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Singularly perturbed Cauchy problem for abstract linear differential equations of second order in Hilbert spaces

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Abstract. We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon\Big(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\Big) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f(t), \quad t > 0, \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1, \end{cases}$$

in the Hilbert space H as $\varepsilon \mapsto 0$, where A_1 and A_0 are two linear selfadjoint operators.

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1 Introduction

Let H be a real Hilbert space endowed with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. Let $V \subset H$ be a real Hilbert space which is endowed with norm $||\cdot||$ such that the inclusion is dense and continuous. Let $V = V_0 \cap V_1$, and $A_i : D(A_i) = V_i \to H$, $i = 0, 1$, be two linear selfadjoint operators such that

$$\Big((A_0 + \varepsilon A_1)u, u\Big) \geq \gamma ||u||^2, \quad u \in V, \quad \gamma > 0, \tag{1}$$

for some $\varepsilon \ll 1$ and εA_1 generates a C_0- semigroup $\{S(t,\varepsilon), t \geq 0\}$ with the following two properties:

$$A_0 S(t,\varepsilon)u = S(t,\varepsilon)A_0 u, \forall u \in V. \tag{2}$$

$$\exists \delta > 0 : |S(t,\varepsilon)u| \geq \delta |u|, u \in V. \tag{3}$$

Consider the following Cauchy problem, which will be called (P_ε):

$$\begin{cases} \varepsilon\Big(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\Big) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f(t), \quad t > 0, \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $u, f : [0,\infty) \to H$. We will investigate the behavior of solutions $u_\varepsilon(t)$ to the perturbed system (P_ε) as $\varepsilon \to 0$. We will establish a relationship between solutions to the problem (P_ε) and the corresponding solutions to the following unperturbed system, which will be called (P_0):

$$\begin{cases} v'(t) + A_0 v(t) = f(t), \quad t > 0, \\ v(0) = u_0. \end{cases}$$

2 A *priori* estimates for solutions to the problems (P_ε) and (P_0)

In this section we remind the existence theorems for the solutions to the problems (P_ε) and (P_0) and give some a *priori* estimations for them.

Definition 1. *We say a function $u \in L^2(0,T,V)$, with $u' \in L^2(0,T,V')$ is a solution of (P_0) if*

$$\langle u', v\rangle + (A_0 u, v) = (f, v)$$

for each $v \in V$ and a.e. time $0 < t < T$, and

$$u(0) = u_0.$$

Definition 2. *We say a function $u \in L^2(0,T,V)$, with $u' \in L^2(0,T,H)$ and $u'' \in L^2(0,T,V')$ is a solution of (P_ε) if*

$$\varepsilon\langle u'', v\rangle + \varepsilon(A_1 u, v) + (u', v) + (A_0 u, v) = (f, v)$$

for each $v \in V$ and a.e. time $0 < t < T$, and

$$u(0) = u_0, \quad u'(0) = u_1,$$

where $\langle , \rangle$ express the pairing between H and H'.

Theorem A [1]. *Let $T > 0$. If condition (1) is fulfilled, $f \in W^{1,1}(0,T;H)$, $u_0 \in V$, then there exists a unique solution $v \in W^{1,\infty}(0,T;H)$ of the problem (P_0) such that*

$$|v(t)| + |v'(t)| \leq C(T, u_0, f, \gamma), \quad t \in [0,T].$$

Theorem B [1, 2]. *Let $T > 0$. If condition (1) is fulfilled, $f \in W^{1,1}(0,T;H)$, $u_0 \in V, u_1 \in H$, then there exists a unique solution of the problem (P_ε) such that $u_\varepsilon \in C(0,T;V)$, $u'_\varepsilon \in C(0,T;H) \cap L^\infty(0,T;V)$, $u''_\varepsilon \in L^\infty(0,T;H)$. Moreover, for u the following estimate*

$$|u_\varepsilon(t)| + |u'_\varepsilon(t)| \leq C(T, u_0, u_1, f, \gamma), \quad t \in [0,T],$$

is true.

3 Relation between solution to the problems (P_ε) and (P_0)

Now we are going to establish the relationship between the solution of the problem (P_ε) and the corresponding solutions of the problem (P_0). This relationship was inspired by the work [2]. To this end we defined the kernel of transformation which realizes this relationship.

