THE CLASS OF THE INFORMATIONAL EXTENDED GAMES

Ludmila Novac
Catedra Informatică şi Optimizare Discretă

În acest articol [1] sunt definite câteva tipuri de jocuri informaţional extinse de n jucători, adică jocuri în care jucătorii aleg strategiile sale simultan, cu presupunerea că ei posedă informaţie despre strategiile pe care le vor alege adversarii [1]. Pentru toate tipurile de jocuri informaţional extinse care sunt definite se presupune că funcţiile de utilitate reprezintă cunoştinţele comune. Pentru aceste jocuri informaţional extinse se analizează mulţimile situaţiilor Nash de echilibru şi sunt determinate condiţiile suficiente de existenţă a situaţiilor Nash de echilibru. Rezultatul esenţial al acestui articol este teorema despre condiţiile de extindere a mulţimilor situaţiilor Nash de echilibru în baza extinderii informatizării jucătorilor.

1 Preliminary facts

We consider the noncooperative game denoted by $\Gamma = \langle I, X_i, H_i \rangle$, where $I = \{1, 2, ..., n\}$ represents the set of players, X_i $(i \in I)$ represents the set of strategies for the player i and $H_i : \prod\limits_{i \in I} X_i \to R, (i \in I)$ is the payoff function for the the player i.

Usually, in the noncooperative games, the players are interested to keep their chosen strategy. The participants of the game can know the sets of strategies of other players and their payoff functions. For the informational extended games it is significant that the players can say which is the chosen strategy, or some players can obtain the chosen strategies of other players in their manner [2].

We will define some informational extended games for the game Γ in which we will consider that some players have some information about the chosen strategies of other players.

We define the game ${}_1\Gamma$ for which we will consider that only the first player has information about chosen strategies of all players. We will define this game by ${}_1\Gamma = \left\langle I, \overline{X}_1, X_j\ (j \in I, j \neq 1), \overline{H}_i \left(i = \overline{1,n}\right) \right\rangle$, where $\overline{X}_1 = \left\{ \varphi : \prod\limits_{j \in I, j \neq 1} X_j \to X_1 \right\}$ is the set of strategies for the first player and the payoff functions are defined by: $\overline{H}_i : \overline{X}_1 \times \prod\limits_{j \in I, j \neq 1} X_j \to R, \left(i = \overline{1,n}\right)$.

Next we will define another informational extend game which will be denoted by ${}_n\Gamma$. For the game ${}_n\Gamma$ we will consider that all players are informed about chosen strategies of all participants of the game. This informational extend game can be represented by:
${}_n\Gamma = \left\langle I, \overline{X}_i, \left(i = \overline{1,n}\right), \overline{H}_i \left(i = \overline{1,n}\right) \right\rangle$, where $\overline{X}_i = \left\{ \varphi_i : \prod\limits_{j \in I, j \neq i} X_j \to X_i \right\}$, for $(\forall i \in I)$ and the payoff functions are $\overline{H}_i : \prod\limits_{j \in I} \overline{X}_j \to R$, for $(\forall i \in I)$.

We will denote by ${}^i_j\Gamma$ the game in which we will consider that the player i has information about the chosen strategy of the player j. We define this informational extended game by
${}^i_j\Gamma = \left\langle I, \overline{X}_i, X_k\ (k \in I, k \neq i), \overline{H}_p \left(p = \overline{1,n}\right) \right\rangle$, where $\overline{X}_i = \{\varphi_i : X_j \to X_i\}$,
and $\overline{H}_p : \overline{X}_i \times \prod\limits_{k \in I, k \neq i} X_k \to R, \left(p = \overline{1,n}\right)$.

A generalization of the previous game is the game ${}^i_J\Gamma$, for which we will consider that the player i is informed about the chosen strategies of the players from the subset J of the players set I.

[1] The research was supported by SCSTD of ASM grant 07411.08 INDF and MRDA/CRDF Grant CERIM 10006-06.

We define the game ${}_J^i\Gamma = \left\langle I, \overline{X}_i, X_k\,(k \in I, k \neq i)\,, \overline{H}_p\left(p = \overline{1,n}\right)\right\rangle$, where $\overline{X}_i = \left\{\varphi_i : \prod\limits_{j \in J} X_j \to X_i\right\}$ and $\overline{H}_p : \overline{X}_i \times \prod\limits_{k \in I, k \neq i} X_k \to R, \left(p = \overline{1,n}\right).$

The game ${}_I^i\Gamma$ is a particular case of the informational extended game ${}_J^i\Gamma$, for which we will consider that the set J is the set of all participants ($J = I$), and the player i is informed about the chosen strategies of the all players of the game.

