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On Strong Stability of Linear Poisson Actions

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Abstract. Linear Poisson actions of the group R^m are considered. Conditions on the joint spectrum of the generators and on the centralizers assuring stability and strong stability of the action are given. We give also some examples of Poisson actions using CAS *"Mathematica"*.

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1 Introduction

The problem of stability and strong stability of the Hamiltonian systems is an old one and begins with the Poincare's and Lyapunov's classical results. Even the linear autonomous case represents an interesting problem and a series of papers has been devoted to these systems [1–7]. Some generalizations for dynamical systems with manydimensional time have been given in [8–10].

In the last decade some bihamiltonian systems as models of phisical problems appeared. In [11] a Poincare type classification of the fixed points of a bihamiltonian system in the dimension four has been purposed. In this connection the problem of stability and strong stability of fixed points, and more generally, of periodical orbits of these systems, arises. This problem is the main subject of the paper. More precisely, the linear parts of the Hamiltonian vector fields near fixed points give us a tuple of pairwise commuting linear Hamiltonian matrices, or, in other words, a linear Poisson action of the abelian group R^m in the vector space with a symplectic structure. We define stability and strong stability for such actions.

It is known that the linear differential equation

$$\dot{x} = Ax, \tag{1}$$

where $x \in R^{2n}$ and $A \in sp(2n, R)$, i.e. $A = JH$, $H^T = H$, $J^2 = -I$, is stable if and only if all the eigenvalues are purely imaginary and A is diagonalizable. Moreover, if the spectrum of A is simple and purely imaginary, then (1) is strongly stable [6,7]. M.G.Krein [5] has shown that strong stability holds even in the case when multiple eigenvalues occur, provided these eigenvalues are "positive definite".

Other criteria of strong stability has been stated (and proved using normal forms) by R.Cushman and R.Kelly ([2]). A geometrical proof of this result has been given by M.Levi ([3]).

Theorem 1. [2, 3] *An infinitesimally stable symplectic matrix A is strongly stable if and only if its centralizer $C(A)$ (in $sp(2n, R)$) consists of stable matrices.*

Another criterion in the language of first integrals has been stated by M.Wójtkowski ([4]). More precisely, remark that $h_k(x) = 1/2(JA^k x, x)$ is a quadratic first integral (if k is even, then $h = 0$); here $h(x) = 1/2(JAx, x)$ denotes the Hamiltonian of the system (1) ($(\cdot, \cdot)$ is the standard scalar product in R^{2n}).

Theorem 2. [4] *A linear Hamiltonian system is strongly stable if and only if some linear combination of the quadratic first integrals h_k, $k = 1, \dots, n$, is a nondegenerate definite quadratic form.*

In what follows we generalize the above mentioned criteria to linear Poisson actions of the abelian group R^m. New problems arise in this context. Firstly, we have no kind of normal form of commuting m-tuples of linear operators, similar to the Jordan normal form of a matrix, or a normal form of Hamiltonian first integrals as those of Williamson [6]. We make use of results of L. Lerman and Ya. Umanskiy [11], who give normal forms of bihamiltonian systems in dimension four.

Another problem, an algebraic geometric one, is the question about the structure of the variety of commuting m-tuples of matrices in the direct product of Lie algebras $gl(n, C)$ or $sl(2n, R)$. For some related results see [12].

2 Basic notions

Let V be a real $2n$-dimensional vector space and let ω be a nondegenerate skew symmetric bilinear form on V. We call the pair (V, ω) a real *symplectic vector space.* The standard example of the *symplectic inner product* ω is $\omega(x, y) = [x, y] = x^T Jy$, where the matrix J has the form:

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

with I_n for the identity matrix. A *symplectic basis* for V is a basis $v_1, \dots, v_{2n}$ such that $\omega(v_i, v_j) = J_{ij}$, the i, jth entry of J.

