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Singular limits of solutions to the Cauchy problem for second order linear differential equations in Hilbert spaces

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Abstract. We study the behavior of solutions to the problem

$$\begin{cases}\varepsilon\Big(u''_\varepsilon(t)+A_1u_\varepsilon(t)\Big)+u'_\varepsilon(t)+A_0u_\varepsilon(t)=f(t), \quad t>0,\\ u_\varepsilon(0)=u_0, \quad u'_\varepsilon(0)=u_1,\end{cases}$$

in the Hilbert space H as $\varepsilon\to 0$, where A_1 and A_0 are two linear selfadjoint operators.

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1 Introduction

Let H be a real Hilbert space endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$. Let $A_i : D(A_i)\to H$, $i=0,1$, be two linear self-adjoint, positive defined operators. Consider the following Cauchy problem:

$$\begin{cases}\varepsilon\left(u''_\varepsilon(t)+A_1u_\varepsilon(t)\right)+u'_\varepsilon(t)+A_0u_\varepsilon(t)=f_\varepsilon(t), \quad t\in(0,T),\\ u_\varepsilon(0)=u_{0\varepsilon}, \quad u'_\varepsilon(0)=u_{1\varepsilon},\end{cases} \qquad (P_\varepsilon)$$

where $\varepsilon>0$ is a small parameter($\varepsilon\ll 1$), $u_\varepsilon, f_\varepsilon : [0,T)\to H$.

We will investigate the behavior of solutions $u_\varepsilon(t)$ to the perturbed system (P_ε) when $\varepsilon\to 0$, $u_{0\varepsilon}\to u_0$ and $f_\varepsilon\to f$. We will establish a relationship between solutions to the problem (P_ε) and the corresponding solutions to the following unperturbed system:

$$\begin{cases}v'(t)+A_0v(t)=f(t), \quad t\in(0,T),\\ v(0)=u_0.\end{cases} \qquad (P_0)$$

In our study we will use the following conditions:

(H1) *The operator $A_0 : D(A_0)\subseteq H\to H$ is self-adjoint and positive defined, i.e. there exists $\omega_0>0$ such that*

$$(A_0u,u)\geq\omega_0\,|u|^2, \quad \forall u\in D(A_0);$$

(H2) *The operator $A_1 : D(A_1)\subseteq H\to H$ is self-adjoint, positive defined and there exists $\alpha>1$ such that*:

(i) $D(A_0^\alpha) \subseteq D(A_1)$;

(ii) $A_1\Big[D(A_0^{2\alpha-1})\Big] \subseteq D(A_0^{\alpha-1})$;

(iii) $A_1 A_0^{\alpha-1}u = A_0^{\alpha-1}A_1 u, \quad \forall u \in D(A_0^{2\alpha-1})$;

(iv) *there exists* $\omega_2 > 0$ *and* $\omega_3 > 0$ *such that*

$$\omega_2 \, |u|^2 \leq (A_1 u, u) \leq \omega_3 \, (A_0^\alpha u, u) \,, \quad \forall u \in D(A_0^{2\alpha-1}).$$

The definition and properties of operator A^α can be found in [2].

If, in some topology, $u_\varepsilon(t)$ tends to the corresponding solutions $v(t)$ of the unperturbed system (P_0) as $\varepsilon \to 0$, then the system (P_0) is called *regularly perturbed.* In the opposite case, the system (P_0) is called *singularly perturbed.* In the last case, a subset of $[0, \infty)$, in which the solution $u_\varepsilon(t)$ has a singular behavior relative to ε, arises. This subset is called *the boundary layer.* The function which defines the singular behavior of the solution $u_\varepsilon(t)$ within the boundary layer is called *the boundary layer function.*

Our approach is based on two key points. The first one is the relationship between the solutions to the problems (P_ε) and (P_0). The second key point consists in obtaining a priori estimates for the solutions to the problems (P_ε), estimates which are uniform with respect to the small parameter ε.

In what follows we will need some notations. Let $k \in \mathbb{N}^\star$, $1 \leq p \leq +\infty$, $(a, b) \subset (-\infty, +\infty)$ and let X be a Banach space. We denote by $W^{k,p}(a, b; X)$ the Banach space of all vectorial distributions $u \in D'(a, b; X)$, $u^{(j)} \in L^p(a, b; X)$, $j = 0, 1, \ldots, k$, endowed with the norm

$$\|u\|_{W^{k,p}(a,b;X)} = \left(\sum_{j=0}^{k} \|u^{(j)}\|^p_{L^p(a,b;X)} \right)^{1/p}$$

for $p \in [1, \infty)$ and

$$\|u\|_{W^{k,\infty}(a,b;X)} = \max_{0 \leq j \leq k} \|u^{(j)}\|_{L^\infty(a,b;X)}$$

for $p = \infty$.

In the particular case $p = 2$, we denote $W^{k,2}(a, b; X) = H^k(a, b; X)$. If X is a Hilbert space, then $H^k(a, b; X)$ is also a Hilbert space with the inner product

$$(u, v)_{H^k(a,b;X)} = \sum_{j=0}^{k} \int_a^b \Big(u^{(j)}(t), v^{(j)}(t) \Big)_X \, dt.$$

For each fixed $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1, \infty]$, we define the Banach space

$$W_s^{k,p}(a, b; H) = \{ f : (a, b) \to H; f^{(l)}(\cdot) e^{-st} \in L^p(a, b; X),\; l = 0, \ldots, k \},$$

with the norm

$$\|f\|_{W_s^{k,p}(a,b;X)} = \|f e^{-st}\|_{W^{k,p}(a,b;X)}.$$

2 Existence of strong solutions to both (P_ε) and (P_0)

Theorem 1. [1] *Let $T > 0$ and let us assume that A_0 satisfies the condition* (**H1**)*. If $u_0 \in D(A_0)$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique strong solution $v \in W^{1,\infty}(0,T;H)$ to the problem (P_0). Moreover, v satisfies*

$$|v(t)| + \left(\int_0^t \left|A_0^{1/2} u(s)\right| ds\right)^{1/2} \leq |u_0| + \int_0^t |f(s)|\, ds, \quad \forall t \in [0,T],$$

$$\left|v'(t)\right| \leq |A_0 u_0 - f(0)| + \int_0^t \left|f'(s)\right| ds, \quad \forall t \in [0,T].$$

Theorem 2. [1] *Let $T > 0$. Let us assume that $A : D(A) \subset H \to H$ is linear self-adjoint and positive defined. If $u_0 \in D(A)$, $u_1 \in H$ and $f \in W^{1,1}(0,T;H)$, then there exists a unique function $u : [0,T] \to H$ such that: $u \in W^{2,\infty}(0,T;H)$, $A^{1/2}u \in W^{1,\infty}(0,T;H)$, $Au \in L^\infty(0,T;H)$, $A^{1/2}u$ and u' are differentiable on the right in H for every $t \in [0,T)$ and*

$$\frac{d^+}{dt}\frac{du}{dt}(t) + \frac{du}{dt}(t) + Au(t) = f(t), \quad t \in [0,T), \tag{1}$$

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2}$$

In what follows this function will be called *the strong solution* to the problem (1), (2).

