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Global attractors for V-monotone nonautonomous dynamical systems*

David N. Cheban, Peter E. Kloeden, Björn Schmalfuß

Abstract. This article is devoted to the study of the compact global atrractors of V-momotone nonautonomous dynamical systems.We give a description of the structure of compact global attractors of this class of systems. Several applications of general results for different classes of differential equations (ODEs, ODEs with impulse, some classes of evolutionary partial differential equations) are given.

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1 Introduction

The differential equations with monotone right-hand side are one of the most studied classes of nonlinear equations (see, for example,[4, 16, 20, 24, 25] and the literature quoted there).

Many authors studied the problem of the existence of almost periodic solutions of monotone nonlinear almost periodic equations (see [12, 13, 15, 18, 19, 24, 25] and others).

Purpose of our article is the study of global attractors of general V-monotone nonautonomous dynamical systems and their applications to different classes of differential equations (ODEs, ODEs with impulse, some classes of evolution partial differential equations).

For autonomous equations the analogous problem was studied before (see, for example,[2, 14, 23]), but for nonautonomous dynamical system this problem is considered in our paper for the first time.

2 Nonautonomous dynamical systems and skew-product flows

Definition 1. *Let $\Theta = \{\theta_t\}_{t\in\mathbb{R}}$ be a group of mappings of Ω into itself, that is a continuous time autonomous dynamical system on a metric space Ω, and let $\mathbb{B}$ be a*

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Banach space. Consider a continuous mapping $\varphi : \mathbb{R}^+ \times \Omega \times \mathbb{B} \to \mathbb{B}$ *satisfying the properties*

$$\varphi(0, \omega, \cdot) = id_{\mathbb{B}}, \ (\varphi(t + \tau, \omega, x) = \varphi(\tau, \theta_t\omega, \varphi(t, \omega, x))$$

for all $s, t \in \mathbb{R}^+$, $\omega \in \Omega$ *and* $x \in \mathbb{B}$. *Such a mapping* φ *(or more explicit* $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{R}, \Theta)\rangle$*) is called [1],[22] a continuous cocycle or nonautonomous dynamical system (NDS) on* $\Omega \times \mathbb{B}$.

Example 1. *As an example, consider a parameterized differential equation*

$$\frac{dx}{dt} = F(\theta_t\omega, x) \quad (\omega \in \Omega)$$

on a Banach space $\mathbb{B}$ *with* $\Omega = C(\mathbb{R} \times \mathbb{B}, \mathbb{B})$. *Define* $\theta_t : \Omega \to \Omega$ *by* $\theta_t\omega(\cdot, \cdot) = \omega(t + \cdot, \cdot)$ *for each* $t \in \mathbb{R}$ *and interpret* $\varphi(t, \omega, x)$ *as the solution of the initial value problem*

$$\frac{d}{dt}x(t) = F(\theta_t\omega, x(t)), \quad x(0) = x. \tag{1}$$

Under appropriate assumptions on $F : \Omega \times \mathbb{B} \to \mathbb{B}$ *(or even* $F : \mathbb{R} \times \mathbb{B} \to \mathbb{B}$*) with* $\omega(t)$ *instead of* $\theta_t\omega$ *in (1) to ensure forwards the existence and uniqueness,* (Θ, φ) *generates a nonautonomous dynamical system on* $\Omega \times \mathbb{B}$.

The usual concept of a global attractor for the autonomous semi-dynamical system π on the state space $X = \Omega \times \mathbb{B}$ can be used here.

Definition 2. *The nonempty compact subset* $\mathcal{A}$ *of* $X = \Omega \times \mathbb{B}$ *is called maximal if it is* π*-invariant, that is*

$$\pi(t, \mathcal{A}) = \mathcal{A} \quad \textit{for all} \quad t \in \mathbb{R}^+,$$

and it attracts all compact subsets of $X = \Omega \times \mathbb{B}$, *that is*

$$\lim_{t\to\infty} \beta\left(\pi(t, \mathcal{D}), \mathcal{A}\right) = 0 \qquad \textit{for all} \quad \mathcal{D} \in \mathcal{K}(\mathbb{X}),$$

where $C(X)$ *is the space of all nonempty compact subsets of* X *and* β *is the Hausdorff semi-metric on* $C(X)$.