For $\varepsilon > 0$ denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3t-2\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3t+6\tau}{4\varepsilon}\Big\}\lambda\Big(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon t}}\Big), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}d\eta.$$

The properties of kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 1 [2]. *The function $K(t,\tau,\varepsilon)$ possesses the following properties*:

- **(i)** *For any fixed $\varepsilon > 0$ $K \in C(\{t \geq 0\} \times \{\tau \geq 0\}) \cap C^\infty(R_+ \times R_+)$;*
- **(ii)** $K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad t > 0, \tau > 0;$
- **(iii)** $K(0,\tau,\varepsilon) = \dfrac{1}{2\varepsilon}\exp\Big\{-\dfrac{\tau}{2\varepsilon}\Big\}, \tau \geq 0; \quad \varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0,\ t \geq 0;$
- **(iv)** *For each fixed $t > 0$, $s,q \in \mathbb{N}$ there exist constants $C_1(s,q,t,\varepsilon) > 0$ and $C_2(s,q,t) > 0$ such that*
$$|\partial_t^s \partial_\tau^q K(t,\tau,\varepsilon)| \leq C_1(s,q,t,\varepsilon)\exp\{-C_2(s,q,t)\tau/\varepsilon\}, \quad \tau > 0;$$
- **(v)** *Let ε be fixed, $0 < \varepsilon \ll 1$ and H is a Hilbert space. For any $\varphi : [0,\infty) \to H$ continuous on $[0,\infty)$ such that $|\varphi(t)| \leq M\exp\{Ct\}$, $t \geq 0$, the relation*
$$\lim_{t\to 0}\int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau = \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau,$$
is valid in H;
- **(vi)** $\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad t \geq 0; \quad K(t,\tau,\varepsilon) > 0, \quad t \geq 0, \quad \tau \geq 0;$
- **(vii)** *Let $f \in W^{1,\infty}(0,\infty;H)$. Then there exists a positive constant C such that*
$$\Big|\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big|\Big|_H \leq C\sqrt{\varepsilon}(1+\sqrt{t})\|f'\|_{L^\infty(0,\infty;H)}, \quad t \geq 0;$$
- **(viii)** *There exists $C > 0$ such that*
$$\int_0^t\int_0^\infty K(\tau,\theta,\varepsilon)\exp\Big\{-\frac{\theta}{\varepsilon}\Big\}d\theta d\tau \leq C\varepsilon, \quad t \geq 0, \quad \varepsilon > 0.$$

Denote by $\mathcal{K}(t,\tau,\varepsilon) = K(t,\tau,\varepsilon)S(t,\varepsilon)$.

Theorem 1. *Suppose that A_1 satisfies condition (2). If $f \in L^\infty(0,\infty;H)$ and $u_\varepsilon \in W^{2,\infty}(0,\infty;H) \cap L^\infty(0,\infty;V)$, is the solution to the problem (P_ε), then the function $v_{0\varepsilon}$ which is defined as*

$$v_{0\varepsilon}(t) = \int_0^\infty \mathcal{K}(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau$$

is the solution to the problem:

$$\begin{cases} v_{0\varepsilon}'(t) + A_0 v_{0\varepsilon}(t) = f_0(t,\varepsilon), \quad t > 0, \\ v_{0\varepsilon}(0) = \varphi_\varepsilon, \end{cases}$$

where

$$f_0(t,\varepsilon) = F_0(t,\varepsilon) + \int_0^\infty \mathcal{K}(t,\tau,\varepsilon) f(\tau) d\tau,$$

$$F_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big] S(t,\varepsilon) u_1,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau} u_\varepsilon(2\varepsilon\tau) d\tau.$$

Moreover, $v_{0\varepsilon} \in W^{2,\infty}(0,\infty;H) \cap L^\infty(0,\infty;V)$.