Next we will define the game ${}_n^J\Gamma = \left\langle I, \overline{X}_j, (j \in J \subset I)\,, X_k\,(k \in I \backslash J)\,, \overline{H}_p\left(p = \overline{1,n}\right)\right\rangle$, in which we will consider that some players (from the subset $J \subseteq I$) are informed about the chosen strategies of the all players of the game, here the new sets of strategies are $\overline{X}_j = \left\{\varphi_j : \prod\limits_{k \in I \backslash \{j\}} X_k \to X_j\right\}, (j \in J)$ and the payoff functions are $\overline{H}_p : \prod\limits_{j \in J} \overline{X}_j \times \prod\limits_{k \in I \backslash J} X_k \to R, \left(p = \overline{1,n}\right).$

An outcome for the game Γ is an action profile $x = (x_1, x_2, ..., x_n)$ and the outcome space is $X = \prod\limits_{i \in I} X_i$, $x \in X$; we will consider that the payoff functions $H_i\,(x_1, x_2, ..., x_n)$ for the game Γ are continuous and are defined on the compactum X.

For the informational extended games ${}_1\Gamma$ and ${}_n\Gamma$ we will make some notations and assumptions for the new strategies of the players, corresponding to their information about the chosen strategies.

For the informational extended game ${}_1\Gamma$ we will denote an outcome by $(\varphi\,(x_2, ..., x_n)\,, x_2, ..., x_n)$,the outcome space will be $\overline{X}_1 \times \prod\limits_{j \in I, j \neq 1} X_j$ and the payoff functions $\overline{H_i}\,(\varphi\,(x_2, ..., x_n)\,, x_2, ..., x_n)$ $(i \in I,$ $i \neq 1)$ which are concave on the set X_i, and the function $\overline{H}_1$ is concave on the set $\overline{X}_1$, where $\overline{X}_1 \subset C\left(\prod\limits_{j \in I, j \neq 1} X_j, X_1\right)$, and $C\left(\prod\limits_{j \in I, j \neq 1} X_j, X_1\right)$ represents the space of all continuous functions defined on the compactum $\prod\limits_{j \in I, j \neq 1} X_j$ with values from the compactum X_1. Evidently, the functions $\overline{H_i}\,(\cdot)$ are continuous functions as composed functions of continuous functions.

For the informational extended game ${}_n\Gamma$ we will make next notations: an outcome for this game will be denoted by $\varphi = (\varphi_1\,(x\,(1))\,, ..., \varphi_i\,(x\,(i))\,, ..., \varphi_n\,(x\,(n)))$ and the outcome space is $\prod\limits_{i \in I} \overline{X}_i$, where $\overline{X}_i \subset C\left(\prod\limits_{j \in I, j \neq i} X_j, X_i\right), (\forall i \in I)$ and $C\left(\prod\limits_{j \in I, j \neq i} X_j, X_i\right)$ represents the space of all continuous functions defined on the compactum $\prod\limits_{j \in I, j \neq i} X_j$ with values from the compactum X_i.

For the game ${}_n\Gamma$ we will make the next notations for the strategies of the players:
$\varphi_1\,(x\,(1)) = \varphi_1\,(x_2, ..., x_n)\,,\ \varphi_2\,(x\,(2)) = \varphi_2\,(x_1, x_3, ..., x_n)\,, ...$
$\varphi_i\,(x\,(i)) = \varphi_i\,(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)\,, ..., \varphi_n\,(x\,(n)) = \varphi_n\,(x_1, ..., x_{n-1})\,.$
Here and later we will use the notation $x_{-i} = x\,(i) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)\,.$

Definition 1. *A Nash equilibrium of the game* Γ *is an action profile* $x^* \in X$ *such that for every* $i \in I$:

$$H_i(x^*) > H_i\left(x_{-i}^*, x_i\right) \text{ for all } x_i \in X_i.$$

Another and sometimes a more convenient way of defining Nash equilibrium is via best response correspondences $Br_i : \underset{j \in I \backslash \{i\}}{\times} X_j \rightrightarrows X_i$ such that

$$Br_i(x_{-i}) = \left\{x_i \in X_i : H_i(x) \geqslant H_i\left(x_{-i}, x_i'\right) \text{ for } \forall x_i' \in X_i\right\}. \qquad (*)$$

Definition 2. *A Nash equilibrium is an action profile* x^* *such that* $x_i^* \in Br_i(x_{-i}^*)$ *for all* $i \in I$.

If the sets X_i are compacts and the functions H_i are continuous, then the best response set (*) for the player i can be represented by:

$$Br_i(x_{-i}) = Arg \max_{x_i \in X_i} H_i\left(x_{-i}, x_i\right).$$

We will denote by $NE\left(\Gamma\right)$ the set of all Nash equilibria for the game Γ.

