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On the Division of Abstract Manifolds in Cubes

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Abstract. We prove that in the class of abstract multidimensional manifolds without borders only torus V_1^n of dimension $n \geq 1$ can be divided in abstract cubes with the property: every face I^m from V_1^n is shared by 2^{n-m} cubes, $m = 0, 1, \dots, n-1$. The abstract torus V_1^n is realized in E^d, $n+1 \leq d \leq 2n+1$, so it results that in the class of all n-dimensional combinatorial manifolds [1] *only* torus respects this propriety. Torus is autodual because of this propriety.

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In paper [7, p.402] the scheme of the main types of n-dimensional manifolds it is presented, but the type of abstract manifolds which have been introduced recently in the papers [3–5] is missing. These abstract n-dimensional manifolds can be isomorphicly represented in E^d, $n+1 \leq d \leq 2n+1$. So we obtain combinatorial manifolds [1] which belong to the scheme mentioned above. We investigate abstract manifolds, which are defined by multi-ary relations and do not investigate directly combinatorial manifolds because we can obtain new results from abstract and more general point of view [6]. The base of an abstract manifold's definition is an abstract simplex S^n, which is defined on the set of $(n+1)$ elements from the $(n+1)$-ary relation of distinct elements.

First let's mention

Definition 1 [3]. *The complex of multi-ary relations,* $K^n = \{S_\lambda^m : \lambda \in \Lambda, card\Lambda < \infty, 0 \leq m \leq n\}$, *denoted* V_Δ^n, *is called an **abstract n-dimensional manifold without borders** if it satisfies the following postulates:*

A. *any abstract simplex* $S^{n-1} \in V_\Delta^n$ *is a common face exactly for two abstract n-dimensional simplexes;*

B. *for any simplexes* $S_i^n, S_j^n \in V_\Delta^n$, $i \neq j$, *there exists a sequence of n-dimensional simplexes* $S_1^n = S_i^n, S_2^n, \dots, S_k^n = S_j^n$, $k \geq 2$, *where* $S_r^n \cap S_{r+1}^n = S_{r,r+1}^{n-1}$, $r \in \{1, 2, \dots, k-1\}$;

C. *for* $\forall S^m \in V_\Delta^n$ *it holds that* $\exists S^n \in V_\Delta^n$, *such that* S^m *is a face of* S^n, $m \in \{0, 1, \dots, n-1\}$;

D. *for any two disjoint simplexes* $\forall S_i^n, S_j^n \in V_\Delta^n$, *where* $S_i^n \cap S_j^n = S^m$, *it holds that* $\exists S_1^n = S_i^n, S_2^n, \ldots, S_k^n = S_j^n$, *such that* $\bigcap_{l=1}^{k} S_l^n = S^m$.

We are interested only in the examination of oriented manifolds [3, 4]. Let's mention

Definition 2 [2, 6]**.** *The cubic complex* $K^n = \{I_\lambda^m : \lambda \in \Lambda, card\Lambda < \infty, 0 \le m \le n\}$, *denoted* $V_\square^n$, *is called an* ***abstract cubic n-dimensional manifold without borders*** *if the following properties are satisfied*:

A. *any* $(n-1)$*-dimensional cube is a common face exactly for two n-dimensional cubes from* K^n*;*

B. *for* $\forall I_i^n, I_j^n \in K^n$, $i \neq j$, *there exists a sequence of cubes from* K^n, $I_{i_1}^n = I_i^n, I_{i_2}^n \ldots, I_{i_q}^n = I_j^n$, *where* $I_r^n \cap I_{r+1}^n = I_{r,r+1}^{n-1}, r \in \{i_1, i_2, \ldots, i_{q-1}\}$*;*

C. *for* $\forall I^p \in K^n$, $0 \le p \le n-1$, *it holds* $\exists I^n \in K^n$, *where* I^p *is a face of* I^n*;*

D. *for any disjoint cubes* $\forall I_i^n, I_j^n \in K^n$, $I_i^n \cap I_j^n = I^p$, $2 \le p < n$, *there exists a sequence of abstract cubes from B.,* $I_{i_1}^n = I_i^n, I_{i_2}^n, \ldots, I_{i_q}^n = I_j^n$, *such that* $\bigcap_{j=1}^{q} I_{i_j}^n = I^p$.

We are interested also in the examination of oriented cubic manifolds [6].

Definition 1 is based on a finite complex of multi-ary relations, but Definition 2 is formulated using a finite number of abstract cubes, which are defined by abstract simplexes. So Definition 1 and 2 are equivalent and in the following we use only the notation V^n.