3 A priori estimates for solutions to the problem (P_ε)

Consider the following problem:

$$\begin{cases} \varepsilon\left(u_\varepsilon''(t) + A_1 u_\varepsilon(t)\right) + u_\varepsilon'(t) + A_0 u_\varepsilon(t) = f(t), \quad t \in (0,T), \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(0) = u_1. \end{cases} \tag{3}$$

Lemma 1. [4] *Let $T > 0$. Suppose that, for each $\varepsilon \in (0,1)$, the operator $A(\varepsilon) = (\varepsilon A_1 + A_0) : D\left(A(\varepsilon)\right) \subseteq H \to H$ is self-adjoint and satisfies*

$$(A(\varepsilon)u, u) \geq \omega\, |u|^2, \quad \forall u \in D\left(A(\varepsilon)\right), \quad \omega > 0, \quad \varepsilon \in (0,1]. \tag{4}$$

If $f \in W^{1,1}(0,T;H), u_0 \in D\left(A(\varepsilon)\right)$, $u_1 \in H$, then the unique strong solution, u_ε, of the problem (3) *satisfies*

$$\|A^{1/2}(\varepsilon)u_\varepsilon\|_{C([0,\,t];\,H)} + \|u_\varepsilon'\|_{L^2(0,\,t;\,H)} \leq C(\omega)\, M(t), \tag{5}$$

for each $t \in [0,T]$ and each $\varepsilon \in (0,1/2]$. If, in addition, $u_1 \in D\left(A^{1/2}(\varepsilon)\right)$, then

$$\|u_\varepsilon'\|_{C([0,\,t];\,H)} + \|A^{1/2}(\varepsilon)\, u_\varepsilon'\|_{L^2(0,\,t;\,H)} \leq C(\omega)\, M_1(t), \tag{6}$$

for each $t \in [0,T]$, *and each* $\varepsilon \in (0,1]$, *and*

$$\|A(\varepsilon)u_\varepsilon\|_{L^\infty(0,\,t;\,H)} \le C(\omega)M_1(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1], \tag{7}$$

where $C(\omega)$ *is a constant depending on* ω,

$$M(t) = M(t, u_0, u_1, f) = \left|A^{1/2}(\varepsilon)u_0\right| + |u_1| + \|f\|_{W^{1,1}(0,t;\,H)} + |f(0)|$$

and

$$M_1(t) = M_1(t, u_0, u_1, f) = \left|A^{1/2}(\varepsilon)u_1\right| + |A(\varepsilon)u_0| + \|f\|_{W^{1,1}(0,t;\,H)} + |f(0)|.$$

Let u_ε be a strong solution of the problem (3) and let us denote by

$$z_\varepsilon(t) = u'_\varepsilon(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = f(0) - u_1 - A(\varepsilon)u_0. \tag{8}$$

Lemma 2. [4] *Let* $T > 0$ *and let us assume that, for each* $\varepsilon \in (0,1)$, *the operator* $A(\varepsilon) = \varepsilon A_1 + A_0$ *is self-adjoint and satisfies* (4). *If* u_1, $f(0) - A(\varepsilon)u_0 \in D(A(\varepsilon))$ *and* $f \in W^{2,1}(0,T;H)$, *then there exists* $C(\omega) > 0$, *such that the function* z_ε, *defined by* (8), *satisfies*

$$\|A^{1/2}(\varepsilon)z_\varepsilon\|_{C([0,\,t];\,H)} + \|z'_\varepsilon\|_{C([0,\,t];\,H)} + \left\|A^{1/2}(\varepsilon)z'_\varepsilon\right\|_{L^2(0,\,t;\,H)}$$
$$\le C(\omega)\, M_2(t), \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,1], \tag{9}$$

where

$$M_2(t) = |A(\varepsilon)f(0) - A^2(\varepsilon)u_0| + \|f\|_{W^{2,1}(0,t;H)} + |A(\varepsilon)u_1| + |f'(0)|.$$

4 The relationship between the solution to (P_ε) and (P_0)

Now we are going to establish the relationship between the solution to the problem (P_ε) and the corresponding solution to the problem (P_0). To this end, we begin by defining the transformation kernel which realizes this relationship.

Namely, for $\varepsilon > 0$, let us denote

$$K(t,\tau,\varepsilon) = \frac{1}{2\,\varepsilon\,\sqrt{\pi}}\left(K_1(t,\tau,\varepsilon) + 3K_2(t,\tau,\varepsilon) - 2K_3(t,\tau,\varepsilon)\right),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\left\{\frac{3t - 2\tau}{4\varepsilon}\right\}\lambda\left(\frac{2t-\tau}{2\sqrt{\varepsilon t}}\right),$$

$$K_2(t,\tau,\varepsilon) = \exp\left\{\frac{3t + 6\tau}{4\varepsilon}\right\}\lambda\left(\frac{2t+\tau}{2\sqrt{\varepsilon t}}\right),$$

$$K_3(t,\tau,\varepsilon) = \exp\left\{\frac{\tau}{\varepsilon}\right\}\lambda\left(\frac{t+\tau}{2\sqrt{\varepsilon t}}\right), \quad \lambda(s) = \int_s^\infty e^{-\eta^2}\,d\eta.$$

Lemma 3. [3]. *The function $K \in C([0,\infty)\times[0,\infty)) \cap C^2((0,\infty)\times(0,\infty))$ has the following properties:*

(*i*) $K(t,\tau,\varepsilon) > 0, \quad \forall t \geq 0, \quad \forall \tau \geq 0\,;$