The Nash equilibrium is a solution concept of a game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only his or her own strategy (i.e., by changing unilaterally). If each player has chosen a strategy and no player can benefit by changing his or her strategy while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.

We can prove that the Nash equilibria sets for all informational extended games defined here are nonempty sets. For proof we can use the Kakutani fixed point theorem for point-to-set mappings (for proof see [3] for the informational extended games with two players).

In the next section we will prove a theorem for the inclusion of the Nash equilibria sets for the informational extended games.

Before giving this theorem we need to recall some theorems.

Theorem (**Kakutani**). (1941). *Let X be a Banach space and K a non-empty, compact and convex subset of X Let $F : K \rightrightarrows 2^K$ be a point-to-set mapping on K with a closed graph and the property that the set $F(x)$ is non-empty and convex for all $x \in K$. Then F has a fixed point.*

Theorem (**Arzelà-Ascoli**). (Compactness criterion). *A set of continuous functions $E \subseteq C(K)$ is compact if and and only if the set E is uniformly bounded: ($|x(t)| \leq M$, $\forall t \in K$, for $\forall x \in E$) and the functions from the set E are equicontinuous (i.e. for $\forall\varepsilon, \exists\delta$ so that if $\rho(t_1;t_2) < \delta$ then $|x(t_1) - |x(t_2)| < \varepsilon$ for $\forall x \in E$).*

Theorem (**Tihonov**). *A product of a family of compact topological spaces $X = \prod\limits_{\alpha \in A} X_\alpha$ is compact.*

2 Main results

Now we will state our base theorem. This theorem gives the conditions for the inclusion of the Nash equilibria sets for the informational extended games with n players define above.

Theorem. *Let us consider that for the game ${}_n\Gamma$ the next conditions hold:*

1) the sets $X_i \neq \emptyset, (\forall i \in I)$ are compact of Banach spaces;

2) the sets of functions $\overline{X}_i \subset C\left(\prod\limits_{j \in I, j \neq i} X_j, X_i\right), (\forall i \in I)$ are uniformly bounded and the functions from the sets $\overline{X}_i$ are equicontinuous;

3) the payoff functions $H_i\left(\cdot\right), (\forall i \in I)$ are continuous on the compactum $\prod\limits_{i \in I} X_i$, and the functions $\overline{H_i}\left(\cdot\right), (\forall i \in I)$ are concave on $\overline{X}_i$ for $\forall \varphi\left(i\right)$, respectively.

Then the next relation $NE\left(\Gamma\right) \subset NE\left({}_n\Gamma\right) \neq \emptyset$ holds.

Proof.

The set of Nash equilibria for the game ${}_n\Gamma$ is a nonempty set $NE\left({}_n\Gamma\right) \neq \emptyset$. For proof we can apply de Kakutani fixed point theorem for point-to-set mappings.

We denote by $\overline{X} = \prod\limits_{i \in I} \overline{X}_i$ the outcome space for the game ${}_n\Gamma$. According to Arzelà-Ascoli theorem, the sets $\overline{X}_i, (\forall i \in I)$ are compact and according to Tihonov theorem the space $\overline{X}$ is compact too.

We will denote by $\varphi = (\varphi_1, ..., \varphi_i, ..., \varphi_n) \in \prod_{i \in I} \overline{X}_i$ an outcome for the game ${}_n\Gamma$, where $\varphi_1(x_2, ..., x_n) \in \overline{X}_1, ..., \varphi_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in \overline{X}_i$, ..., $\varphi_n(x_1, ..., x_{n-1}) \in \overline{X}_n$.

We will use the notation $\varphi(i) = (\varphi_1, \varphi_2, ..., \varphi_{i-1}, \varphi_{i+1}, ..., \varphi_n) \in \overline{X}(i) = \prod_{j \in I \setminus \{i\}} \overline{X}_j$.

Since the functions $H_i(\cdot), (i = \overline{1, n})$ for the game Γ are continuous on the compactum $\prod_{i \in I} X_i$ (from the third condition of the theorem) and because the functions $\varphi_i \in \overline{X}_i$ are continuous on the compactum $\prod_{j \in I, j \neq i} X_j$, then it follows that the functions $\overline{H}_i, i = \overline{1, n}$ are continuous on the compactum $\prod_{i \in I} \overline{X}_i$ as composed functions of continuous functions $\overline{H}_i(\varphi) = H_i(\varphi(x))$.

We define the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$, such that $B(\varphi) = (B_1(\varphi_{-1}), B_2(\varphi_{-2}), ..., B_n(\varphi_{-n}))$, where $B_i(\varphi_{-i}), (i \in I)$ represents the best response set for the player i for the chosen strategies of all players $j \in I \setminus \{i\}$.