3 Global attractors of V- monotone NDS.

Let Ω be a compact topological space , (E, h, Ω) be a locally trivial Banach stratification [3] and $|\cdot|$ be a norm on (E, h, Ω) co-ordinated with the metric ρ on E (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$).

Definition 3. *Let us remember [8],[5],[6] that the triplet* $\langle (E, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ *is called a (general) nonautonomous dynamical system if* $h : E \to \Omega$ *is a homomorphism of the dynamical system* $(E, \mathbb{T}_1, \pi)$ *on* $(\Omega, \mathbb{T}_2, \Theta)$, *where* $\mathbb{T}_1$ *and* $\mathbb{T}_2$ $(\mathbb{T}_1 \subseteq \mathbb{T}_2)$ *are two subsemigroups of the group* $\mathbb{T}$.

Example 2. *Let $\mathbb{T}_2$ be a subsemigroup of $\mathbb{T}$, $(\Omega, \mathbb{T}_2, \Theta)$ be a dynamical system on Ω and $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}_2, \Theta)\rangle$ be a cocycle over $(\Omega, \mathbb{T}_2, \Theta)$ with the fiber $\mathbb{B}$, $X := \Omega \times \mathbb{B}$, $\mathbb{T}_1 \subseteq \mathbb{T}_2$ be a subsemigroup of $\mathbb{T}_2$, $(X, \mathbb{T}_1, \pi)$ be a semi-group dynamical system on X defined by the equality $\pi = (\varphi, \theta)$ (i.e. $\pi(t, (\omega, u)) := (\varphi(t, \omega, u), \theta_t \omega)$ for all $t \in \mathbb{T}_1$ and $(\omega, u) \in X$), then the triple $\langle (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \Theta), h\rangle$ ($h = pr_2$) will be a nonautonomous dynamical system, generated by cocycle φ.*

Definition 4. *The cocycle $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}, \Theta)\rangle$ is called compact dissipative if there is a nonempty compact $K \subseteq W$ such that*

$$\lim_{t\to+\infty} \sup\{\beta(\varphi(t, \omega)M, K) \mid \omega \in \Omega \} = 0 \tag{2}$$

for any $M \in C(\mathbb{B})$, where $\varphi(t, \omega) := \varphi(t, \omega, \cdot)$.

If $M \subseteq \mathbb{B}$, then suppose

$$\Omega_\omega(M) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, \theta_{-\tau}\omega, M)}$$

for every $\omega \in \Omega$.

Definition 5. *We will say that the space X possesses the (S)-property if for any compact $K \subseteq X$ there is a connected set $M \subseteq X$ such that $K \subseteq M$.*

Theorem 1. *[9] Let Ω be a compact metric space, $\langle \mathbb{B}, \varphi, (\Omega, \mathbb{T}, \Theta)\rangle$ be a compact dissipative cocycle and K be the nonempty compact appearing in the equality (2), then :*

1. $I_\omega = \Omega_\omega(K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and $\lim_{t\to+\infty} \beta(\varphi(t, \theta_{-t}\omega)K, I_\omega) = 0$ for every $\omega \in \Omega$;

2. $\varphi(t, \omega)I_\omega = I_{\theta_t\omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$;

3. $\lim_{t\to+\infty} \beta(\varphi(t, \theta_{-t})M, I_\omega) = 0$ for all $M \in C(\mathbb{B})$ and $\omega \in \Omega$;

4. $\lim_{t\to+\infty} \sup\{\beta(\varphi(t, \omega_{-t})M, I) \mid \omega \in \Omega \} = 0$ for any $M \in C(\mathbb{B})$, where $I = \cup\{I_\omega \mid \omega \in \Omega \}$;

5. $I_\omega = pr_1 I_\omega$ for all $\omega \in \Omega$, where J is a Levinson centre of $(X, \mathbb{T}^+, \pi)$, and, hence, $I = pr_1 J$;

6. the set I is compact;

7. the set I is connected if one of the following two conditions is fulfilled :

a. $\mathbb{T}^+ = \mathbb{R}^+$ and the spaces $\mathbb{B}$ and Ω are connected;

b. $\mathbb{T}^+ = \mathbb{Z}^+$ and the space $\Omega \times \mathbb{B}$ possesses the (S)-property or it is connected and locally connected.