Proof. Integrating by parts,using the properties of C_0- semigroups, (ii), (iii) from Lemma 1 and (2) we get:

$$v_{0\varepsilon}'(t) = \Big(\int_0^\infty \mathcal{K}(t,\tau,\varepsilon) u_\varepsilon(\tau) d\tau\Big)' = \int_0^\infty K_t(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau =$$

$$= \int_0^\infty [\varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon)] S(t,\varepsilon) u_\varepsilon(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau = \varepsilon K_\tau(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon(\tau)\big|_0^\infty -$$

$$- \int_0^\infty \varepsilon K_\tau(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau) d\tau - K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon(\tau)\big|_0^\infty +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau = [\varepsilon K_\tau(t,\tau,\varepsilon) - K(t,\tau,\varepsilon)] S(t,\varepsilon) u_\varepsilon(\tau)\big|_0^\infty -$$

$$-\varepsilon K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau)\big|_0^\infty + \int_0^\infty \varepsilon K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon''(\tau) d\tau +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S(t,\varepsilon) u_\varepsilon'(\tau) d\tau + \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau =$$

$$= [\varepsilon K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon)] S(t,\varepsilon) u_\varepsilon(0) + \varepsilon K(t,0,\varepsilon) S(t,\varepsilon) u_1 +$$

$$+ \int_0^\infty K(t,\tau,\varepsilon) S(t,\varepsilon)(\varepsilon u_\varepsilon''(\tau) + u'(\tau)) d\tau + \int_0^\infty K(t,\tau,\varepsilon) S'(t,\varepsilon) u_\varepsilon(\tau) d\tau =$$

$$= \varepsilon K(t,0,\varepsilon) S(t,\varepsilon) u_1 +$$

$$+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)(f(\tau)-A_0u_\varepsilon(\tau)-\varepsilon A_1u_\varepsilon(\tau))d\tau+$$

$$+\int_0^\infty K(t,\tau,\varepsilon)S'(t,\varepsilon)u_\varepsilon(\tau)d\tau=$$

$$=\varepsilon K(t,0,\varepsilon)S(t,\varepsilon)u_1+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)f(\tau)d\tau-A_0v_{0\varepsilon}(t)+$$

$$+\int_0^\infty K(t,\tau,\varepsilon)[S'(t,\varepsilon)u_\varepsilon(\tau)-\varepsilon A_1S(t,\varepsilon)u_\varepsilon(\tau)]d\tau=$$

$$=\varepsilon K(t,0,\varepsilon)S(t,\varepsilon)u_1+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)f(\tau)d\tau-A_0v_{0\varepsilon}(t)=$$

$$=F_0(t,\varepsilon)+\int_0^\infty K(t,\tau,\varepsilon)S(t,\varepsilon)f(\tau)d\tau-A_0v_{0\varepsilon}(t).$$

Thus $v_{0\varepsilon}(t)$ satisfies the equation from Theorem 1.

The initial condition is a simple consequence of property (iii) from Lemma 1.

Theorem 1 is proved.

4 The limit of solutions to the problem (P_ε) as $\varepsilon \mapsto 0$

In this section we will study the behavior of solutions to the problem (P_ε) as $\varepsilon \mapsto 0$.

Lemma 2. *Let A_0 and A_1 satisfy the conditions (1) and (2). If $u_0 \in V, u_1, f \in W^{1,\infty}(0,T;H)$ then the estimate:*

$$|S(t,\varepsilon)u_\varepsilon(t)-v_{0\varepsilon}(t)|\le C(T,u_0,u_1,f,\gamma,\gamma_1)\sqrt{\varepsilon},\quad t\in[0,T],$$

is true.

Proof. According to the C_0-semigroup theory there exists a constant $\gamma_1>0$ such that

$$|S(t,\varepsilon)|\le\gamma_1(T,\varepsilon). \tag{4}$$

Using the last mentioned property of $S(t,\varepsilon)$, Theorem B and the **property**(**vii**) of Lemma 1 we can easy obtain:

$$|S(t,\varepsilon)u_\varepsilon(t)-\int_0^\infty \mathcal{K}(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau|\le|S(t,\varepsilon)||u_\varepsilon(t)-\int_0^\infty K(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau|\le$$

$$\le\gamma_1|u_\varepsilon(t)-\int_0^\infty K(t,\tau,\varepsilon)u_\varepsilon(\tau)d\tau|\le\widetilde{c}(1+\sqrt{t})\parallel f'\parallel_{L^\infty(0,T:H)}\le C(T,u_0,u_1,f,\gamma,\gamma_1).$$

Lemma 2 is proved.