Definition 3. *The property of n-dimensional abstract manifold without borders* V_p^n, *which is determined of a cubic complex* K^n, *such that* ***every*** *m-dimensional cube,* $0 \le m \le n$, *belongs to* 2^{n-m} *n-dimensional cubes, is called a* ***normal cubiliaj***[1] *of* V_p^n.

Let's define now a finite product of edges (abstract 1-dimensional cubes [4]) analogous with cartesian product.

Definition 4. *Let* $I_1^1, I_2^1, \ldots, I_r^1$ *be some 1-dimensional oriented abstract cubes. By induction*

1. $I_1^1 \otimes I_2^1 = I^2$, *where* I^2 *is a 2-dimensional abstract cube* [4] *and* $\overset{\circ}{I}{}^2 = \overset{\circ}{I}{}_1^1 \otimes \overset{\circ}{I}{}_2^1$ [4] *is his vacuum.*

(r-1). *Let's consider that* $I^{r-1} = I^{r-2} \otimes I_{r-1}^1$ *is defined, where* I^{r-1} *is an* $(r-1)$*-dimensional abstract cube* [4] *and* $\overset{\circ}{I}{}^{r-1} = \overset{\circ}{I}{}_1^{r-2} \otimes \overset{\circ}{I}{}_{r-2}^1$ *is its vacuum.*

[1] The notion of cubilaj was borrowed from the papers [10, 11].

r. *Inductively we define r-dimensional abstract cube* I^r *in the following way:* $I^r = I^{r-1} \otimes I^1_r$, *where* $\overset{\circ}{I^r} = \overset{\circ}{I^{r-1}} \otimes \overset{\circ}{I^1_r}$. *The cube* I^r *is called a* **cartesian product** *of cubes* $I^1_1, I^1_2, \dots, I^1_r$ *and will be denoted by*

$$I^r = \prod_{i=1}^{r} I^1_i \tag{1}$$

Let's consider n abstract oriented circumferences (1-dimensional manifolds): $V^1_1, V^1_2, \dots, V^1_n$ with the length (the number of 1-dimensional cubes) $d_1, d_2, \dots, d_n$.

Using (1), we consider the cartesian product:[2]

$$K^n = \prod_{i=1}^{n} V^1_i \tag{2}$$

In accordance with Definition 2, it is obvious that (2) establishes an n-dimensional abstract manifold without borders which posseses a normal cubiliaj. Moreover, the Euler characteristic of V^1_i is $\chi(V^1_i) = 0, i \in \{1, 2, \dots, n\}$, so [7]:

$$\chi(K^n) = \prod_{i=1}^{n} \chi(V^1_i) = 0. \tag{3}$$

Consequently we have

Corollary 1. *The product* (2) *establishes an abstract torus* V^n_1 *(see Figure 1).*

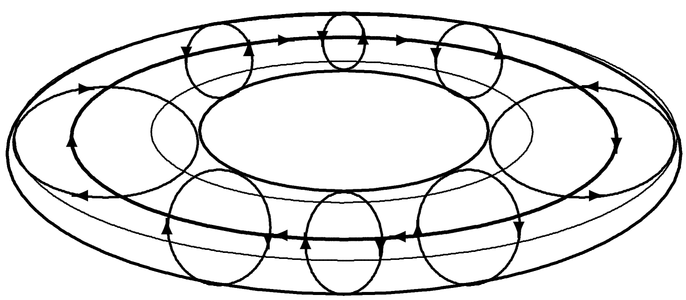

Figure 1

This corollary results from the fact that for every n (odd or even) (3) is true.

It holds

Theorem 1. *An abstract oriented manifold without borders which has a normal cubiliaj is a torus* V^n_1 *if and only if* V^n *is established by the cartesian product* (2).

[2]The formula (2) is realized in E^{2n+1} [1], so it define an n-dimensional torus as a cartesian product of n circumferences [13].

Proof. *The sufficiency* is obvious because of Corollary 1.

The necessity is simple. Let V^n be an abstract manifold which has a normal cubiliaj and I^n an abstract cube of V^n. $a_1, a_2, \dots, a_n$ are n oriented arcs with common origin which determine the manifold V^n. Let's consider the class of equivalence of "parallel" arcs A_1 ([8 − 10, 12]) and the class of $(n-1)$-dimensional cubes of V^n which are determined by elements from the class A_1.