Definition 6. *A nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h\rangle$ is said to be uniformly stable in the positive direction on compacts of X [7] if, for arbitrary $\varepsilon > 0$ and $K \subseteq X$, there is $\delta = \delta(\varepsilon, K) > 0$ such that the inequality $\rho(x_1, x_2) < \delta$ ($h(x_1) = h(x_2)$) implies that $\rho(\pi^t x_1, \pi^t x_2) < \varepsilon$ for $t \in \mathbb{T}^+$.*

Definition 7. *A set $M \subset X$ is called minimal with respect to a dynamical system $(X, \mathbb{T}^+, \pi)$ if it is nonempty, closed and invariant and if no proper subset of M has these properties.*

Definition 8. *Denote by $X \dot{\times} X = \{(x_1, x_2) \in X \times X \mid h(x_1) = h(x_2)\ \}$. If there exists the function $V : X \dot{\times} X \to \mathbb{R}_+$ with the following properties:*

a. V is continuous.

b. V is positive defined, i.e. $V(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

c. $V(x_1 t, x_2 t) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X \dot{\times} X$ and $t \in \mathbb{T}_+$,

then the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ is called (see [12],[13] and [19],[25]) V - monotone.

Theorem 2. *Every V - monotone compact dissipative nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ is uniformly stable in the positive direction on compacts from X.*

Corollary 1. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ be a V - monotone compact dissipative nonautonomous dynamical system and Ω be minimal, then:*

1. J is uniformly orbitally stable in the positive direction, i.e., for $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that the inequality $\rho(x, J_{h(x)}) < \delta$ implies that $\rho(\pi^t x, J_{h(\pi^t x)}) < \varepsilon$ for $t \geq 0$;

2. J is an attractor of compact sets from X, i.e., for $\varepsilon > 0$ and a compact $K \subseteq X$, there is $L(\varepsilon, K) > 0$ such that $\pi^t K_\omega \subseteq \tilde{B}(J_{\theta_t \omega}, \varepsilon)$ for $\omega \in \Omega$ and $t \geq L(\varepsilon, K)$;

3. any motion on J can be continued to the left and J is bilaterally distal;

4. $J_\omega = X_\omega \bigcap J$ for $\omega \in \Omega$, is a connected set if X_ω is connected, and for distinct ω_1 and ω_2 the sets J_{ω_1} and J_{ω_2} are homeomorphic;

5. J is formed of recurrent trajectories, and two arbitrary points $x_1, x_2 \in J_\omega$ $(\omega \in \Omega)$ are mutually recurrent.

Theorem 3. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ be a V - monotone compact dissipative nonautonomous dynamical system, Ω be minimal and J be its Levinson center, then*

$$V(x_1 t, x_2 t) = V(x_1, x_2) \tag{3}$$

for all $x_1, x_2 \in J$ such that $h(x_1) = h(x_2)$.

Corollary 2. *Under the conditions of Theorem 3 if the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\mathbb{B}, \mathbb{T}, \Theta), h \rangle$ is strictly monotone, i.e. $V(x_1 t, x_2 t) < V(x_1, x_2)$ for all $t > 0$ and $(x_1, x_2) \in X \dot{\times} X$ $(x_1 \neq x_2)$, then $J_\omega = J \bigcap X_\omega$ consists of a single point for all $\omega \in \Omega$.*

Theorem 4. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ be a $V-$monotone compact dissipative nonautonomous dynamical system with compact minimal base Ω and J be its Levinson center, then for every point $x \in X_y$ there exists a unique recurrent point $p \in J_\omega$ such that*

$$\lim_{t \to +\infty} \rho(xt, pt) = 0, \tag{4}$$

i.e. every trajectory of this system is asymptotically recurrent.

