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Queuing system evolution in phase merging scheme*

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Abstract. We study asymptotic average scheme for semi-Markov queuing systems using compensating operator of the corresponding extended Markov process. The peculiarity of our queuing system is that the series scheme is considered with phase merging procedure.

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1 Introduction

The queuing system (QS) of $[SM|M|1|\infty]^N$ type means that the input flow is described by a semi-Markov process, the service time is exponentially distributed, there are N servers connected by a route probability matrix. So the queuing networks is considered with a semi-Markov flow. The peculiarity of our queuing system is that the series scheme is considered with phase merging procedure [1]. The average algorithm is established for the queuing process (QP) described the number of claims at every node. Analogously problem was investigated in work [1].

2 Preliminaries

The regular semi-Markov process $k^\varepsilon(t)$, $t \geq 0$ on the standard phase space (E, E) in the series scheme, with the small series parameter $\varepsilon \to 0$ ($\varepsilon > 0$), given by the semi-Markov kernel [1, 3, 4].

$$Q^\varepsilon(\kappa, B, t) = P^\varepsilon(\kappa, B) G_\kappa(t), \kappa \in E, B \in e, t \geq 0. \tag{1}$$

The stochastic kernel

$$P^\varepsilon(\kappa, B) = P(\kappa, B) + \varepsilon P_1(\kappa, B). \tag{2}$$

The stochastic kernel $P(\kappa, B)$ is coordinated with the split phase space

$$E = \bigcup_{k=1}^{N} E_k, E_k \bigcap E_k = \oslash, k \neq k', \tag{3}$$

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as follows

$$P(\kappa, E_k) = \delta_k(\kappa) := \begin{cases} 1, \kappa \in E_k \\ 0, \kappa \notin E_k \end{cases} \tag{4}$$

The perturbing kernel $P_1(\kappa, B)$ provides the transition probabilities of the embedded Markov chain k_n^ε, $n \geq 0$, between classes of states E_k, $1 \leq k \leq N$, which tend to zero as $\varepsilon \to 0$.

The renewal moments τ_n, $n \geq 0$, are defined by the distribution functions

$$G_\kappa(t) = P(\theta_{n+1} \leq t | k_n^\varepsilon = \kappa) =: P(\theta_\kappa \leq t), \tag{5}$$

here $\theta_{n+1} = \tau_{n+1} - \tau_n$, $n \geq 0$, are the sojourn times. For more details of semi-Markov process see monograph [1, Ch 1].

Introduce the mean values of sojourn time

$$g(\kappa) := E\theta_\kappa = \int_0^\infty \bar{G}_\kappa(t)dt, \bar{G}_\kappa(t) := 1 - G_\kappa(t), \tag{6}$$

and the average intensities

$$q(\kappa) = 1/g(\kappa), \kappa \in E. \tag{7}$$

In what follows the associated Markov process $k^0(t)$, $t \geq 0$, given by the generator

$$Q\varphi(\kappa) = q(\kappa) \int_E P(\kappa, dy)[\varphi(y) - \varphi(\kappa)], \tag{8}$$

is uniformly ergodic in every class E_k, $k \in \widehat{E}$, $\hat{E} = \{1, 2, \ldots, N\}$ with the stationary distributions $\pi_k(d\kappa)$, $k \in \widehat{E}$. The corresponding embedded Markov chain $k_n^0 = k^0(\tau_n)$, $n \geq 0$, is uniformly ergodic also with the stationary distributions $\rho_k(d\kappa)$, $k \in \widehat{E}$. Note that the following relations are valid:

$$\pi_k(d\kappa)q(\kappa) = q_k\rho_k(d\kappa), q_k = \int_{E_1} \pi_k(d\kappa)q(\kappa). \tag{9}$$

According to Theorem 4.1 [1, § 4.2.1, p.108] the merged process $\nu(k^\varepsilon(t/\varepsilon))$ converges weakly as $\varepsilon \to 0$, to the Markov process $\hat{k}(t)$, $t \geq 0$, on the merged phase space $\hat{E} = \{1, 2, \ldots, N\}$, given by the generative matrix $\hat{Q} = [\hat{q}_{kr}; k, r \in \hat{E}]$.

We assume that the merged Markov process $\hat{k}(t)$, $t \geq 0$, is ergodic with the stationary distribution $\hat{\pi} = (\hat{\pi}_k, k \in \hat{E})$.

3 Queuing process in the networks

The evolution of claims in the networks on $\hat{E} = \{1, 2, \ldots, N\}$ is defined by the route matrix P_0 and the intensity vector of exponential service time $\mu = (\mu_k, k \in \hat{E})$.