To prove the following result we need to remember an important inequality:

Lemma A [4]. *Let* $\psi \in L^1(a,b)(-\infty < a < b < \infty)$ *with* $\psi \geq 0$ *a. e. on (a,b) and* c *be a fixed real constant. If* $h \in C[a,b]$ *verifies*

$$\frac{1}{2}h^2(t) \leq \frac{1}{2}c^2 + \int_a^t \psi(s)h(s)ds, \forall t \in [a,b],$$

then

$$h(t) \leq |c| + \int_a^t \psi(s)ds, \forall t \in [a,b]$$

also holds.

Lemma 3. *Let the operators* A_0, A_1 *satisfy conditions (1)- (3).*
If $u_0, A_1u_0 \in V, u_1 \in H, f, A_1 f \in W^{1,\infty}(0,T;H)$ *then the estimate*:

$$|S(t,\varepsilon)v(t) - v_{0\varepsilon}(t)| \leq C(T, u_0, u_1, f, \gamma, \gamma_1,)\sqrt{\varepsilon}, \quad t \in [0,T]$$

is true.

Proof. Let $v(t)$ be the solution to the problem (P_0). We will denote by $w(t) = S(t,\varepsilon)v(t)$. Thus

$$w'(t) = S'(t,\varepsilon)v(t) + S(t,\varepsilon)v'(t) = \varepsilon A_1 S(t,\varepsilon)v(t)+$$

$$+S(t,\varepsilon)v'(t) = \varepsilon A_1 w(t) + S(t,\varepsilon)[f(t) - A_0 v(t)] =$$

$$= \varepsilon A_1 w(t) + S(t,\varepsilon)f(t) - A_0 S(t,\varepsilon)v(t) = \varepsilon A_1 w(t) + S(t,\varepsilon)f(t) - A_0 w(t),$$

and
$w(0) = S(0,\varepsilon)v(0) = v(0) = u_0$.

So we obtained the following Cauchy problem for $w(t)$:

$$\begin{cases} w'(t) + (A_0 - \varepsilon A_1)w(t) = S(t,\varepsilon)f(t), \\ w(0) = u_0. \end{cases}$$

To estimate $|S(t,\varepsilon)v(t) - v_{0\varepsilon}(t)|$ we denote by $R_\varepsilon(t)$=$w(t) - v_{0\varepsilon}(t)$. Then for $R_\varepsilon(t)$ we get the following Cauchy problem:

$$\begin{cases} R'_\varepsilon(t) + A_0 R_\varepsilon(t) = \varepsilon A_1 w(t) + S(t,\varepsilon)f(t) - f_0(t), \quad t > 0 \\ R_\varepsilon(0) = u_0 - \varphi_\varepsilon \end{cases}$$

Then taking scalar product of last equation with $R_\varepsilon(t)$ and integrating on [0,t], by Lemma A we get:

$$|R_\varepsilon(t)| \leq C(T)\Big[|u_0 - \varphi_\varepsilon| + 1/C(T)\int_0^t \Big|\varepsilon A_1 w(\tau) + S(\tau,\varepsilon)f(\tau) - f_0(\tau)\Big|d\tau\Big] \leq$$

$$\leq C(T)\Big[|u_0 - \varphi_\varepsilon| + 1/C(T)\int_0^t |\varepsilon A_1 w(\tau)|d\tau + 1/C(T)\int_0^t |F_0(\tau,\varepsilon)|d\tau +$$

$$+1/C(T)\int_0^t \Big|S(\tau,\varepsilon)f(\tau)-\int_0^\infty K(\tau,\mu,\varepsilon)S(\tau,\varepsilon)f(\mu)d\mu\Big|d\tau\Big],\quad 0\le t\le T. \tag{5}$$

Now step by step we will estimate all terms in the right of inequality (5).

In what follows we will denote by C all constants depending on $T, u_0, u_1, f, \gamma, \gamma_1$.