Corollary 3. *Under the conditions of Theorem 4 the following assertions hold:*

a. $\omega-$limit set ω_x of every point $x \in X$ is a compact minimal set.

b. if $x_1, x_2 \in X_\omega$ ($\omega \in \Omega$) then $\omega_{x_1} = \omega_{x_2}$ or $\omega_{x_1} \bigcap \omega_{x_2} = \emptyset$.

4 On the structure of Levinson center of V-monotone NDS with minimal base

Definition 9. *(X, ρ) is called [18] a metric space with segments if for any $x_1, x_2 \in X$ and $\alpha \in [0, 1]$, the intersection of $B[x_1, \alpha r]$ (the closed ball centered at x with radius αr, where $r = \rho(x_1, x_2)$) and $B[x_2, (1-\alpha)r]$ has a unique element $S(\alpha, x_1, x_2)$.*

Definition 10. *The metric space (X, ρ) is called [18] strict-convex if (X, ρ) is a metric space with segments, and for any $x_1, x_2, x_3 \in X$, $x_2 \neq x_3$, and $\alpha \in (0, 1)$, the inequality $\rho(x_1, S(a, x_2, x_3)) < \max\{\rho(x_1, x_2), \rho(x_1, x_3)\}$ holds.*

Definition 11. *Let X be a strict metric-convex space. A subset M of X is said to be metric-convex if $S(\alpha, x_1, x_2) \in M$ for any $\alpha \in (0, 1)$ and $x_1, x_2 \in M$.*

We note that every convex closed subset X of the Hilbert space H equipped with the metric $\rho(x_1, x_2) = |x_1 - x_2|$ is strictly metric-convex.

Let $x \in X$, denote by Φ_x the family of all entire trajectories of dynamical system $(X, \mathbb{T}^+, \pi)$ passing through the point x for $t = 0$, i.e. $\gamma \in \Phi_x$ if and only if $\gamma : \mathbb{T} \to X$ is a continuous mapping with the properties: $\gamma(0) = x$ and $\pi^t \gamma(\tau) = \gamma(t + \tau)$ for all $t \in \mathbb{T}^+$ and $\tau \in \mathbb{T}$.

Theorem 5. *Let $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ be a V - monotone compact dissipative nonautonomous dynamical system, J is its Levinson center and the following conditions hold:*

1. $V(x_1, x_2) = V(x_2, x_1)$ for all $(x_1, x_2) \in X \dot{\times} X$.

2. $V(x_1, x_2) \leq V(x_1, x_3) + V(x_3, x_2)$ for all $x_1, x_2, x_3 \in X$ with the condition $h(x_1) = h(x_2) = h(x_3)$.

3. the space (X_ω, V_ω) is strict metric-convex for all $\omega \in \Omega$, where $X_\omega = h^{-1}(\omega) = \{x \in X | h(x) = \omega\}$ $(\omega \in \Omega)$ and $V_\omega = V|_{X_\omega \times X_\omega}$.

If $\gamma_{x_i} \in \Phi_{x_i}$ ($i = 1, 2$) and $x_1, x_2 \in I_\omega$ $(\omega \in \Omega)$, then the function $\gamma : \mathbb{T} \to X$ $(\gamma(t) = S(\alpha, \gamma_{x_1}(t), \gamma_{x_2}(t))$ for all *$t \in \mathbb{T})$ defines an entire trajectory of dynamical system $(X, \mathbb{T}^+, \pi)$.*

We denote by $\mathcal{K} = \{a \in C(\mathbb{T}_+, \mathbb{R}_+) \mid a(0) = 0,\ a \text{ is strictly increasing}\}$.