(*ii*) *For every continuous $\varphi : [0,\infty) \to H$, with $|\varphi(t)| \leq M \exp\{\gamma\, t\}$, we have:*

$$\lim_{t\to 0} \left\| \int_0^\infty K(t,\tau,\varepsilon)\varphi(\tau)d\tau - \int_0^\infty e^{-\tau}\varphi(2\varepsilon\tau)d\tau \right\|_H = 0,$$

for every $\varepsilon \in \left(0, (2\,\gamma)^{-1}\right)$;

(*iii*)

$$\int_0^\infty K(t,\tau,\varepsilon)d\tau = 1, \quad \forall t \geq 0.$$

(*iv*) *For every $q \in [0,1]$, there exists $C > 0$ and $\varepsilon_0 > 0$, depending on q, such that:*

$$\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^q\, d\tau \leq C\,\varepsilon^{q/2}\,\left(1+\sqrt{t}\right)^q,\ \forall t \geq 0, \forall \varepsilon \in (0,1]\,;$$

(*v*) *Let $p \in (1,\infty]$ and $f : [0,\infty) \to H$, $f \in W^{1,p}(0,\infty;H)$. There exist $C > 0$, and ε_0 depending on p, such that*

$$\left\| f(t) - \int_0^\infty K(t,\tau,\varepsilon) f(\tau)d\tau \right\|_H$$

$$\leq C\,\|f'\|_{L^p(0,\infty;H)}\,\left(1+\sqrt{t}\right)^{\frac{p-1}{p}}\ \varepsilon^{(p-1)/2p},\ \forall t \geq 0,\ \forall \varepsilon \in (0,1].$$

(*vi*) *For every $q > 0$ and $\alpha \geq 0$, there exists $C(q,\alpha) > 0$ such that*

$$\int_0^t \int_0^\infty K(\tau,\theta,\varepsilon)\, e^{-q\,\theta/\varepsilon}\,|\tau-\theta|^\alpha\, d\theta\, d\tau \leq C(q,\alpha)\,\varepsilon^{1+\alpha},$$

for each $t \geq 0$, and each $\varepsilon > 0$.

Theorem 3. [4] *Suppose that $A(\varepsilon)$ satisfies* **(H1)***, let $f \in L^\infty_c(0,\infty;H)$ and let $u_\varepsilon \in W^{2,\infty}_c(0,\infty;H)$ be the strong solution to the problem* (3)*, with $Au_\varepsilon \in L^\infty_c(0,\infty;H)$, for some $c \geq 0$. Then the function w_ε, defined by*

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,u_\varepsilon(\tau)\,d\tau,$$

is the strong solution to the problem

$$\begin{cases} w'_\varepsilon(t) + A(\varepsilon)w_\varepsilon(t) = F_0(t,\varepsilon), & t > 0, \\ w_\varepsilon(0) = \varphi_\varepsilon, \end{cases} \tag{10}$$

where

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}u_\varepsilon(2\varepsilon\tau)d\tau, \quad F_0(t,\varepsilon) = f_0(t,\varepsilon)u_1 + \int_0^\infty K(t,\tau,\varepsilon)\,f(\tau)\,d\tau,$$

$$f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\left[2\exp\left\{\frac{3t}{4\varepsilon}\right\}\lambda\left(\sqrt{\frac{t}{\varepsilon}}\right) - \lambda\left(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\right)\right].$$

5 The limit of the solutions to the problem (P_ε) as $\varepsilon \to 0$

In this section we will study the behavior of the solutions to the problem (P_ε) as $\varepsilon \to 0$.

Theorem 4. *Let $T > 0$ and $p \in (1, \infty]$. Suppose that the operators A_0 and A_1 satisfy conditions* **(H1)** *and* **(H2)**. *If*

$$u_0 \in D(A_0), u_{0\,\varepsilon} \in D(A_0^{2\alpha-1}), \quad u_{1\,\varepsilon} \in D(A_0^{\alpha-1}), f, A_0^{\alpha-1} f_\varepsilon \in W^{1,p}(0,T;H),$$

then there exist constants $\varepsilon_0 = \varepsilon_0(\omega_0) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_2, \omega_3, \alpha) > 0$ such that

$$||u_\varepsilon - v||_{C([0,T];H)} \leq C \left(\mathcal{M}_{3\varepsilon}\, \varepsilon^\beta + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)} \right), \tag{11}$$

for all $\varepsilon \in (0, \varepsilon_0]$, where u_ε and v are the strong solutions to problems (P_ε) and (P_0) respectively,
$\beta = \min\{1/4, (p-1)/2p\}$ *and*

$$\mathcal{M}_{3\varepsilon} = \big|A_0^{(3\alpha-2)/2} u_{0\,\varepsilon}\big| + |A_0^{\alpha-1} u_{1\,\varepsilon}| + ||A_0^{\alpha-1} f_\varepsilon||_{W^{1,p}(0,T;H)},$$

If in addition, $u_{1\,\varepsilon} \in D\big(A_0^{\alpha/2}\big)$, then

$$||u_\varepsilon - v||_{C([0,T];H)} \leq C \left(\mathcal{M}_{4\,\varepsilon}\, \varepsilon^{(p-1)/2p} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)} \right), \tag{12}$$

$\varepsilon \in (0, \varepsilon_0]$ and

$$||A_0^{1/2} u_\varepsilon - A_0^{1/2} v||_{L^2(0,\,T;\,H)}$$
$$\leq C \left(\mathcal{M}_{4\,\varepsilon}\, \varepsilon^\beta + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)} \right), \tag{13}$$

$\varepsilon \in (0, \varepsilon_0]$ where
$\beta = \min\{1/4, (p-1)/2p\}$ and

$$\mathcal{M}_{4\,\varepsilon} = \big|A_0^{(3\alpha-2)/2} u_{0\,\varepsilon}\big| + |A_0 u_{0\,\varepsilon}| + |A_1 u_{0\,\varepsilon}| + \big|A_0^{\alpha/2} u_{1\,\varepsilon}\big|$$
$$+ \big|A_0^{\alpha-1} u_{1\,\varepsilon}\big| + ||A_0^{\alpha-1} f_\varepsilon||_{W^{1,p}(0,\,T;\,H)}.$$

Proof. During the proof of this theorem, we will agree to denote all constants $\varepsilon_0(\omega_0)$ and $C = C(T, p, \omega_0, \omega_2, \omega_3, \alpha) > 0$ be ε_0 and C respectively.