A linear map $T : V \to V$ is called *symplectic* if $[Tx, Ty] = [x, y]$ for all $x, y \in V$. The group of all real symplectic operators on (V, ω) is denoted by $Sp(2n, R)$.

A linear operator $L : V \to V$ is called *Hamiltonian* if the condition

$$[Lx, y] + [x, Ly] = 0$$

holds for all $x, y \in V$. A matrix A is called Hamiltonian or *infinitesimally symplectic* if $A^T J + JA = 0$. The Lie algebra of all Hamiltonian matrices is denoted by $sp(2n, R)$.

Let $\mathcal{T} = \{T_1, \dots, T_m\}$ be an m-tuple of bounded linear operators in a Hilbert space H. One says [13] that a point $\Lambda = \{\lambda_1, \dots, \lambda_m\} \in C^{m^*}$ belongs to the *left joint spectrum* $\sigma_l(\mathcal{T})$ (respectively, to the *right joint spectrum* $\sigma_r(\mathcal{T})$) if an m-tuple

$\mathcal{R} = \{R_1, \ldots, R_m\}$ of linear bounded operators in H such that $\sum_{k=1}^{m} R_k(T_k - \lambda_k I) = I$ (respectively, such that $\sum_{k=1}^{m} (T_k - \lambda_k I)R_k = I$) does not exist. The *joint spectrum* of a polyoperator $\mathcal{T}$ is defined as a sum of its left and right joint spectra. It is denoted by $\sigma(\mathcal{T})$.

In the case of $n = 1$ the above mentioned definition is equivalent to the common definition of the operator's spectrum. (In the case of finite dimension, the notions of the left and the right joint spectra coincide.)

Another way to define the joint spectrum in the finite-dimensional case is based on the known fact from linear algebra that any family of commuting complex linear operators possesses a *joint eigenvector*, i.e. for any $\mathcal{A} = \{A_1, \ldots, A_m\}$, $A_i \circ A_j = A_j \circ A_i$, there exists a vector $h \neq 0$ such that $A_j h = \lambda_j h$ for any $j = 1, \ldots, m$ and some $\{\lambda_1, \ldots, \lambda_m\} \in C^{m^*}$. Then $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ is called the *eigenfunctional* corresponding to the joint eigenvector h. The set of all eigenfunctionals creates the joint spectrum $\sigma(\mathcal{A})$. Some details concerning the properties of joint spectra can be found in [8, 13].

We mention that for m-tuples of commuting hamiltonian matrices the joint spectrum has symmetry properties similar to those of a single hamiltonian matrix.

Let $\Phi : R^m \times V \to V$ be a continuous action of the group R^m on V such that for any fixed $t \in R^m$ the transformation $\Phi^t = \Phi(t, \cdot)$ is a linear symplectic transformation of the space V. An action of this type is called [11] a linear Poisson action.

Consider a Hamiltonian polyoperator $\mathcal{A} = \{A_1, \ldots, A_m\}$. Remark that for the linear completely integrable system

$$\frac{\partial x}{\partial t_j} = A_j x \quad (x \in R^{2n},\ t_j \in R,\ j = 1, \ldots, m) \tag{2}$$

the fundamental matrix is $\exp(\mathcal{A}, t) := \exp(A_1 t_1 + \cdots + A_m t_m)$. The system (2) is called *stable* if $\exists M > 0$ such that $\|\exp(\mathcal{A}, t)\| < M$ for all $t \in R^m$. It is called *strongly stable* if there exists $\varepsilon > 0$ such that for any polyoperator $\mathcal{B} = \{B_1, \ldots, B_m\} \in (sp(2n, R))^m$, $B_i \circ B_j = B_j \circ B_i$, $\|B_i - A_i\| < \varepsilon$ $(i, j = 1, \ldots, m)$, the inequality $\|\exp(\mathcal{B}, t)\| < M$ holds for some $M > 0$ and all $t \in R^m$.