Theorem 6. *Under the conditions of Theorem 5 if in addition the nonautonomous dynamical system $\langle (X, \mathbb{T}^+, \pi), (\Omega, \mathbb{T}, \Theta), h \rangle$ is bounded k - dissipative and there exists a function $a \in \mathcal{K}$ with the property $\lim\limits_{t \to +\infty} a(t) = +\infty$ such that $a(\rho(x_1, x_2)) \leq V(x_1, x_2)$ for all $(x_1, x_2) \in X \dot{\times} X$, then J_ω will be metric-convex for all $\omega \in \Omega$, where $J_\omega = J \bigcap X_\omega$ and J is the Levinson center of $(X, \mathbb{T}^+, \pi)$.*

5 Almost periodic solutions of V - monotone almost periodic dissipative systems

Definition 12. *Let (X,ρ) be a metric space. A function $\phi:\mathbb{T}\to X$ is called almost periodic (in the sense of Bohr) if for every $\varepsilon>0$ there exists a relatively dense subset A_ε of $\mathbb{T}$ such that*

$$\rho(\phi(t+\tau),\phi(t))<\varepsilon$$

for all $t\in\mathbb{T}$ and $\tau\in A_\varepsilon$.

Definition 13. *A point $x\in X$ is said to be almost periodic if there is an entire trajectory $\gamma_x\in\Phi_x$ such that the function $\gamma_x:\mathbb{T}\to X$ is almost periodic.*

Definition 14. *The compact invariant set M of nonautonomous dynamical system $\langle(X,\mathbb{T}_+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is called [19],[5] distal on the invariant set M in the negative direction if $\inf\limits_{t\in\mathbb{T}_-}\rho(\gamma_{x_1}(t),\gamma_{x_2}(t))>0$ for all $x_1,x_2\in M(h(x_1)=h(x_2)$ and $x_1\neq x_2)$ and $\gamma_{x_i}\in\Phi_{x_i}(i=1,2)$, where Φ_x is the set of all entire trajectories of $(X,\mathbb{T}_+,\pi)$ passing through the point $x\in X$.*

Lemma 1. *[19] Let Ω be a compact minimal set and $M\subseteq X$ be a compact invariant set of $(X,\mathbb{T}^+,\pi)$. If the nonautonomous dynamical system $\langle(X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is distal on M in negative direction, then the mapping $\omega\longmapsto M_\omega:=M\bigcap X_\omega$ is continuous with respect to Hausdorff metric.*

Lemma 2. *Let $M\subseteq X$ be a compact invariant set of $(X,\mathbb{T}^+,\pi)$. If the nonautonomous dynamical system $\langle(X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is uniformly stable in the positive direction on compacts from X, then $\langle(X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ is distal on the invariant set M in the negative direction .*

Corollary 4. *Under the conditions of Lemma 2 if Ω is a compact minimal set, then the mapping $\omega\longmapsto J_\omega$ is continuous with respect to Hausdorff metric.*

Lemma 3. *Let (M,ρ) be a compact, strictly metric-convex space and E be a compact subsemigroup of isometries of semigroup M^M (i.e. $E\subseteq M^M$ and $\rho(\xi x_1,\xi x_2)=\rho(x_1,x_2)$ for all $x_1,x_2\in M$). Then there exists a common fixed point $\bar{x}\in M$ of E, i.e. $\xi(\bar{x})=\bar{x}$ for all $\xi\in E$.*

Theorem 7. *Let $\langle(X,\mathbb{T}^+,\pi),(\Omega,\mathbb{T},\Theta),h\rangle$ be a V - monotone bounded k - dissipative NDS, J be its Levinson center and the following conditions hold:*

1. $V(x_1,x_2)=V(x_2,x_1)$ for all $(x_1,x_2)\in X\dot{\times}X$.

2. $V(x_1,x_2)\leq V(x_1,x_3)+V(x_3,x_2)$ for all $x_1,x_2,x_3\in X$ with the condition $h(x_1)=h(x_2)=h(x_3)$.

3. the space (X_ω,V_ω) is strictly metric-convex for all $\omega\in\Omega$, where $X_\omega=h^{-1}(\omega)=\{x\in X\mid h(x)=\omega\ \}$ $(\omega\in\Omega)$ and $V_\omega=V|_{X_\omega\times X_\omega}$.