Using **(H1)** and the properties of A_0^α proved in [2], we can state that there exists a constant $C(\omega_0, \alpha)$ such that:

$$\left(A_0^\alpha u, u \right) \geq C(\omega_0, \alpha) \parallel u \parallel^2, \quad u \in D(A_0^\alpha). \tag{14}$$

Using **(H1)**, **(H2)** and (14), since $u_{0\varepsilon} \in D(A_0^{2\alpha-1})$, $u_{1\varepsilon} \in D(A_0^{\alpha-1})$

$$(A_0^{3\alpha-2}u_{0\varepsilon}, u_{0\varepsilon}) = (A_0^{2\alpha-2}A_0^{\alpha/2}u_{0\varepsilon}, A_0^{\alpha/2}u_{0\varepsilon}) \geq C(\omega_0,\alpha)(A_0^{\alpha}u_{0\varepsilon}, u_{0\varepsilon})$$

$$= C(\omega_0,\alpha)(A_0^{\alpha-1}A_0^{1/2}u_{0\varepsilon}, A_0^{1/2}u_{0\varepsilon}) \geq C^2(\omega_0,\alpha)(A_0u_{0\varepsilon}, u_{0\varepsilon});$$
$$(A_0^{\alpha-1}u_{1\varepsilon}, A_0^{\alpha-1}u_{1\varepsilon}) = (A_0^{\alpha-1}A_0^{(\alpha-1)/2}u_{1\varepsilon}, A_0^{(\alpha-1)/2}u_{1\varepsilon})$$
$$\geq C(\omega_0,\alpha)(A_0^{\alpha-1}u_{1\varepsilon}, u_{1\varepsilon}) \geq C^2(\omega_0,\alpha)(u_{1\varepsilon}, u_{1\varepsilon}). \tag{15}$$

Let us also observe that, for $\alpha > 1$, we have $D(A_0^{2\alpha-1}) \subset D(A_0^{\alpha})$. Thus, from (**H2**), we get

$$(\lambda I + A_0^{\alpha-1})A_1u = A_1(\lambda I + A_0^{\alpha-1})u, \quad u \in D(A_0^{2\alpha-1}), \quad \lambda \geq 0,$$

which implies

$$(\lambda I + A_0^{\alpha-1})^{-1}A_1^{-1}u = A_1^{-1}(\lambda I + A_0^{\alpha-1})^{-1}u, \quad \forall u \in D(A_0^{2\alpha-1}), \quad \forall \lambda \geq 0.$$

Since A_1^{-1} is bounded and commutes with the resolvent of $A_0^{\alpha-1}$, we can state that

$$\left[A_0^{\alpha-1}\right]^{1/2}A_1^{-1}u = A_1^{-1}\left[A_0^{\alpha-1}\right]^{1/2}u, \quad \forall u \in D(A_0^{\alpha-1}).$$

So, if $u \in D(A_0^{\alpha-1})$, then $A_1^{-1}\left[A_0^{\alpha-1}\right]^{1/2}u \in D(A_1)$. Thus

$$A_1\left[A_0^{(\alpha-1)/2}A_1^{-1}\right]u = A_0^{(\alpha-1)/2}u, \quad \forall u \in D(A_0^{\alpha-1}).$$

Taking $u \in D(A_0^{2\alpha-1})$, from (ii) of (**H2**), we get $A_1u \in D(A_0^{\alpha-1})$, which finally implies

$$A_1A_0^{(\alpha-1)/2}u = A_0^{(\alpha-1)/2}A_1u, \quad \forall u \in D(A_0^{2\alpha-1}).$$

Using (iv) of (**H2**) and the last inequality, we get

$$|(A_1u,v)| = |(A_0^{(\alpha-1)/2}A_1u, A_0^{-(\alpha-1)/2}v)| = |(A_1A_0^{(\alpha-1)/2}u, A_0^{-(\alpha-1)/2}v)|$$

$$\leq \sqrt{(A_1A_0^{(\alpha-1)/2}u, A_0^{(\alpha-1)/2}u)(A_1A_0^{-(\alpha-1)/2}v, A_0^{-(\alpha-1)/2}v)}$$
$$\leq \omega_3\sqrt{(A_0^{\alpha}A_0^{(\alpha-1)/2}u, A_0^{(\alpha-1)/2}u)(A_0^{\alpha}A_0^{-(\alpha-1)/2}v, A_0^{-(\alpha-1)/2}v)}$$
$$= \omega_3|A_0^{\alpha-1/2}u||A_0^{1/2}v|, \quad \forall u,v \in D(A_0^{2\alpha-1}). \tag{16}$$

If $f_\varepsilon \in W^{l,p}(0,T;H)$ with $p \in (1,\infty]$ and $l \in \mathbb{N}^\star$, we have that $f_\varepsilon \in C([0,T];H)$ and there exists an extension $\tilde{f}_\varepsilon \in W^{l,p}(0,\infty;H)$ such that

$$||\tilde{f}_\varepsilon||_{C([0,\infty);H)} + ||\tilde{f}_\varepsilon||_{W^{l,p}(0,\infty;H)} \le C(T,p,l)\,||f_\varepsilon||_{W^{l,p}(0,T;H)}. \tag{17}$$

Let us denote by $\tilde{u}_\varepsilon$ the unique strong solution to the problem (P_ε) and by $\tilde{v}$ the unique strong solution to the problem (P_0), substituting $(0,T)$ by $(0,\infty)$ and f_ε by $\tilde{f}_\varepsilon$. From Theorem 2, we have

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,T;H),\ A^{1/2}(\varepsilon)\tilde{u}_\varepsilon \in W^{1,\infty}(0,T;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,T;H),\ \forall T \in (0,\infty). \end{cases}$$

From Lemma 1 and (15), it follows that

$$\begin{cases} \tilde{u}_\varepsilon \in W^{2,\infty}(0,\infty;H), \quad A_0^{1/2}\tilde{u}_\varepsilon \in W^{1,2}(0,\infty;H), \\ A(\varepsilon)\tilde{u}_\varepsilon \in L^\infty(0,\infty;H). \end{cases}$$

Moreover, due to this lemma and inequalities (15) and (17), we get

$$||A_0^{1/2}\tilde{u}_\varepsilon||_{C([0,t];\,H)} + ||\tilde{u}'_\varepsilon||_{L^2(0,t;\,H)} \le C\,\mathcal{M}_{3\varepsilon}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{18}$$