3 Results. Stability and strong stability of linear Poisson actions

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a Hamiltonian polyoperator, i.e. the matrices A_j are Hamiltonian and pairwise commuting.

Theorem 3. *A linear constant completely integrable Hamiltonian system*

$$\frac{\partial x}{\partial t_j} = A_j x \quad (x \in R^{2n}, \quad t_j \in R \ \ \forall j \in \{1, \ldots, m\}) \tag{3}$$

is stable if and only if for any $j = 1, 2, \ldots, m$ the Hamiltonian system

$$\frac{dx}{ds} = A_j x \quad (x \in R^{2n}, s \in R) \tag{4}$$

is stable.

Proof. Assume that the systems (4) are stable for $j = 1, \ldots, m$. Hence, for each fixed j there exists $M_j > 0$ such that $\|\exp(A_j t_j)\| < M_j$ $(t_j \in R)$. Then, there is $M = M_1 \cdot \cdots \cdot M_m > 0$ such that for all $(t_1, \ldots, t_m) \in R^m$:

$$\|\exp(A_1 t_1 + \cdots + A_m t_m)\| = \|\exp(A_1 t_1) \cdot \cdots \cdot \exp(A_m t_m)\| \leq M.$$

Let (3) be stable. So, there exists $M > 0$, for which $\|\exp(A_1 t_1 + \cdots + A_m t_m)\| < M$ $((t_1, \ldots, t_m) \in R^m)$. In particular, the inequality holds for all $(t_1, 0, \ldots, 0)$, $(0, t_2, 0, \ldots, 0)$ and so on. So, one has $\|\exp(A_j t_j)\| < M$ for $j = 1, \ldots, m$. Hence all systems (4) are stable.

The following result shows that this is not the case for strong stability.

Theorem 4. *Let (2) be stable and assume that there exists a strongly stable element* $\exp(\mathcal{A}, t_0)$ *for some* $t_0 \in R^m$. *Then the system (2) is strongly stable.*

Proof. Choose $\varepsilon > 0$ such that for each $B \in sp(2n, R)$ satisfying $\|B - (\mathcal{A}, t_0)\| < \varepsilon$, one has $\|\exp B\tau\| < \infty$ $(\tau \in R)$. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a Hamiltonian polyoperator ε - close to $\mathcal{A}$, i.e. $\|B_i - A_i\| < \varepsilon$ and $B_i \circ B_j = B_j \circ B_i$ $(i, j = 1, 2, \ldots, m)$. Then $\|(\mathcal{B}, t_0) - (\mathcal{A}, t_0)\| = \|(\mathcal{B} - \mathcal{A}, t_0)\| \leq \|\mathcal{B} - \mathcal{A}\| \cdot \|t_0\| < \varepsilon$ and $(\mathcal{B}, t_0)$ is strongly stable if $\|\mathcal{B}\| = \frac{\varepsilon}{\|t_0\|}$. On the other hand, $B_j \in C((\mathcal{B}, t_0))$, so, by Theorem 1 $\dot{x} = B_j x$ are stable (for every $j = 1, 2, \ldots$), which implies that $\mathcal{B}$ is also stable.

Remark 1. *It is worth noting that at least formally, strong stability of the polyoperator is weaker than the condition of existence of a strongly stable element: a neighbourhood of a point in $sl(2n, R)$ is larger than a neighbourhood of a polyoperator in the subvariety of commuting m-tuples from $sl(2n, R)^m$. It is a problem whether this subvariety is irreducible or not.*

The following result reduces the problem of strong stability of a polyoperator on the whole phase space to the problem of such stability on the invariant symplectic subspaces. The main idea of the proof uses the fact that the centralizer of a block-diagonal matrix with spectrally separated blocks coincides with the direct sum of centralizers of the blocks.