Then the set-valued mapping $\omega\to J_\omega$ admits at least one continuous invariant section, i.e. there exists a continuous mapping $\nu:\Omega\to J$ with the properties: $h(\nu(\omega))=\omega$ and $\nu(\theta(t,y))=\pi(t,\nu(\omega))$ for all $t\in\mathbb{T}$ and $\omega\in\Omega$.

Corollary 5. *Under the conditions of Theorem 7 the Levinson center of dynamical system $(X, \mathbb{T}_+, \pi)$ contains at least one stationary (τ ($\tau > 0$) - periodic, quasiperiodic, almost periodic) point, if the minimal set Ω consists a stationary (τ ($\tau > 0$) - periodic, quasiperiodic, almost periodic) point.*

6 Applications

6.1 Finite-dimensional systems

Denote by $\mathbb{R}^n$ the real $n-$dimensional Euclidean space with the scalar product $\langle , \rangle$ and the norm $|\cdot|$, generated by the scalar product. Let $[\mathbb{R}^n]$ be the space of all the linear mappings $A : \mathbb{R}^n \to \mathbb{R}^n$, equipped with the operational norm.

Theorem 8. *Let Ω be a compact minimal set, $F \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n),\ W \in C(\Omega, [\mathbb{R}^n])$ and the following conditions hold:*

1. The matrix-function W is positively defined, i.e. $\langle W(\omega)u, u\rangle \in \mathbb{R}$ for all $\omega \in \Omega,\ u \in \mathbb{R}^n$ and there exists a positive constant a such that $\langle W(\omega)u, u\rangle \geq a|u|^2$ for all $\omega \in \Omega$ and $u \in \mathbb{R}^n$.

2. The function $t \to W(\theta_t\omega)$ is differentiable for every $\omega \in \Omega$ and $\dot{W}(\omega) \in C(\Omega, [\mathbb{R}^n])$, where $\dot{W}(\omega) = \frac{d}{dt}W(\theta_t\omega)|_{t=0}$.

3. $\langle \dot{W}(\omega)(u - v) + W(\omega)(F(\omega, u) - F(\omega, v)), u - v\rangle \leq 0$ for all $\omega \in \Omega$ and $u, v \in \mathbb{R}^n$.

4. There exist a positive constant r and the function $c : [r, +\infty) \to (0, +\infty)$ such that $\langle \dot{W}(\omega)u + W(\omega)F(\omega, u), u\rangle \leq -c(|u|)$ for all $|u| > r$.

Then the equation

$$u' = F(\theta_t\omega, u) \tag{5}$$

generates a cocycle φ on $\mathbb{R}^n$ which admits a compact global attractor $I = \{I_\omega \mid \omega \in \Omega\}$ with the following properties:

a. I_ω is a nonvoid, compact and convex subset of $\mathbb{R}^n$ for every $\omega \in \Omega$.

b. $I = \bigcup\{I_\omega \mid \omega \in \Omega\}$ is connected.

c. The mapping $\omega \to I_\omega$ is continuous with respect to Hausdorff metric.

d. $I = \{I_\omega \mid \omega \in \Omega\}$ is invariant, i.e. $\varphi(t, \omega, I_\omega) = I_{\theta_t\omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}_+$.

e. $\lim\limits_{t\to+\infty} \beta(\varphi(t, \theta_t\omega)M, I_\omega) = 0$ for all $M \in C(\mathbb{R}^n)$ and $\omega \in \Omega$;

f. $\lim\limits_{t\to+\infty} \sup\{\beta(\varphi(t, \theta_t\omega)M, I) \mid \omega \in \Omega\ \} = 0$ for any $M \in C(\mathbb{R}^n)$, where $I = \bigcup\{I_\omega \mid \omega \in \Omega\ \}$.

g. $I = \{I_\omega \mid \omega \in \Omega\ \}$ is a uniform forward attractor, i.e.

$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, \omega)M, I_{\theta_t\omega}) = 0$$

for any $M \in C(\mathbb{R}^n)$.

h. The equation (5) admits at least one stationary (τ - periodic, quasiperiodic, almost periodic) solution if the point $\omega \in \Omega$ is stationary (τ - periodic, quasiperiodic, almost periodic).