If, in addition, $u_{1\,\varepsilon} \in D\big(A_0^{\alpha/2}\big)$, then

$$||\tilde{u}'_\varepsilon||_{C([0,t];\,H)} + ||A_0^{1/2}\,\tilde{u}'_\varepsilon||_{L^2(0,t;\,H)} \le C\,\mathcal{M}_{4\,\varepsilon}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{19}$$

Proof of (11). According to Theorem 3, the function

$$w_\varepsilon(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau, \tag{20}$$

is the strong solution to the problem

$$\begin{cases} w'_\varepsilon(t) + A(\varepsilon)w_\varepsilon(t) = F(t,\varepsilon), \quad t > 0, \quad \text{în} \quad H, \\ w_\varepsilon(0) = w_0, \end{cases} \tag{21}$$

for $0 < \varepsilon \le \varepsilon_0$, where

$$\begin{cases} F(t,\varepsilon) = f_0(t,\varepsilon)\,u_{1\,\varepsilon} + \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau, \\ f_0(t,\varepsilon) = \frac{1}{\sqrt{\pi}}\Big[2\exp\Big\{\frac{3t}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big) - \lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big], \\ w_0 = \int_0^\infty e^{-\tau}\tilde{u}_\varepsilon(2\varepsilon\tau)d\tau. \end{cases} \tag{22}$$

Using Holder's inequality, properties (i)-(v) of Lemma 3 and (18), we obtain

$$\big|\big|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\big|\big|_H = \Big|\Big|\tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau\Big|\Big|_H$$

$$\le \int_0^\infty K(t,\tau,\varepsilon)\big|\big|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\big|\big|_H\,d\tau \le \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||\tilde{u}'_\varepsilon(s)||_H\,ds\Big|\,d\tau$$

$$\leq \|\widetilde{u}_\varepsilon'\|_{L^2(0,\,\infty;\,H)} \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,d\tau \leq C\,\mathcal{M}_{3\varepsilon}\,\varepsilon^{1/4}, t\in[0,T], \quad \varepsilon\in(0,\varepsilon_0].$$

Then it follows

$$||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,T];\,H)} \leq C\,\mathcal{M}_{3\varepsilon}\,\varepsilon^{1/4}, \quad \varepsilon\in(0,\varepsilon_0]. \tag{23}$$

Let us denote by $R(t,\varepsilon) = \tilde{v}(t) - w_\varepsilon(t)$, which clearly is the strong solution in H to the problem

$$\begin{cases} R'(t,\varepsilon) + A_0 R(t,\varepsilon) = \varepsilon\, A_1 w_\varepsilon(t) + \mathcal{F}(t,\varepsilon), \quad t>0, \\ R(0,\varepsilon) = R_0, \end{cases} \tag{24}$$

where $R_0 = u_0 - w_0$ and

$$\mathcal{F}(t,\varepsilon) = \tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon)\tilde{f}_\varepsilon(\tau)\,d\tau - f_0(t,\varepsilon)\,u_{1\,\varepsilon}. \tag{25}$$

Taking the inner product by R in the equation (24) and then integrating, we obtain

$$|R(t,\varepsilon)|^2 + 2\int_0^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds$$

$$= \left|R_0\right|^2 + 2\int_0^t \left|\mathcal{F}(s,\varepsilon)\right|\left|R(s,\varepsilon)\right| ds + 2\,\varepsilon\int_0^t \big(A_1 w_\varepsilon(s), R(s,\varepsilon)\big)\,ds, \quad t\geq 0.$$

Using (16), from the last equality, we get

$$|R(t,\varepsilon)|^2 + \int_0^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds$$

$$\leq \left|R_0\right|^2 + 2\int_0^t \left|\mathcal{F}(s,\varepsilon)\right|\left|R(s,\varepsilon)\right| ds + \varepsilon^2\int_0^t \left|A_0^{\alpha-1/2}w_\varepsilon(s)\right|^2 ds, \quad t\geq 0. \tag{26}$$

From (26), we obtain

$$|R(t,\varepsilon)| + \Big(\int_0^t \left|A_0^{1/2}R(s,\varepsilon)\right|^2 ds\Big)^{1/2}$$

$$\leq \left|R_0\right| + \int_0^t \left|\mathcal{F}(s,\varepsilon)\right| ds + \varepsilon\Big(\int_0^t \left|A_0^{\alpha-1/2}w_\varepsilon(s)\right|^2 ds\Big)^{1/2}, \quad t\geq 0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{27}$$

From (18), it follows that

$$\left|R_0\right| \leq \int_0^\infty e^{-s}\left|\tilde{u}_\varepsilon(2\varepsilon s) - u_{0\,\varepsilon}\right| ds + |u_{0\varepsilon} - u_0|$$

$$\le \int_0^\infty e^{-s} \int_0^{2\varepsilon s} \left|\tilde{u}'_\varepsilon(\tau)\right| d\tau\, ds + |u_{0\,\varepsilon} - u_0|$$
$$\le C\,\mathcal{M}_{3\varepsilon}\,\varepsilon^{1/2} + |u_{0\,\varepsilon} - u_0|, \quad \varepsilon \in (0, \varepsilon_0]. \tag{28}$$

Using property (v) of Lemma 3, from (17), we have

$$\Big|\tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}_\varepsilon(\tau)\,d\tau\Big| \le \left|\tilde{f}(t) - \tilde{f}_\varepsilon(t)\right|$$
$$+ \int_0^\infty K(t,\tau,\varepsilon)\left|\tilde{f}_\varepsilon(t) - \tilde{f}_\varepsilon(\tau)\right| d\tau$$

$$\le \left|\tilde{f}(t) - \tilde{f}_\varepsilon(t)\right| + C(T,p)\,\|f'_\varepsilon\|_{L^p(0,T;\,H)}\,\varepsilon^{(p-1)/2\,p}, \quad t \ge 0, \quad \varepsilon \in (0, \varepsilon_0]. \tag{29}$$

As $e^{\tau}\lambda(\sqrt{\tau}) \le C, \quad \tau \ge 0$, we have

$$\int_0^t \exp\Big\{\frac{3\tau}{4\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\tau}{\varepsilon}}\Big)d\tau \le C\,\varepsilon \int_0^{\frac{t}{\varepsilon}} e^{-\tau/4}\,d\tau \le C\,\varepsilon \int_0^\infty e^{-\tau/4}\,d\tau \le C\varepsilon, \quad t \ge 0,$$

$$\int_0^t \lambda\Big(\frac{1}{2}\sqrt{\frac{\tau}{\varepsilon}}\Big)\,d\tau \le \varepsilon \int_0^\infty \lambda\Big(\frac{1}{2}\sqrt{\tau}\Big)\,d\tau \le C\,\varepsilon, \quad t \ge 0.$$

