infty;H)}$, and C_1 and C_2 are independent of M_2 and ε.

Proof. Under the conditions of the theorem from (3) follows the estimate

$$|u'(t)| \leq CM, \quad t \geq 0. \tag{50}$$

According to Theorem 1 the function w which is defined by

$$w(t) = \int_0^\infty K(t,\tau)u(\tau)d\tau$$

is a solution of the problem

$$\begin{cases} w'(t) + Aw(t) = F(t,\varepsilon), \\ w(0) = w_0, \end{cases} \tag{P.w}$$

where

$$F(t,\varepsilon) = F_0(t,\varepsilon) + \int_0^\infty K(t,\tau)f(\tau)d\tau,$$

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]u_1, \quad w_0 = \int_0^\infty e^{-\tau}u(2\varepsilon\tau)d\tau.$$

Using the property (**x**) of Lemma 2 and the estimate (50) we get

$$|u(t) - w(t)| \leq C_1 M e^{c_2 t}\sqrt{\varepsilon}, \quad t \geq 0. \tag{51}$$

Let us denote $R(t) = v(t) - w(t)$, where v is the solution of the problem $(P.v)$ and w is the solution of the problem $(P.w)$. Then $R(t)$ is the solution of the problem

$$\begin{cases} R'(t) + AR(t) = \mathcal{F}(t,\varepsilon), \quad t \geq 0, \\ R(0) = R_0, \end{cases}$$

where $R_0 = u_0 - w_0$ and

$$\mathcal{F}(t,\varepsilon) = f(t) - \int_0^\infty K(t,\tau)f(\tau)d\tau - F_0(t,\varepsilon).$$

As

$$\frac{d}{dt}|R(t)|^2 = -2\Big(AR(t), R(t)\Big) + 2\Big(\mathcal{F}(t,\varepsilon), R(t)\Big) \leq$$
$$\leq -2\omega|R(t)|^2 + 2|\mathcal{F}(t,\varepsilon)||R(t)|, \quad t \geq 0,$$

and hence

$$\frac{1}{2}|R(t)|^2 e^{2\omega t} \leq \frac{1}{2}|R_0|^2 + \int_0^t |\mathcal{F}(\tau,\varepsilon)||R(\tau)|e^{2\omega\tau}d\tau, \quad t \geq 0,$$

then using Lemma A we obtain the estimate

$$|R(t)| \leq e^{-\omega t}\Big(|R_0| + \int_0^t |\mathcal{F}(\tau,\varepsilon)|e^{\omega\tau}d\tau\Big), \quad t \geq 0. \tag{52}$$

From (50) follows the estimate

$$|R_0| \leq \int_0^\infty e^{-\tau}|u(2\varepsilon\tau) - u_0|d\tau \leq \int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}|u'(s)|dsd\tau \leq CM\varepsilon \tag{53}$$

for $0 < \varepsilon \ll 1$. Now let us estimate $|\mathcal{F}(t,\varepsilon)|$. Using the property $(\mathbf{x})$ of Lemma 2 we have

$$\Big|f(t) - \int_0^\infty K(t,\tau)f(\tau)d\tau\Big| \leq C_1M\sqrt{\varepsilon}e^{C_2t}, \quad t \geq 0. \tag{54}$$

As

$$\int_0^t \exp\Big\{\frac{3\tau}{4\varepsilon} + \omega\tau\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau = \varepsilon\int_0^{\frac{t}{\varepsilon}} \exp\Big\{\frac{3\tau}{4} + \omega\tau\Big\}\lambda\Big(\sqrt{\tau}\Big)d\tau$$
$$\leq C\int_0^\infty e^{\tau}\lambda(\sqrt{\tau}) \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1,$$

and

$$\int_0^t e^{\omega\tau}\lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau \leq C\varepsilon, \quad t \geq 0, \quad 0 < \varepsilon \ll 1,$$

then

$$\int_0^t e^{\omega\tau}|F_0(\tau,\varepsilon)|d\tau \ \leq C\varepsilon|u_1| \leq C\varepsilon M, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{55}$$

From (54) and (55) follows the estimate

$$\int_0^t e^{\omega\tau}|\mathcal{F}(\tau,\varepsilon)|d\tau \leq C_1Me^{\omega t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{56}$$

From (52), using the estimates (53) and (56) we get

$$|R(t)| \leq C_1Me^{C_2t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{57}$$

Finally from estimates (51) and (57) we have

$$|u(t) - v(t)| \leq |u(t) - w(t)| + |R(t)| \leq C_1Me^{C_2t}\sqrt{\varepsilon}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1.$$

The estimate (45) is proved.