Example 3. *As an example which illustrates this theorem we can consider the following equation*

$$u' = g(u) + f(\theta_t \omega),$$

where $f \in C(\Omega, \mathbb{R})$ *and*

$$g(u) = \begin{cases} (u+1)^2 & : \quad u < -1 \\ 0 & : \quad |u| \le 1 \\ -(u-1)^2) & : \quad u > 1. \end{cases}$$

Example 4. *We consider the equation*

$$x'' + p(x)x' + ax = f(\theta_t \omega),$$

where $p \in C(\mathbb{R}, \mathbb{R})$, $f \in C(\Omega, \mathbb{R})$ *and* a *is a positive number. Denote by* $y = x' + F(x)$, *where* $F(x) = \int_0^x p(s)ds$, *then we obtain the system*

$$\begin{cases} x' = y - F(x) \\ y' = -ax + f(\theta_t \omega). \end{cases} \tag{6}$$

Theorem 9. *Suppose the following conditions hold:*

1. $p(x) \ge 0$ *for all* $x \in \mathbb{R}$.

2. There exist positive numbers r *and* k *such that* $p(x) \ge k$ *for all* $|x| \ge r$.

Then the nonautonomous dynamical system generated by (6) is compact dissipative and $V-$ *monotone.*

6.2 Evolution equations with monotone operators

Let H be a real Hilbert space with inner product $\langle , \rangle, |\cdot| = \sqrt{\langle , \rangle}$ and $\mathbb{B}$ be a reflexive Banach space contained in H algebraically and topologically. Furthermore, let $\mathbb{B}$ be dense in H in which case H can be identified with a subspace of the dual $\mathbb{B}'$ of $\mathbb{B}$ and $\langle , \rangle$ can be extended by continuity to $\mathbb{B}' \times \mathbb{B}$.

We consider the initial value problem

$$u'(t) + Au(t) = f(\theta_t \omega) \tag{7}$$

$$u(0) = u, \tag{8}$$

where $A : \mathbb{B} \to \mathbb{B}'$ is (generally nonlinear) bounded,

$$|Au|_{\mathbb{B}'} \le C|u|_{\mathbb{B}}^{p-1} + K, u \in \mathbb{B}, p > 1,$$

coercive,

$$\langle Au, u \rangle \ge a|u|_{\mathbb{B}}^p, u \in \mathbb{B}, a > 0,$$

monotone,

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle \ge 0, u_1, u_2 \in \mathbb{B},$$

and hemicontinuous (see [20]).

The nonlinear "elliptic" operator

$$Au = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(\frac{\partial u}{\partial x_i}) \quad \text{in } D \subset \mathbb{R}^n,$$

$$u = 0 \text{ on } \partial D,$$

where D is a bounded domain in $\mathbb{R}^n$, $\phi(\cdot)$ is an increasing function satisfying

$$\phi|_{[-1,1]} = 0,\ c|\xi|^p \le \sum_{i=1}^{n} \xi_i \phi(\xi_i) \le C|\xi|^p \text{ (for all } |\xi| \ge 2 \text{)},$$

provides an example with $H = L^2(D), \mathbb{B} = W_0^{1,p}(D),\ \mathbb{B}' = W^{-1,p'}(D),\ p' = \frac{p}{p-1}$.

The following result is established in [20] (Ch.2 and Ch.4). If $x \in H$ and $f \in C(\Omega, \mathbb{B}'),\ p' = \frac{p}{p-1}$, then there exists a unique solution $\varphi \in C(\mathbb{R}_+, H)$ of (7) and (8).

We denote by $\varphi(\cdot, \omega, u)$ the unique solutions of (7) and (8). According to [21] $\varphi(\cdot, \omega, u)$ is a continuous cocycle on H.