Theorem 5. *Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a Hamiltonian polyoperator with multiple eigenfunctionals*

$$\Lambda_1 = \{i\lambda_1^1, \ldots, i\lambda_m^1\}, -\Lambda_1, \ldots, \Lambda_k = \{i\lambda_1^k, \ldots, i\lambda_m^k\}, -\Lambda_k,$$

$m_1, \ldots, m_k$ denoting corresponding multiplicities and V_r - the subspace of $(R^{2n})^m$ corresponding to the eigenfunctionals Λ_r and $-\Lambda_r$ with multiplicity m_r. Besides, let $\mathcal{A}/V_r$ stand for the polyoperator $\mathcal{A}$ restricted to this subspace. Then, $\mathcal{A}$ is strongly stable if and only if $\mathcal{A}/V_r$ is strongly stable for all r.

Proof. Assume that $\mathcal{A}$ is strongly stable, i.e. there is $\varepsilon > 0$ such that for any polyoperator $\mathcal{B} = \{B_1, \dots, B_m\}$ such that $\|B_j - A_j\| < \varepsilon$ the inequality:

$$\|\exp(B_1 t_1 + \dots + B_m t_m)\| < M$$

holds for some $M > 0$ and for all $(t_1, \dots, t_m) \in R^m$. We shall show that $\mathcal{A}/V_r$ are strongly stable for $r = 1, \dots, k$.

Recall that a subspace U of a symplectic space (V, ω) is called [1] symplectic if ω restricted to this subspace is nondegenerate. (Obviously, such U is of even dimension, hence (U, ω) is a symplectic space.) Choose a polyoperator $\mathcal{B}_r = \{B_1^r, \dots, B_m^r\}$ on the symplectic subspace V_r such that $\|\mathcal{A}/V_r - \mathcal{B}_r\| \le \varepsilon$ and consider a polyoperator $\mathcal{B} = \oplus_{s \ne r} \mathcal{A}/V_s \oplus \mathcal{B}_r$ on R^{2n} (here $\oplus$ stands for the direct sum of operators). Then $\|\mathcal{B} - \mathcal{A}\| = \|\mathcal{B}_r - \mathcal{A}/V_r\| \le \varepsilon$, since $\mathcal{B}/V_s = \mathcal{A}/V_s$ for $s \ne r$.
Hence one has: $\exp(\mathcal{B}, t) = \oplus_{s \ne r} \exp(\mathcal{A}/V_s, t) \oplus \exp(\mathcal{B}_r, t)$ and

$$M \ge \|\exp(\mathcal{B}, t)\| = \prod_{s \ne r} \|\exp(\mathcal{A}/V_s, t)\| \, \|\exp(\mathcal{B}_r, t)\| \, . \tag{5}$$

Using the Banach-Steinhaus Theorem one can easily prove that

$$p = \inf_{t \in R^m} \prod_{s \ne r} \|\exp(\mathcal{A}/V_s, t)\| > 0.$$

From (5) we obtain

$$\|\exp(\mathcal{B}_r, t)\| \le \frac{M}{p}.$$

So, $\mathcal{A}/V_r$ is strongly stable.

Assume now that $\mathcal{A}/V_s$ are strongly stable for $s = 1, \dots, k$ and suppose that $\mathcal{A}$ is not strongly stable. That means that there exists a sequence $\{\mathcal{B}_k\}_{k=1}^{\infty} \to \mathcal{B}$ of nonstable polyoperators. Due to the upper semicontinuity of the joint spectrum, $\{\mathcal{B}_k\}$ have a spectral decomposition close to V_r and $\left\| \mathcal{B}_r/U_r^{(k)} - \mathcal{A}/V_r \right\| \to 0$ as $k \to \infty$. The latest implies that there is r such that $\mathcal{A}/V_r$ is not strongly stable. This contradiction proves the theorem.

Following [5, 7], we call an eigenfunctional $\Lambda \in C^{m*}$ definite if there exists an element $t_0 \in R^m$ such that $\exp(\Lambda, t_0)$ is a positive definite eigenvalue for the symplectic operator $\exp(\mathcal{A}, t_0)$.