In conditions of Theorem B we can estimate the difference

$$|u_0-\varphi_\varepsilon| = \Big|\int_0^\infty e^{-\tau}(u_\varepsilon(2\varepsilon\tau)-u_0)d\tau\Big| \le \int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}|u'_\varepsilon(\mu)|d\mu \le C\varepsilon. \tag{6}$$

Using the property (vii) from Lemma 1 we have

$$\Big|S(\tau,\varepsilon)f(\tau)-\int_0^\infty K(\tau,\mu,\varepsilon)S(\tau,\varepsilon)f(\mu)d\mu\Big| \le$$

$$\le |S(\tau,\varepsilon)|\Big|f(\tau)-\int_0^\infty K(\tau,\mu,\varepsilon)f(\mu)d\mu\Big| \le$$

$$\le \gamma_1\sqrt{\varepsilon}(1+\sqrt{t})\|f'\|_{L^\infty(0,\infty;H}=C\sqrt{\varepsilon}. \tag{7}$$

In [2] it is also shown that

$$\int_0^t e^{\gamma\tau}|F_0(\tau,\varepsilon)|d\tau \le \widetilde{C}\varepsilon|u_1| \le C\varepsilon. \tag{8}$$

To estimate $|A_1 w(t)|$ we will consider now the (P_0) problem and will apply to it the operator A_1 to obtain:

$$\begin{cases} A_1v'(t)+A_1A_0v(t)=A_1f(t), t>0 \\ A_1v(0)=A_1u_0. \end{cases} \tag{9}$$

In condition (2) we can observe that

$$\varepsilon A_1A_0v(t)=\lim_{h\mapsto 0}\frac{S(h,\varepsilon)A_0v(t)-A_0v(t)}{h}=\lim_{h\mapsto 0}\frac{A_0S(h,\varepsilon)v(t)-A_0v(t)}{h}=$$

$$=\lim_{h\mapsto 0}A_0\frac{S(h,\varepsilon)v(t)-v(t)}{h}=\varepsilon A_0A_1v(t)$$

Thus, denoting by $y(t)=A_1v(t)$ we can write the problem for y

$$\begin{cases} y'(t)+A_0y(t)=A_1f(t), t>0 \\ y(0)=A_1u_0. \end{cases}$$

If $A_1u_0\in V, A_1f\in W^{1,1}(0,T,H)$, then by Theorem B we obtain the estimate

$$|y(t)|\le C(T,u_0,f,\gamma).$$

But

$$|A_1w(t)|=|A_1S(t,\varepsilon)v(t)|\le\gamma_1|A_1v(t)|=\gamma_1|y(t)|\le C. \tag{10}$$

From estimates (5)-(10) we finally obtain the estimate

$$|R_\varepsilon(t)| \leq \sqrt{\varepsilon} C(T, u_0, u_1, f, \gamma, \gamma_1).$$

Lemma 3 is proved.

Theorem 2. *Let $T > 0$. If $u_0, A_1 u_0 \in V, u_1 \in H, f, A_1 f \in W^{1,\infty}(0,T;H)$ and A_0, A_1 satisfies conditions (1)-(3) then the estimate*:

$$|u_\varepsilon(t) - v(t)| \leq C(u_0, u_1, f, \gamma, \gamma_1, \delta)\sqrt{\varepsilon}, \quad t \in [0,T], \quad 0 < \varepsilon \ll 1$$

is true.

The proof of this theorem is a simple consequence of Lemmas 1 and 2. Indeed

$$|u_\varepsilon(t) - v(t)| \leq \frac{1}{\delta} |S(t,\varepsilon) u_\varepsilon(t) - S(t,\varepsilon) v(t)| \leq$$

$$\leq \frac{1}{\delta}[|S(t,\varepsilon) u_\varepsilon(t) - v_{0\varepsilon}(t)| + |S(t,\varepsilon) v(t) - v_{0\varepsilon}(t)|] \leq \sqrt{\varepsilon} C(T, u_0, u_1, f, \gamma, \gamma_1, \delta).$$

Theorem 3. *Let $T > 0$. If*

$$u_0, A_0 u_0, A_1 u_0, A_1 A_0 u_0, u_1, f(0), A_1 f(0) \in V, \quad f, A_1 f \in W^{2,\infty}(0,T;H)$$

and A_0, A_1 satisfies conditions (1)-(3), then the estimate

$$|u'_\varepsilon(t) - v'(t) + h e^{-\frac{t}{\varepsilon}}| \leq \sqrt{\varepsilon} C(u_0, u_1, f, \gamma, \gamma_1, \delta),$$

is true, where $h = f(0) - u_1 - A_0 u_0$.