Theorem 10. *Suppose that the operator A satisfies the conditions above and the cocycle φ, generated by equation (7), is asymptotically compact, then it admits a compact global attractor $I = \{I_\omega \mid \omega \in \Omega\ \}$ possessing the following properties:*

a. I_ω is a nonvoid, compact and convex subset of H for every $\omega \in \Omega$.

b. $I = \bigcup\{I_\omega \mid \omega \in \Omega\ \}$ is connected.

c. The mapping $\omega \to I_\omega$ is continuous with respect to Hausdorff metric.

d. $I = \{I_\omega \mid \omega \in \Omega\ \}$ is invariant, i.e. $\varphi(t, \omega, I_\omega) = I_{\sigma_t \omega}$ for all $\omega \in \Omega$ and $t \in \mathbb{T}^+$.

e. $\lim_{t\to+\infty} \beta(\varphi(t, \theta_{-t}\omega)M, I_\omega) = 0$ for all $M \in C(H)$ and $\omega \in \Omega$;

f. $\lim_{t\to+\infty} \sup\{\beta(\varphi(t, \theta_t\omega)M, I)| \ \omega \in \Omega\ \} = 0$ for any $M \in C(H)$, where $I = \bigcup\{I_\omega \mid \omega \in \Omega\ \}$.

g. $I = \{I_\omega \mid \omega \in \Omega\ \}$ is a uniform forward attractor, i.e.

$$\lim_{t\to+\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, \omega)M, I_{\theta_t\omega}) = 0$$

for any $M \in C(H)$.

h. The equation (7) admits at least one stationary (τ - periodic, quasiperiodic, almost periodic) solution if the point $\omega \in \Omega$ is stationary (τ - periodic, quasiperiodic, almost periodic).

Remark 1. *If the injection of $\mathbb{B}$ into H is compact, then the cocycle φ generated by equation (7), evidently, is asymptotically compact.*

Example 5. *A typical example of equation of type (7) is the equation*

$$\frac{\partial}{\partial t} u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\varphi(\frac{\partial u}{\partial x_i}) + f(\theta_t\omega),\ u|_{\partial D} = 0 \tag{9}$$

with "nonlinear Laplacian" $Au = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(\frac{\partial u}{\partial x_i})$ *provides an example of equation of type (7) with* $H = L^2(D), \mathbb{B} = W_0^{1,p}(D),\ \mathbb{B}' = W^{-1,p'}(D)$ *and* $p' = \frac{p}{p-1}$, *where* $\phi(\cdot)$ *is an increasing function satisfying the condition*

$$c|\xi|^p \le \sum_{i=1}^{n} \xi_i \phi(\xi_i) \le C|\xi|^p$$

for all $|\xi| \ge 2$ *and* $\phi|_{[-1,1]} = 0$. *It is possible to verify (see, for example,[4, 20] and [2]) that the "nonlinear Laplacian" verifies all the conditions of Theorem 10 and, consequently, (9) admits a compact global attractor with the properties a.-h.. We note that the attractor of equation (9) is not trivial, i.e. the set* I_ω *is not a single point set at least for certain* $\omega \in \Omega$.

Remark 2. *If the operator* $A = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\phi(\frac{\partial u}{\partial x_i})$ *is uniformly elliptic, i.e.* $c|\xi|^p \le \sum_{i=1}^{n} \xi_i \varphi(\xi_i) \le C|\xi|^p$ *(for all* $\xi \in \mathbb{R}^n$*), then the set* I_ω *is a single point set for all* $\omega \in \Omega$ *(for autonomous system see [23], Ch.III), because in this case the nonautonomous dynamical system generated by equation (9) is strictly monotone.*

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David N. Cheban
Faculty of Mathematics and Informatics,
State University of Moldova,
MD–2009 Chişinău, Moldova
E-mail: cheban@usm.md

Peter E. Kloeden
Fachbereich Mathematik,
Johann Wolfgang Goethe Universität,
D–60054 Frankfurt am Main, Germany
E-mail: kloeden@math.uni-frankfurt.de

Björn Schmalfuß
University of Applied Sciences,
D–06217 Merseburg, Germany
E-mail: schmalfuss@in.fh-merseburg.de