Hence

$$\Big|\int_0^t f_0(\tau,\varepsilon)\,d\tau\, u_{1\,\varepsilon}\Big| \le C\,\varepsilon\,|u_{1\,\varepsilon}|, \quad t \ge 0. \tag{30}$$

Using (29) and (30), we get

$$\int_0^t \left|\mathcal{F}(s,\varepsilon)\right| ds$$
$$\le C\,\Big(\mathcal{M}_{3\varepsilon}\,\varepsilon^{(p-1)/2p} + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\Big), \quad t \in [0,T], \quad \varepsilon \in (0, \varepsilon_0]. \tag{31}$$

Let us denote by $\tilde{y}_\varepsilon = A_0^{\alpha-1}\tilde{u}_\varepsilon$. Since $A_0^{\alpha-1}u_{0\,\varepsilon} \in D(A_0^{\alpha})$, $A_0^{\alpha-1}u_{1\,\varepsilon} \in H$, $A_0^{\alpha-1}f_\varepsilon \in W^{1,p}(0,T;H)$, from Lemma 1, we can state:

$$||A_0^{1/2}\tilde{y}_\varepsilon||_{C([0,t];\,H)} + ||\tilde{y}'_\varepsilon||_{L^2(0,t;\,H)} \le C\,\mathcal{M}_{3\varepsilon}, \quad t \ge 0, \quad \varepsilon \in (0, 1/2]. \tag{32}$$

As the operator $A_0^{\alpha-1/2}$ is closed, then, using (32), we obtain

$$\left|A_0^{\alpha-1/2}w_\varepsilon(t)\right|$$
$$\le \int_0^\infty K(t,\tau,\varepsilon)\left|A_0^{1/2}\tilde{y}_\varepsilon(\tau)\right| d\tau \le C\,\mathcal{M}_{3\varepsilon}, t \ge 0, \varepsilon \in (0, \varepsilon_0]. \tag{33}$$

Thanks to (28), (31) and (33), from (27) it follows that

$$||R||_{C([0,T];H)} + \big|\big|A_0^{1/2}R\big|\big|_{L^2(0,T;H)}$$
$$\leq \Big(\mathcal{M}_{3\varepsilon}\,\varepsilon^{(p-1)/2\,p} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\Big), \quad \varepsilon \in (0, \varepsilon_0]. \tag{34}$$

Finally, from (23) and (34), it follows that

$$||\tilde{u}_\varepsilon - \tilde{v}||_{C([0,T];H)} \leq ||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,T];H)} + ||R||_{C([0,T];H)}$$
$$\leq C\Big(\mathcal{M}_{3\varepsilon}\,\varepsilon^{\beta} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\Big), \quad \varepsilon \in (0, \varepsilon_0]. \tag{35}$$

According to Theorems 1 and 2, we have that $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$ and $\tilde{v}(t) = v(t)$ for $t \in [0,T]$. Therefore, from (35), we deduce (11).

Proof of (12). If $u_{1\,\varepsilon} \in D\big(A_0^{\alpha/2}\big)$, from (19), we get

$$\big|\big|\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\big|\big|_H = \Big|\Big|\tilde{u}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{u}_\varepsilon(\tau)\,d\tau\Big|\Big|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\big|\big|\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\big|\big|_H\,d\tau \leq \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||\tilde{u}'_\varepsilon(s)||_H\,ds\Big|\,d\tau$$

$$\leq \|\tilde{u}'_\varepsilon\|_{C([0,\infty);H)}\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|\,d\tau \leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/2}, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0].$$

This yields

$$||\tilde{u}_\varepsilon - w_\varepsilon||_{C([0,T];H)} \leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/2}, \quad \varepsilon \in (0,\varepsilon_0].$$

As, for $p \in (1;\infty]$, we have $(p-1)/2p \leq 1/2$, the proof of (12) follows in the same way as the proof of (11).

Proof of (13). Using properties (i), (iii) and (iv) of Lemma 3 and (19), we get

$$\big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t) - w_\varepsilon(t)\big)\big| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\big|A_0^{1/2}\big(\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(\tau)\big)\big|\,d\tau$$
$$\leq \int_0^\infty K(t,\tau,\varepsilon)\,\Big|\int_\tau^t \big|\big|A_0^{1/2}\tilde{u}'_\varepsilon(s)\big|\big|_H\,ds\Big|\,d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,\Big|\int_\tau^t \big|\big|A_0^{1/2}\tilde{u}'_\varepsilon(s)\big|\big|_H^2\,ds\Big|^{1/2}\,ds\Big|\,d\tau$$

$$\leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/4}, \quad t \geq 0, \quad \varepsilon \in (0, \varepsilon_0].$$

As $u_\varepsilon(t) = \tilde{u}_\varepsilon(t)$, $t \in [0, T]$, therefore

$$\big|\big|A_0^{1/2}\big(u_\varepsilon - w_\varepsilon\big)\big|\big|_{C([0,T];\,H)} \leq C\,\mathcal{M}_{4\,\varepsilon}\,\varepsilon^{1/4}, \quad \varepsilon \in (0, \varepsilon_0]. \tag{36}$$

From (34), it follows that

$$\big|\big|A_0^{1/2} R\big|\big|_{L^2(0,T)\,;H)}$$
$$\leq \Big(\mathcal{M}_{4\varepsilon}\,\varepsilon^{(p-1)/2\,p} + \big|u_{0\,\varepsilon} - u_0\big| + \big|\big|f_\varepsilon - f\big|\big|_{L^p(0,T;H)}\Big), \quad \varepsilon \in (0, \varepsilon_0]. \tag{37}$$