Proof. Denote by $z_\varepsilon(t) = u'_\varepsilon(t) + h e^{-\frac{t}{\varepsilon}}$. Then for $z_\varepsilon(t)$ we get the following Cauchy problem:

$$\begin{cases} \varepsilon z''_\varepsilon(t) + z'_\varepsilon(t) + (A_0 + \varepsilon A_1) z_\varepsilon(t) = f'(t) + e^{-\frac{t}{\varepsilon}}(A_0 + \varepsilon A_1)h, \quad t > 0 \\ z(0) = f(0) - A_0 u_0, \quad z'(0) = -A_1 u_0. \end{cases} \tag{11}$$

As $A_0 u_0, f(0) \in V, \quad f \in W^{2,\infty}(0,T;H)$, according to Theorem 1 the function

$$w_{1\varepsilon}(t) = \int_0^\infty \mathcal{K}(t,\tau,\varepsilon) z_\varepsilon(\tau) d\tau$$

is the solution to the problem:

$$\begin{cases} w'_{1\varepsilon}(t) + A_0 w_{1\varepsilon}(t) = F_1(t,\varepsilon), \quad t > 0, \\ w_{1\varepsilon}(0) = \int_0^\infty e^{-\tau} z_\varepsilon(2\varepsilon\tau) d\tau, \end{cases} \tag{12}$$

where

$$F_1(t,\varepsilon) = \int_0^\infty \mathcal{K}(t,\tau,\varepsilon)[f'(\tau) d\tau + e^{-\frac{\tau}{\varepsilon}}(A_0 + \varepsilon A_1)h] d\tau -$$

$$-\frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big]S(t,\varepsilon)A_1u_0.$$

Denoting by $v_1(t)=v'(t),(P_0)$, for v_1 we have the problem (Pv_1):

$$\begin{cases} v_1'(t)+A_0v_1(t)=f'(t), \quad t>0,\\ v_1(0)=f(0)-A_0u_0.\end{cases}$$

If $w_{2\varepsilon}(t)=S(t,\varepsilon)v_1(t)$, then (Pv_1) becomes

$$\begin{cases} w_{2\varepsilon}'(t)+[A_0-\varepsilon A_1]w_{2\varepsilon}(t)=S(t,\varepsilon)f'(t), \quad t>0,\\ w_{2\varepsilon}(0)=f(0)-A_0u_0.\end{cases} \tag{13}$$

Let $R_{1\varepsilon}(t)=w_{1\varepsilon}(t)-w_{2\varepsilon}(t)$. Then, using (12) and (13) we get the following Cauchy problem for it:

$$\begin{cases} R_{1\varepsilon}'(t)+A_0R_{1\varepsilon}(t)=F_1(t,\varepsilon)-S(t,\varepsilon)f'(t)-\varepsilon A_1w_{2\varepsilon}(t), \quad t>0,\\ R_{1\varepsilon}(0)=\int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}z_\varepsilon'(\theta)d\theta d\tau.\end{cases} \tag{14}$$

Taking scalar product of (14) with $R_{1\varepsilon}(t)$, integrating on $[0,t]$ and using Lemma A we get the estimate

$$|R_{1\varepsilon}(t)|\le e^{-\gamma t}\Big(|R_{1\varepsilon}(0)|+\int_0^t e^{\gamma\tau}\Big|F_1(\tau,\varepsilon)-S(\tau,\varepsilon)f'(\tau)-\varepsilon A_1w_{2\varepsilon}(\tau)\Big|d\tau\Big). \tag{15}$$