Let us prove the estimate (47). Denote by $z(t) = u'(t) + h\exp\Big\{-\dfrac{t}{\varepsilon}\Big\}$. If u_0, u_1 and f satisfy the conditions (46) and A satisfies the condition (1), then $z(t)$ is a solution of the problem

$$\begin{cases} \varepsilon z''(t) + z'(t) + Az(t) = f'(t) + \exp\Big\{-\dfrac{t}{\varepsilon}\Big\}h, & t \geq 0, \\ z(0) = f(0) - Au_0, \quad z'(0) = 0. \end{cases}$$

According to Theorem 1 the function $w_1(t)$ which is defined by

$$w_1(t) = \int_0^\infty K(t,\tau)z(\tau)d\tau$$

is a solution of the problem

$$\begin{cases} w_1'(t) + Aw_1(t) = \mathcal{F}_1(t,\varepsilon), \quad t \geq 0, \\ w_1(0) = \displaystyle\int_0^\infty \exp\Big\{-\tau\Big\} z(2\varepsilon\tau)d\tau, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \int_0^\infty K(t,\tau)\Big[f'(\tau) - \exp\Big\{-\frac{t}{\varepsilon}\Big\}Ah\Big]d\tau.$$

Further denote by $v_1(t) = v'(t)$, where $v(t)$ is the solution of the problem $(P.v)$. Then $v_1(t)$ is the solution of the problem

$$\begin{cases} v_1'(t) + Av_1(t) = f'(t), \quad t \geq 0, \\ v_1(0) = f(0) - Au_0. \end{cases}$$

Let $R_1(t) = w_1(t) - v_1(t)$. Then $R_1(t)$ is the solution of the problem

$$\begin{cases} R_1'(t) + AR(t) = \mathcal{F}_1(t,\varepsilon) - f'(t), \quad t \geq 0, \\ R_1(0) = \displaystyle\int_0^\infty \exp\Big\{-\tau\Big\} \int_0^{2\varepsilon\tau} z'(\theta)d\theta d\tau. \end{cases}$$

Using Theorem B we obtain the estimate

$$|R_1(t)| \leq e^{-\omega t}\Big(|R_1(0)| + \int_0^t e^{\omega t}|\mathcal{F}_1(\tau,\varepsilon) - f'(\tau)|d\tau\Big), \quad t \geq 0. \tag{58}$$

Using the estimate (3) we get

$$|z'(t)| \leq C_1\Big(|f'(0) + Ah| + \int_0^t \Big|f''(t) - \frac{1}{\varepsilon}\exp\Big\{-\frac{t}{\varepsilon}\Big\}Ah\Big|d\tau\Big) \leq C_1 e^{C_2 t} M_1 \tag{59}$$

for $t \geq 0$. Then from (59) follows the estimate

$$|R(0)| \leq C_1\varepsilon, \quad 0 < \varepsilon \ll 1. \tag{60}$$

Due to the property (**x**) of Lemma 2 we get the estimate

$$|f'(t) - \int_0^\infty K(t,\tau)d\tau| \leq C_1 e^{C_2 t}\sqrt{\varepsilon}||f''||_{L^\infty_C(0,\infty;H)}, \quad t \geq 0, \quad 0 < \varepsilon \ll 1. \tag{61}$$

Further using the property (**xi**) of Lemma 2 we have

$$\Big|\int_0^t \int_0^\infty K(\tau,\theta)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}Ahd\theta d\tau\Big| \leq C\varepsilon M_1, \quad t \geq 0. \tag{62}$$

Using the estimates (60), (61) and (62) from (58) follows the estimate

$$|R_1(t)| \leq C_1 e^{C_2 t}\sqrt{\varepsilon}M_1, \quad t \geq 0, 0 < \varepsilon \ll 1. \tag{63}$$