Finally, (36) and (37) imply (13) and this completes the proof. □

Theorem 5. *Let $T > 0$ and $p \in (1, \infty]$. Suppose that the operators A_0 and A_1 satisfy* **(H1)** *and* **(H2)**. *If $u_0, A_0 u_0, f(0) \in D(A_0)$, $\quad u_{1\varepsilon}, A_0 u_{0\varepsilon}, A_1 u_{0\varepsilon}, f_\varepsilon(0) \in D(A_0^{2\alpha-1}), f, A_0^{\alpha-1} f_\varepsilon \in W^{2,p}(0,T;H)$, then there exist constants $\varepsilon_0 = \varepsilon_0(\omega_0) \in (0,1)$ and $C = C(T, p, \omega_0, \omega_2, \omega_3, \alpha) > 0$ such that*

$$\big|\big|u_\varepsilon' - v' + h_\varepsilon\, e^{-\frac{t}{\varepsilon}}\big|\big|_{C([0,T];\,H)} \leq C\,\Big(\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p} + D_\varepsilon\Big), \tag{38}$$

$$\big|\big|A_0^{1/2}\big(u_\varepsilon' - v' + h_\varepsilon\, e^{-\frac{t}{\varepsilon}}\big)\big|\big|_{L^2(0,T;\,H)} \leq C\,\Big(\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{\beta} + D_\varepsilon\Big), \tag{39}$$

where v and u_ε are the strong solutions to problems (P_0) and (P_ε) respectively, $\beta = \min\{1/4, (p-1)/2p\}$, $h_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}$,

$$D_\varepsilon = \big|\big|f_\varepsilon - f\big|\big|_{W^{1,\,p}(0,T;\,H)} + \big|A_0(u_{0\varepsilon} - u_0)\big|,$$

$$\mathcal{M}_{5\,\varepsilon} = \big|A_0^\alpha h_\varepsilon\big| + \big|A_0^{\alpha-1} A_1 h_\varepsilon\big| + \big|A_0^\alpha u_{1\varepsilon}\big|$$
$$+ \big|A_0^{\alpha-1} A_1 u_{1\varepsilon} + |A_1 u_{0\varepsilon}| + \big|\big|A_0^{\alpha-1} f_\varepsilon\big|\big|_{W^{2,p}(0,T;\,H)}.$$

Proof. During this proof, for $\tilde{u}_\varepsilon$, $\tilde{v}$, $\tilde{f}$ and $\tilde{f}_\varepsilon$ we will use the same notations as in the proof of Theorem 4. Let us denote by

$$\tilde{z}_\varepsilon(t) = \tilde{u}_\varepsilon'(t) + \alpha_\varepsilon e^{-\frac{t}{\varepsilon}}, \quad \alpha_\varepsilon = f_\varepsilon(0) - u_{1\,\varepsilon} - A(\varepsilon) u_{0\,\varepsilon}.$$

If $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\big(A_0^{2\alpha-1})\big) \subseteq D(A_0^\alpha)$ and $f \in W^{2,1}(0,T;H)$, then, due to (15) and (17), $u_{1\,\varepsilon} + \alpha_\varepsilon \in D\big(A(\varepsilon)\big)$ and $\tilde{f} \in W^{2,1}(0,\infty;H)$. According to Theorem 2, $\tilde{z}_\varepsilon$ is the strong solution in H to the problem

$$\begin{cases} \varepsilon \tilde{z}_\varepsilon''(t) + \tilde{z}_\varepsilon'(t) + A(\varepsilon)\tilde{z}_\varepsilon(t) = \tilde{\mathcal{F}}(t,\varepsilon), \quad t > 0, \\ \tilde{z}_\varepsilon(0) = f_\varepsilon(0) - A(\varepsilon) u_{0\,\varepsilon}, \quad \tilde{z}_\varepsilon'(0) = 0, \end{cases} \tag{40}$$

where

$$\tilde{\mathcal{F}}(t,\varepsilon) = \tilde{f}'_\varepsilon(t) + e^{-t/\varepsilon} A(\varepsilon)\alpha_\varepsilon$$

and

$$\tilde{z}_\varepsilon \in W^{2,\infty}(0,\infty;H), \quad A_0^{1/2}\tilde{z}_\varepsilon \in W^{1,2}(0,\infty;H), \quad A(\varepsilon)\tilde{z}_\varepsilon \in L^\infty(0,\infty;H).$$

From Lemma 2 it follows that

$$||A_0^{1/2}\tilde{z}_\varepsilon||_{C([0,\,\infty];\,H)} + ||\tilde{z}'_\varepsilon||_{C([0,\,\infty);\,H)}$$
$$+\big|\big|A_0^{1/2}\tilde{z}'_\varepsilon\big|\big|_{L^2(0,\,\infty;\,H)} \le C\,\mathcal{M}_{5\,\varepsilon}, \quad \varepsilon \in (0,\varepsilon_0]. \tag{41}$$

According to Theorem 3 the function

$$w_{1\,\varepsilon}(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)d\tau$$

is a strong solution to the problem

$$\begin{cases} w'_{1\,\varepsilon}(t) + A(\varepsilon)w_{1\varepsilon}(t) = \mathcal{F}_1(t,\varepsilon), \quad t>0, \quad \varepsilon\in(0,\varepsilon_0] \\ w_{1\,\varepsilon}(0) = \int_0^\infty e^{-\tau}\,\tilde{z}_\varepsilon(2\,\varepsilon\,\tau)d\tau, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\Big(\tilde{f}'_\varepsilon(\tau)\,d\tau + e^{-\frac{\tau}{\varepsilon}}\,A(\varepsilon)\alpha_\varepsilon\Big)\,d\tau.$$

Moreover,

$$\big|A_0^{1/2}w_{1\,\varepsilon}(t)\big| \le \int_0^\infty K(t,\tau,\varepsilon)\,\big|A_0^{1/2}\tilde{z}_\varepsilon(\tau)\big|\,d\tau \le C\mathcal{M}_{5\,\varepsilon}, \quad t\ge 0. \tag{42}$$

Using properties (i), (iii)-(v) of Lemma 3 and(41), we get

$$\big|\big|\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t)\big|\big|_H = \Big|\Big|\tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}_\varepsilon(\tau)\,d\tau\Big|\Big|_H$$
$$\le \int_0^\infty K(t,\tau,\varepsilon)\big|\big|\tilde{z}_\varepsilon(t) - \tilde{z}_\varepsilon(\tau)\big|\big|_H\,d\tau$$
$$\le \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||\tilde{z}'_\varepsilon(s)||_H\,ds\Big|\,d\tau$$
$$\le \|\tilde{z}'_\varepsilon\|_{C([0,\infty);\,H)}\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|\,d\tau \le C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/2}, \quad t\in[0,T], \quad \varepsilon\in(0,\varepsilon_0]$$