The queuing process in average scheme is considered in the following normalizing form:

$$U^{\varepsilon}(t) = \varepsilon^2 \rho^{\varepsilon}(t/\varepsilon^2), t \geq 0, \varepsilon > 0, \tag{10}$$

where $\rho^{\varepsilon}(t) = (\rho^{\varepsilon}_k(t),\ k \in \hat{E})$ is the vector with the components $\rho^{\varepsilon}_k(t)$ – number of claims at node $k \in \hat{E}$ at time t.

The queuing process $U^{\varepsilon}(t)$ in average scheme is considered under the following assumptions.

A1: The queuing networks is open, that means the route matrix satisfies the condition:

$$p^0_{k0} := 1 - \sum_{r=1}^{N} p^0_{kr}, \max_{k \in \hat{E}} p^0_{k0} > 0. \tag{11}$$

A2: There exists nonnegative solution of the evolutionary equation

$$dU^0(t)/dt = C(U^0(t)), U^0(0) = u_0, \tag{12}$$

where the velocity vector

$$C(u) = (C_k(u), k \in \hat{E}), \tag{13}$$

is defined by its components

$$C_k(u) = \gamma_k(u) + \lambda_k, \quad \gamma_k(u) = \sum_{r=1}^{N} \mu_r u_r [p_{rk} - \delta_{rk}], \lambda_k = \hat{\pi}_k q_k.$$

Theorem 1. *Under the assumptions A1-A2 the weak convergence* $U^{\varepsilon}(t) \Rightarrow U^0(t), \varepsilon \to 0,$ *takes place.*

Corollary 1. *Let exist an equilibrium point* $u^0 \geq 0$ *satisfying*

$$C(u^0) = 0. \tag{14}$$

Then under initial condition $U^{\varepsilon}(0) \Rightarrow u_0, \varepsilon \to 0,$ *the weak convergence* $U^{\varepsilon}(t) \Rightarrow u_0, \varepsilon \to 0,$ *takes place.*

Remark 1. The vector $\tilde{\pi} = (\tilde{\pi}_k := q\hat{\pi}_k q_k,\ k \in \hat{E})$, $q^{-1} = \sum_{k \in \hat{E}} \hat{\pi}_k q_k$ describes the stationary distribution of the Markov process $\tilde{k}(t)$, $t \geq 0$, defined by the generating matrix (see [1, Theorem 4.1])

$$\tilde{Q} = [p_{kr}, k, r \in \hat{E}], p_{kr} = \int_{E_1} \rho_k(d\kappa) P_1(\kappa, E_r). \tag{15}$$

Indeed (see [1,(4.17) and (4.19)],

$$\sum_k \hat{\pi}_k q_k p_{kr} = \sum_k \hat{\pi}_k q_k \hat{p}_k \hat{p}_{kr} = \sum_k \hat{\pi}_k \hat{q}_k \hat{p}_{kr} = \sum_k \hat{\pi}_k \hat{q}_{kr} = 0. \tag{16}$$

4 Proof of Theorem. Compensating operator

The extended Markov renewal process

$$u_n^\varepsilon = u^\varepsilon(\tau_n^\varepsilon), k_n^\varepsilon = k^\varepsilon(\tau_n^\varepsilon), \tau_n^\varepsilon = \varepsilon^2 \tau_n, n \geq 0, \tag{17}$$

is characterized by the compensating operator (CO) (see [1,Ch 1, 2])

$$L^\varepsilon \varphi(u,\kappa) = \varepsilon^{-2} q(\kappa) E[\varphi(u_{n+1}^\varepsilon, k_{n+1}^\varepsilon) - \varphi(u,\kappa)] u_n^\varepsilon = u, k_n^\varepsilon = \kappa. \tag{18}$$

The key step in asymptotic analysis of the QS is to construct an asymptotic expansion of the CO (18).

Lemma 1. *The CO (18) can be represented in the following form*

$$L^\varepsilon \varphi(u,\kappa) = \varepsilon^{-2} q(\kappa)[G^\varepsilon(\kappa) P^\varepsilon D^\varepsilon(k) - I, \tag{19}$$

where

$$G^\varepsilon(\kappa) = \int_0^\infty G_\kappa(dt) \Gamma_t^\varepsilon. \tag{20}$$

The semigroup Γ_t^ε *is defined by the generator*

$$\Gamma^\varepsilon \varphi(u) = \sum_{k,r=1}^{N} \gamma_{kr}(u)[\varphi(u + \varepsilon^2 e_{rk}) - \varphi(u)], \tag{21}$$

$$e_{kr} := e_r - e_k, e_k := (\delta_{kl}, l \in \hat{E}).$$

The operators $D^\varepsilon(k)$, $k \in \hat{E}$, *are defined by*

$$D^\varepsilon(k)\varphi(u) = \phi(u + \varepsilon^2 e_{rk}), k \in \hat{E}. \tag{22}$$