From the property (**xi**) of Lemma 2 and the estimates (59) we get

$$|w_1(t) - z(t)| \leq \int_0^\infty K(t,\tau)\Big|\int_\tau^t z'(\theta)d\theta\Big|d\tau \leq$$

$$\leq C_1 e^{C_2 t}\sqrt{\varepsilon}M_1, \quad t \geq 0, 0 < \varepsilon \ll 1. \tag{64}$$

Finally, from the estimates (63) and (64) we obtain

$$|z(t) - v_1(t)| \leq |z(t) - w_1(t)| + |R_1(t)| \leq C_1 e^{C_2 t}\sqrt{\varepsilon}M_1, \quad t \geq 0, 0 < \varepsilon \ll 1,$$

i. e. the estimate (47).

Let us prove the estimate (49). Denote by $y(t) = Au(t)$, $y_1(t) = Av(t)$. Then under conditions (48) $y(t)$ is the solution of the problem

$$\begin{cases} \varepsilon y''(t) + y'(t) + Ay(t) = Af(t), \quad t \geq 0, \\ y(0) = Au_0, \quad y'(0) = Au_1, \end{cases}$$

and $y_1(t)$ is the solution of the problem

$$\begin{cases} y_1'(t) + Ay_1(t) = Af(t), \\ y_1(0) = Au_0. \end{cases}$$

From (45) follows the estimate

$$|Au(t) - Av(t)| \leq C_1 e^{C_2 t}\sqrt{\varepsilon}M_2, \quad t \geq 0, 0 < \varepsilon \ll 1. \tag{65}$$

As from (1) it follows that

$$|Au(t) - Av(t)| \geq \omega||u(t) - v(t)||,$$

then using (65) we obtain the estimate (48). Theorem 2 is proved.

Remark 1. *The relation (47) shows that the function $u'(t)$ possesses the boundary function in the neighborhood of the line $t = 0$. But, if $h = 0$, then the function $u'(t)$ like $u(t)$ does not have a boundary function.*

Finally let us give one simple example. Consider the following initial boundary problems

$$\begin{cases} \varepsilon u_{tt}(x,t) + u_t(x,t) + L(x,\partial_x)u(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in \overline{\Omega}, \\ u(x,t) = 0, \quad (x,t) \quad \text{on} \quad \partial\Omega \times [0,\infty), \end{cases} \tag{66}$$

$$\begin{cases} v_t(x,t) + L(x,\partial_x)v(x,t) = f(x,t), \quad x \in \Omega, t > 0, \\ v(x,0) = u_0(x), \quad x \in \overline{\Omega}, \\ u(x,t) = 0, \quad (x,t) \quad \text{on} \quad \partial\Omega \times [0,\infty), \end{cases} \tag{67}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$. The operator

$$L(x, \partial_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\cdot\Big) + a(x)\cdot$$

is uniformly elliptic in $\overline{\Omega}$, i.e. $a, a_{ij} : \overline{\Omega} \to \mathbb{R}$, $a, a_{ij} \in C(\overline{\Omega})$, $a_{ij}(x) = a_{ji}(x)$, and

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \omega|\xi|^2, \quad \xi \in \mathbb{R}^n, x \in \overline{\Omega},$$

where $\omega > 0, a(x) \geq 0$ for $x \in \overline{\Omega}$. Let us put $H = L^2(\Omega), V = H_0^1(\Omega)$. In this conditions the problems (P_ε) and $(P.v)$ represent the functional analytical statement of the problems (66) and (67) respectively, where A is the closure of the operator L in $L^2(\Omega)$. Under suitable conditions on the functions u_0, u_1 and f which follow from conditions (46) and (48) from Theorem 2 for the variational solutions of the problems (66), (67) we get

$$u = v + O(\sqrt{\varepsilon}) \quad \text{in} \quad C(0,T;L^2(\Omega)), \quad \varepsilon \to 0,$$

$$u_t = v_t + h\exp\Big\{-\frac{t}{\varepsilon}\Big\} + O(\sqrt{\varepsilon}) \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \quad \varepsilon \to 0,$$

$$u = v + O(\sqrt{\varepsilon}) \quad \text{in} \quad L^\infty(0,T;H_0^1(\Omega)), \quad \varepsilon \to 0,$$

where $h(x) = u_1(x) + L(x,\partial_x)u_0(x) - f(x,0)$.

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