and

$$\big|\big|A_0^{1/2}\big(\tilde{z}_\varepsilon(t) - w_{1\,\varepsilon}(t)\big)\big|\big|_H = \Big|\Big|A_0^{1/2}\tilde{z}_\varepsilon(t) - \int_0^\infty K(t,\tau,\varepsilon)\,A_0^{1/2}\tilde{z}_\varepsilon(\tau)\,d\tau\Big|\Big|_H$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\big|\big|A_0^{1/2}\big(\tilde z_\varepsilon(t)-\tilde z_\varepsilon(\tau)\big)\big|\big|_H\,d\tau$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_t^\tau ||A_0^{1/2}\tilde z'_\varepsilon(s)||_H\,ds\Big|\,d\tau$$

$$\leq \|A_0^{1/2}\tilde z'_\varepsilon\|_{L^2(0,\,\infty;\,H)}\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^{1/2}\,d\tau$$

$$\leq C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/4},\quad t\in[0,T],\quad \varepsilon\in(0,\varepsilon_0],$$

from which it follows that

$$||\tilde z_\varepsilon - w_{1\,\varepsilon}||_{C([0,T];\,H)}\leq C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/2},\quad \varepsilon\in(0,\varepsilon_0], \tag{43}$$

$$\big|\big|A_0^{1/2}\big(\tilde z_\varepsilon - w_{1\,\varepsilon}\big)\big|\big|_{L^2(0,\,T;\,H)}\leq C\,\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{1/4},\quad \varepsilon\in(0,\varepsilon_0]. \tag{44}$$

Let $R_1(t,\varepsilon)=\tilde v'(t)-w_{1\varepsilon}(t)$. If $f(0)-A_0u_0\in D(A_0)$ and $f\in W^{2,1}(0,T;H)$, then, according to Theorem 1, $\tilde v\in W^{2,\infty}(0,\infty;H)$, $A_0^{1/2}\tilde v\in W^{1,2}(0,\infty;H)$. Therefore $R_1\in W^{1,\infty}(0,\infty;H)$ and

$$\begin{cases} R_1'(t,\varepsilon)+A_0R_1(t,\varepsilon)=\tilde f'(t)-\mathcal{F}_1(t,\varepsilon)+\varepsilon A_1 w_{1\,\varepsilon}(t), & t>0,\\ R_1(0,\varepsilon)=f(0)-A_0u_0-w_{1\,\varepsilon}(0). \end{cases}$$

Similarly to (27), we deduce inequality

$$\big|R_1(t,\varepsilon)\big|+\Big(\int_0^t \big|A_0^{1/2}R_1(s,\varepsilon)\big|^2\,ds\Big)^{1/2}\leq \big|R_1(0,\varepsilon)\big|$$

$$+\int_0^t \big|\tilde f'(s)-\mathcal{F}_1(s,\varepsilon)\big|\,ds+\,\varepsilon\Big(\int_0^t \big|A_0^{\alpha-1/2}w_{1\,\varepsilon}(s)\big|^2\,ds\Big)^{1/2},\quad t\geq 0. \tag{45}$$

Using (41), for $R_1(0,\varepsilon)$, we get

$$\big|R_1(0,\varepsilon)\big|\leq\big|f(0)-f_\varepsilon(0)\big|+\big|A_0(u_0-u_{0\,\varepsilon})\big|$$

$$+\varepsilon\,\big|A_1u_{0\,\varepsilon}\big|+\int_0^\infty e^{-s}\big|\tilde z_\varepsilon(2\,\varepsilon\,s)-\tilde z_\varepsilon(0)\big|\,ds$$

$$\leq C\,D_\varepsilon+\varepsilon\,\big|A_1u_{0\,\varepsilon}\big|+\mathcal{M}_{5\,\varepsilon}\,\varepsilon\leq C\,D_\varepsilon+\mathcal{M}_{5\,\varepsilon}\,\varepsilon,\quad \varepsilon\in(0,\varepsilon_0]. \tag{46}$$

As

$$\big|\tilde f'(s)-\mathcal{F}_1(s,\varepsilon)\big|\leq\big|\tilde f'(s)-\tilde f'_\varepsilon(s)\big|+\int_0^\infty K(s,\tau,\varepsilon)\,\big|\tilde f'_\varepsilon(\tau)-\tilde f'_\varepsilon(s)\big|\,d\tau$$

$$+\int_0^\infty K(s,\tau,\varepsilon)\,e^{-\frac{\tau}{\varepsilon}}\,d\tau\,\big|\,A(\varepsilon)\alpha_\varepsilon\big|,$$

then, due to property (iv) and (vi) of Lemma 3, it follows:

$$\int_0^t \left|\tilde{f}'(s) - \mathcal{F}_1(s,\varepsilon)\right| ds \leq C\Big(D_\varepsilon + \mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p} + \left|\,A(\varepsilon)\alpha_\varepsilon\right|\varepsilon\Big)$$

$$\leq C\Big(D_\varepsilon + \mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p}\Big), \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0]. \tag{47}$$

Let us denote by $\tilde{y}_{1\varepsilon} = A_0^{\alpha-1}\tilde{z}_\varepsilon$. Since $A_0^{\alpha-1} z_\varepsilon(0) \in D(A_0^\alpha)$, $A_0^{\alpha-1} f_\varepsilon \in W^{1,p}(0,T;H)$, from Lemma 1, we can state the estimate:

$$||A_0^{1/2}\tilde{y}_{1\varepsilon}||_{C([0,t];\,H)} + ||\tilde{y}'_{1\varepsilon}||_{L^2(0,t;\,H)} \leq C\,\mathcal{M}_{5\varepsilon}, \quad t \geq 0, \quad \varepsilon \in (0,1/2]. \tag{48}$$

As the operator $A_0^{\alpha/2}$ is closed, then, using (48), we obtain

$$\left|A_0^{\alpha-1/2} w_{1\varepsilon}(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\left|A_0^{1/2}\tilde{y}_\varepsilon(\tau)\right| d\tau$$

$$\leq C\,\mathcal{M}_{5\varepsilon}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{49}$$

Using (42), (46), (47), from (45), we get

$$\big|\big|R_1\big|\big|_{C([0,T];\,H)} + \big|\big|A_0^{1/2}R_1\big|\big|_{L^2(0,T;\,H)} \leq C\Big(D_\varepsilon$$

$$+\mathcal{M}_{5\,\varepsilon}\,\varepsilon^{(p-1)/2p}\Big), \quad \varepsilon \in (0,1]. \tag{50}$$

Finally, as (43), (44) and (45) imply (38) and (39), the proof is complete. □

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