Remark 2. *Mention that a simple purely imaginary eigenfunctional is definite and that, in this case, the system (2) is strongly stable.*

Theorem 6. *If the joint spectrum of the polyoperator $\mathcal{A}$ is purely imaginary and definite, then the differential system (2) is strongly stable.*

Proof. Due to Theorem 5, it is enough to consider the case when the polyoperator $\mathcal{A}$ has a single-point joint spectrum

$$\Lambda = \{i\omega_1, i\omega_2, \dots, i\omega_n, -i\omega_1, -i\omega_2, \dots, -i\omega_n\}$$

of some multiplicity s.

Let Λ be definite and let $t_0 \in R^m$ be such that $\exp(\Lambda, t_0)$ is definite. Then the element $(\mathcal{A}, t_0) \in sp(2n, R)$ is strongly stable because it has a positive definite first integral. From Theorem 4 it follows that the system (2) is strongly stable.

In what follows we give some generalizations of the strong stability criteria of Cushman-Kelly [2], M. Levi [3] and M. Wójtkowski [4].

Recall that for a given $A \in sp(2n, R)$, $C(A)$ denotes the center of A in $sp(2n, R)$, i.e. $C(A) = \{X \in sp(2n, R) : AX = XA\}$.

Theorem 7. *If $\bigcup_{j=1}^m C(A_j)$ consists of stable linear Hamiltonian operators, then* (2) *is strongly stable.*

Proof. Let $C(\mathcal{A})$ contain only stable operators and let $\{B_1, \dots, B_m\}$ be close enough to $\mathcal{A} = \{A_1, \dots, A_m\}$. By [3], each B_j can be written under the form $B_j = \exp(-T_j) \circ (A_j + D_j) \circ \exp(T_j)$ for some $D_j \in C(A_j)$ and $T_j \in sp(2n, R)$. Since A_j are stable and $D_j \in C(A_j)$, then D_j are stable, as well as $A_j + D_j$, and hence $\exists M > 0$ such that $\|\exp(B\tau)\| \le M$ for all $\tau \in R$.

If, in addition, $B_i \circ B_j = B_j \circ B_i$, then $\|\exp(B_1 t_1 + B_2 t_2 + \cdots + B_m t_m)\| < M^m$ for all $(t_1, t_2, \dots, t_m) \in R^m$.

Theorem 8. *If the system* (2) *is strongly stable, then $\bigcap_{j=1}^m C(A_j)$ consists of stable operators.*

Proof. Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be strongly stable and let $B_1 \in \bigcap_{j=1}^m C(A_j)$. For $\mathcal{B} := \{B_1, 0, \dots, 0\} \in sp(2n, R)^m$ take $\varepsilon > 0$ small enough to assure stability of $\mathcal{A} + \varepsilon\mathcal{B}$. So we have: $\|\exp(\mathcal{A} + \varepsilon\mathcal{B}, t)\| < M$, $\|\exp(-\mathcal{A}, t)\| < M$ for some $M > 0$ and for all $t \in R^m$. Since $B_1 \in \bigcap_{j=1}^m C(A_j)$, one has: $\|\exp(\varepsilon B_1 t_1)\| = \|\exp(-\mathcal{A}, t) \exp(\mathcal{A} + \varepsilon\mathcal{B}, t)\| \le M^2 \quad (t \in R^m)$.

Remark 3. *So, if $\bigcup_{j=1}^m C(A_j)$ consists of stable linear Hamiltonian operators, then $\bigcap_{j=1}^m C(A_j)$ consists also of stable operators. A natural question if the inverse implication is true arises. In what follows we give a counterexample to this hypothesis.*

Proposition 1. *There exist polyoperators $\mathcal{A}$ such that $\bigcap_{j=1}^m C(A_j)$ consists of stable operators, but $\bigcup_{j=1}^m C(A_j)$ cointains unstable operators.*

Proof. The authors of [11] give (see Appendix A) the list of normal forms of all possible quadratic Hamilton functions in the case of two degrees of freedom and also of the quadratic functions that are additional integrals of the corresponding linear Hamiltonian system. There are 15 different possible cases. We use this classification to give the counterexample we need.