Let's denote the last class by V_1^{n-1}. It is obvious that the last one is an abstract submanifold of V^n which possesses hereditarily a normal cubiliaj. Coherently let's move along the arc a_1. The end of the arc a_1 belongs to another abstract submanifold V_2^{n-1} of V^n which is "parallel" with V_1^{n-1}. Suppose that the manifold V_2^{n-1} is "perpendicular" to another arc b_1, coherent to a_1 (otherwise we give a new orientation to it). In the same reasoning we can obtain another manifold V_3^n which has a normal cubiliaj. By induction we can construct a 1-dimensional contour without cross points because of the finite number of cubes from V^n. If the intersections exists then V^n doesn't have a normal cubiliaj. So we obtain the first oriented abstract circumference V_1^1. By induction of the index i of a_i, considering the class of equivalence A_i of arcs "parallel" to a_i for V_i^{n-i}, $i \in \{1, 2, \dots, n-1\}$, we construct the $(n-1)$ oriented abstract circumference. So we have the abstract circumferences $V_1^1, V_2^1, \dots, V_{n-1}^1$. The submanifold V^1 of V^n which is perpendicular to $a_1, a_2, \dots, a_{n-1}$ (see figure 1, the thick meridian) possesses hereditarily a normal cubiliaj. So we have $V^1 = V_n^1$. Using the formula (2) we obtain the proof of Theorem 1.

It holds

Theorem 2. *Let V_p^n, $p \neq 1$, be a coherent oriented abstract manifold without borders* [1]*. This manifold does not possess a normal cubiliaj.*

Proof. By contradiction. We consider a submanifold V_p^{n-1} of V_p^n, $p \neq 1$, which can be obtain in the same way as in the proof of Theorem 2, using the arc $a_1 \in I^n$. Analogously to the proofs' procedure of Theorem 2, we can obtain n n-dimensional contours without autointersection, $V_1^1, V_2^1, \dots, V_n^1$, which cartesian product is

$$V_p^n = \prod_{i=1}^{n} V_i^i. \tag{4}$$

In accordance with Theorem 1, the product (4) represents a torus V_1^n which possesses a normal cubiliaj. This contradiction (for $p \neq 1$) results from a false assumption. Theorem 2 is proved.

Form Theorems 1 and 2 we obtain

Fundamental theorem. *A unique abstract n-dimensional manifold without borders V_p^n, where $n \geq 0$, which possesses a normal cubiliaj is the torus V_1^n.*

Remark 1. *In parer* [9] *was established that the sphere $S^2 \subset E^2$ does not possess a normal cubiliaj.*

It holds

Theorem 3. *Only the abstract torus, V_1^p, which possesses a normal cubiliaj, represents an autodual manifold corresponding to this cubiliaj.*

Proof. Let $\alpha_0, \alpha_1, \dots, \alpha_n$ be the numbers of abstract cubes of V_1^n, with respective dimension $0, 1, 2, \dots, n$. So we have:

$$\chi(V_1^n) = \sum_{i=1}^{n} (-1)^i \alpha_i = 0. \tag{5}$$

Considering the cubic complex K_d^n with the class of the cubes $\alpha_n, \alpha_{n-1}, \dots, \alpha_0$ having the dimensions $0, 1, \dots, n$ respectively and invariant incidences, we obtain that K_d^n is isomorphic to the complex $K^n = V_1^n$. So V_1^n is determined uniquely by K_d^n. From (5) it results:

$$\chi(V_1^n) = \chi(K_d^n) = \sum_{i=0}^{n} (-1)^{n-i} \alpha_{n-i} = 0. \tag{6}$$

So the initial normal cubiliaj of V_1^n is isomorphic to the normal cubiliaj which is established by K_d^n (see Figure 2.)

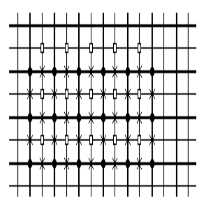

Figure 2

So this fact represents the *autoduality* of the torus V_1^n. Only this one is represented by a normal cubiliaj. In accordance with the Fundamental theorem such kind of autodualism has only the abstract torus V_1^n. Theorem 3 is proved.

Remark 2. *When the above results were obtained as something additional in the solving of application problems, we were informed about the papers* $[11-13]$. *This helped us to change the terms' names. The problems formulated by the famous mathematician Serghei Novikov inspired us to additional examinations. We do this with gratitude.*

In the following paper we will indicate the value of the Fundamental theorem in the transmission, receiving and picking up of information.

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