As we can see in (11) $z_\varepsilon(t)$ is the solution to a second order Cauchy problem which is similar to (P_ε). So, in conditions of this theorem, using Theorem B, the following estimate is true:

$$|z_\varepsilon(t)|\le C(|f|_{W^{2,\infty}(0,T;H)},|A_0u_0|,|A_1u_0|,\gamma)=C.$$

Then

$$|R_{1\varepsilon}(0)|=\Big|\int_0^\infty e^{-\tau}\int_0^{2\varepsilon\tau}z_\varepsilon'(\theta)d\theta d\tau\Big|\le C\varepsilon.$$

From properties (vii), (viii) and (4) it follows:

$$\int_0^t e^{\gamma\tau}\Big|F_1(\tau,\varepsilon)-S(\tau,\varepsilon)f'(\tau)\Big|d\tau\le\int_0^t e^{\gamma\tau}\Big|\int_0^\infty\mathcal{K}(\tau,\mu,\varepsilon)f'(\mu)d\mu-S(\tau,\varepsilon)f'(\tau)\Big|d\tau+$$

$$+\int_0^t\int_0^\infty\mathcal{K}(\tau,\mu,\varepsilon)e^{-\frac{\mu}{\varepsilon}}|(A_0+\varepsilon A_1)h|d\mu d\tau+$$

$$+\int_0^t\frac{1}{\sqrt{\pi}}\Big|2\exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\Big|\Big|S(\tau,\varepsilon)A_1u_0\Big|d\tau\le$$

$$\le C\gamma_1\Big[\sqrt{\varepsilon}(1+\sqrt{t})\|f''\|_{L^\infty(0,T;H)}+\varepsilon+\sqrt{\varepsilon}|A_0u_0|\Big]\le C\sqrt{\varepsilon}.$$

To estimate $|\varepsilon A_1 w_{2\varepsilon}(t)|$ we will apply A_1 to (Pv_1) and denote by $y_1(t) = A_1 v_1(t)$ to get

$$\begin{cases} y_1'(t) + A_0 y_1(t) = A_1 f'(t), & t > 0 \\ y_1(0) = A_1 f(0) + A_1 A_0 u_0. \end{cases}$$

As $A_1 A_0 u_0, A_1 f(0) \in V, A_1 f \in W^{2,\infty}(0,T;H)$, Theorem A implies the estimate

$$|y_1(t)| \leq C(T, \gamma, A_1 A_0 u_0, A_1 f).$$

Consequently,

$$|\varepsilon A_1 w_{2\varepsilon}(t)| = \varepsilon |A_1 S(t,\varepsilon) v_1(t)| \leq \varepsilon \gamma_1 |A_1 v_1(t)| = \varepsilon \gamma_1 |y_1(t)| \leq \varepsilon C.$$

Using the last three inequalities from (15) follows the estimate

$$|R_{1\varepsilon}(t)| \leq C\sqrt{\varepsilon}, \quad 0 \leq t \leq T. \tag{16}$$

From property (vii) from Lemma 1 and (4) it follows:

$$|S(t,\varepsilon) z_\varepsilon(t) - w_{1\varepsilon}(t)| = |S(t,\varepsilon) z_\varepsilon(t) - \int_0^\infty \mathcal{K}(t,\tau,\varepsilon) z_\varepsilon(\tau) d\tau| \leq$$

$$\leq \gamma_1 C(1 + \sqrt{t}) \parallel z' \parallel_{L^\infty(0,T:H)} \leq \sqrt{\varepsilon} C. \tag{17}$$

Finally, using condition (3) and estimates (16), (17) we get

$$|u_\varepsilon'(t) - v'(t) - he^{-\frac{t}{\varepsilon}}| = |z_\varepsilon(t) - v_1(t)| \leq \frac{1}{\delta} |S(t,\varepsilon) z_\varepsilon(t) - S(t,\varepsilon) v_1(t)| \leq$$

$$\leq \frac{1}{\delta} \Big[|S(t,\varepsilon) z_\varepsilon(t) - w_{1\varepsilon}(t)| + |w_{1\varepsilon}(t) - S(t,\varepsilon) v_1(t)| \Big] \leq \sqrt{\varepsilon} C(u_0, u_1, f, \gamma, \gamma_1, \delta).$$

Theorem 3 is proved.

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