The operator

$$P^\varepsilon = P + \varepsilon P_1, \tag{23}$$

where

$$P\varphi(\kappa) = \int_E P(\kappa, dy)\varphi(y), P_1\varphi(\kappa) = \int_E P_1(\kappa, dy)\varphi(y). \tag{24}$$

Proof of Lemma 1. The representation (19) is direct conclusion of the equality

$$u_{n+1}^\varepsilon - u_n^\varepsilon = \beta^\varepsilon(\theta_{n+1}) + \varepsilon^2 e_{n+1},$$

where $\beta^\varepsilon(t)$, $t \geq 0$, is the Markov process given by the generator (21).

Lemma 2. *The CO (19) admits the following asymptotic expansion on the test-function $\phi(u,\kappa) \in C^3(R^d)$ uniformly in $\kappa \in E$:*

$$L^\varepsilon(k)\varphi(u,\kappa) = [\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2(\kappa) + \theta_L^\varepsilon(\kappa)]\varphi(u,\kappa), \tag{25}$$

where

$$Q\varphi(\kappa) = q(\kappa)\int_E P(\kappa,dy)[\varphi(y) - \varphi(\kappa)], \tag{26}$$

$$Q_1\varphi(\kappa) = q(\kappa)\int_E P_1(\kappa,dy)\varphi(y), \tag{27}$$

$$\lambda(\kappa) = (\lambda_k(\kappa), k \in \hat{E}), \lambda_k(\kappa) = q(\kappa)\delta_k(\kappa), Q_2(\kappa)\varphi(u) = [\gamma(u) + \lambda(\kappa)]\varphi'(u) \tag{28}$$

and the negligible term $\theta_L^\varepsilon(\kappa)\varphi(u) \to 0$ as $\varepsilon \to 0$, $\varphi(u) \in C^3(R^N)$.

Proof of Lemma 2. The following identity is used below:

$$GD - I = G - I + D - I + (G - I)(D - I),$$

and asymptotic expansion on the test- function $\varphi(u) \in C^3(R^N)$

$$\varepsilon^{-2}q(\kappa)[G^\varepsilon(\kappa) - I]P\varphi(u) = [q(\kappa)G(\kappa)P + \theta_g^\varepsilon(\kappa)P]\varphi(u),$$

$$\varepsilon^{-2}q(\kappa)P[D^\varepsilon(k) - I]\varphi(u) = [q(\kappa)PD(k) + \theta_d^\varepsilon(\kappa)P]\varphi(u),$$

$$\varepsilon^{-2}q(\kappa)\varepsilon P_1[D^\varepsilon(k) - I]\varphi(u) = [\varepsilon q(\kappa)P_1D(k) + \varepsilon\theta_{dl}^\varepsilon(\kappa)P_1]\varphi(u),$$

$$\varepsilon^{-2}q(\kappa)[G^\varepsilon(\kappa) - I]P^\varepsilon[D^\varepsilon(k) - I]\varphi(u) = \theta_{gd}^\varepsilon(\kappa)P^\varepsilon\varphi(u)$$

is a negligible term.

The limit operator in the theorem is defined by a solution of singular perturbation problem for the truncated operator

$$L_0^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2(\kappa). \tag{29}$$

Lemma 3. *The limit operator L in the theorem is defined by formulae (see [1, Proposition 5.3., p.146]:*

$$L = \hat{\Pi}\Pi Q_2(\kappa)\Pi\hat{\Pi}, \tag{30}$$

where the projectors Π and $\hat{\Pi}$ act as follows:

$$\Pi\varphi(\kappa) = \sum_{k=1}^N \hat{\varphi}_k l_k(\kappa), \hat{\varphi}_k = \int_{E_1} \pi_k(d\kappa)\varphi(\kappa), k \in \hat{E},$$

$$\hat{\Pi}\hat{\varphi}(\kappa) = \sum_{k=1}^N \hat{\pi}_k\hat{\varphi}_k.$$

Corollary 2. *The limit operator L in Theorem is defined as follows*

$$L\varphi(u) = C(u)\varphi'(u) = \sum_{k=1}^{N} C_k(u)\varphi'_k(u), \varphi'_k(u) := \partial\varphi(u)/\partial u_k, \tag{31}$$

where $C(u) = \gamma(u) + \lambda, C_k(u) = \gamma_k(u) + \hat{\pi}_k q_k, \lambda = (\hat{\pi}_k q_k, k \in \hat{E})$.

The last step of the proof of theorem is realized by using Theorem 6.6 from [1, Ch. 6, p.202].

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