Consider the case 3 which is given through the following conditions:
the eigenvalues are $(\pm i\omega_1, \pm i\omega_2)$, $\omega_1 \ne \omega_2$, $\omega_1, \omega_2 \in R$, $\omega_1, \omega_2 \ne 0$,

$$H = \frac{\omega_1}{2}(p_1^2 + q_1^2) + \frac{\omega_2}{2}(p_2^2 + q_2^2),$$

$$K = \frac{\nu_1}{2}(p_1^2 + q_1^2) + \frac{\nu_2}{2}(p_2^2 + q_2^2).$$

The condition for the algebra to be two-dimensional is $\omega_1\nu_2 - \omega_2\nu_1 \neq 0$. In this case for fixed ν_1 and ν_2 the centralizer C coincides with the algebra generated by the pair H, K. Put $\omega_1 = 2$, $\omega_2 = 2$, $\nu_1 = 2$, $\nu_2 = 0$. Then we obtain the particular case where $H = p_1^2 + p_2^2 + q_1^2 + q_2^2$, $K = p_1^2 + q_1^2$ and the condition $\omega_1\nu_2 - \omega_2\nu_1 \neq 0$ is satisfied. It is obvious that for $\alpha_1 = 1$ and $\alpha_2 = 1$ the linear combination $\alpha_1 H + \alpha_2 K = 2p_1^2 + p_2^2 + 2q_1^2 + q_2^2$ is a positively definite quadratic form. So, the polyoperator $\{A_1, A_2\}$ is strongly stable (see [14]). In this case the matrices corresponding to the integrals H and K have the form:

$$A_1 = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The following computations have been done with the help of CAS *"Mathematica"*. The centralizers of the matrices A_1 and A_2 are:

$$C(A_1) = \{C_1 = \begin{pmatrix} 0 & -k_3 & -n_1 & -n_2 \\ k_3 & 0 & -n_2 & -n_3 \\ n_1 & n_2 & 0 & -k_3 \\ n_2 & n_3 & k_3 & 0 \end{pmatrix} : k_3, n_1, n_2, n_3 \in R\},$$

$$C(A_2) = \{C_2 = \begin{pmatrix} 0 & 0 & -t_1 & 0 \\ 0 & r_4 & 0 & s_3 \\ t_1 & 0 & 0 & 0 \\ 0 & t_3 & 0 & -r_4 \end{pmatrix} : r_4, s_3, t_1, t_3 \in R\}.$$

So,

$$C(A_1) \cap C(A_2) = \{C_3 = \begin{pmatrix} 0 & 0 & -t_1 & 0 \\ 0 & 0 & 0 & s_3 \\ t_1 & 0 & 0 & 0 \\ 0 & -s_3 & 0 & 0 \end{pmatrix} : r_4, s_3, t_1, t_3 \in R\},$$

$$JordanForm(C_3) = \begin{pmatrix} -is_3 & 0 & 0 & 0 \\ 0 & is_3 & 0 & 0 \\ 0 & 0 & -it_1 & 0 \\ 0 & 0 & 0 & it_1 \end{pmatrix}.$$

Hence, $C(A_1) \cap C(A_2)$ consists of stable operators. Remark that some matrices in $C(A_2)$ possess real eigenvalues. Let, for example, $t_1 = 3$, $r_4 = 2\sqrt{2}$, $s_3 = 4$ and $t_3 = 2$. Then we get $C_2 \in C(A_2)$ with the eigenvalues ± 4, $\pm 3i$. So, $C(A_1) \cup C(A_2)$ cointains at least one unstable operator.

Remark 4. *The main results have been announced in [15].*

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