Because the sets $\overline{X}_i; (i \in I)$ are compacts and $\overline{H}_i$, for $i = \overline{1, n}$ are continuous functions, then according to Weierstrass theorem we can write:

$$B_i(\varphi_{-i}) = Arg \max_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1, \varphi_2, ..., \varphi_n),$$

i.e.:

$$B_i(\varphi_{-i}) = \left\{ \varphi_i \in \overline{X}_i : \overline{H}_i(\varphi_1, \varphi_2, ..., \varphi_n) = \max_{\varphi_i' \in \overline{X}_i} \overline{H}_i(\varphi_1, \varphi_2, ..., \varphi_2', ..., \varphi_n) \right\}, \ (i = \overline{1, n}).$$

In order to use the Kakutani theorem we need to prove that:

1) $\overline{X} = \prod_{i \in I} \overline{X}_i \neq \emptyset$ is non-empty convex compact set;

2) for the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$ the next conditions hold:

a) for $\forall \varphi_i \in \overline{X}_i$, $(i = \overline{1, n})$ the set $B(\varphi) \neq \emptyset$ is a convex subset of $\overline{X}$;

b) the point-to-set mapping B is closed.

(For complete proof see [3] for the informational extended game with two players).

According to Kakutani theorem there is a fixed point for the point-to-set mapping $B : \overline{X} \rightrightarrows 2^{\overline{X}}$. Let $\varphi^* = (\varphi_1^*, \varphi_2^*, ..., \varphi_n^*) \in \overline{X}$ be a fixed point for the point-to-set mapping B; i.e. $(\varphi_1^*, \varphi_2^*, ..., \varphi_n^*) \in B(\varphi_1^*, \varphi_2^*, ..., \varphi_n^*) = \prod_{i \in I} B_i(\varphi_{-i})$, so the relation $\overline{H}_i(\varphi_1^*, ..., \varphi_i^*, ..., \varphi_n^*) = \max_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1^*, ..., \varphi_i, ..., \varphi_n^*)$ holds for all $i = \overline{1, n}$, thus by definition of the Nash equilibrium it follows that $\overline{H}_i(\varphi_1^*, ..., \varphi_i^*, ..., \varphi_n^*) \in NE({}_n\Gamma) \neq \emptyset$.

Next we will prove that the inclusion $NE(\Gamma) \subset NE({}_n\Gamma)$ holds.

The functions φ_i from the sets $\overline{X}_i \subset C\left(\prod_{j \in I, j \neq i} X_j, X_i\right)$ are defined on the compactum $\prod_{j \in I, j \neq i} X_j$ and their values are from the sets X_i (for $i = \overline{1, n}$).

The problem $\sup_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1^*, ..., \varphi_i, ..., \varphi_n^*)$ is an optimization problem, and its arguments are elements from the compactum $\overline{X}_i$ (for $i = \overline{1, n}$).

On the other hand the solution of this problem will be an element from X_i, because $\varphi_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) = x_i' \in X_i, (\forall i \in I)$ (according to the definition of the sets $\overline{X}_i$). Thus the solution of the problem $\sup_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1^*, ..., \varphi_i, ..., \varphi_n^*)$, $(i = \overline{1, n})$ will be an element from X, for which the functions $\overline{H}_i(\varphi_1(x(1)), ..., \varphi_i(x(i)), ..., \varphi_n(x(n))), (\forall i \in I)$ will have the maximum values.

Let us consider the element $x^* \in NE(\Gamma)$.

According to the definition of the Nash equilibrium it follows that for $\forall i \in I$ the relation:

$H_i(x^*) = H_i(x_1^*, x_2^*, ..., x_n^*) = \sup\limits_{x_i \in X_i} H_i(x_i, x^*(i)) =$

$= \sup\limits_{\varphi_i \in \overline{X}_i} \overline{H}_i(\varphi_1^*, ..., \varphi_i, ..., \varphi_n^*) = \overline{H}_i(\varphi^*(x^*))$

holds, then it follows that $\varphi^*(x^*) \in NE({}_n\Gamma)$.

Thus if $x^* \in NE(\Gamma)$, then $\varphi^*(x^*) \in NE({}_n\Gamma)$, so the inclusion $NE(\Gamma) \subset NE({}_n\Gamma)$ holds.■

From this theorem it follows the next corollary.

Corollary. *We consider that for the informational extended games ${}_n\Gamma$ and ${}_1\Gamma$ the conditions 1)-3) (from the previous theorem) hold. Then the relation*

$$NE(\Gamma) \subset NE({}_1\Gamma) \subset NE({}_n\Gamma)$$

holds.

In a similar manner we can prove the next.

Corollary. *We consider that for all informational extended games defined above the conditions 1)-3) (from the previous theorem) hold. Then for those informational extended games the next relation*

$$NE(\Gamma) \subset NE\left({}_j^i\Gamma\right) \subset ... \subset NE\left({}_J^i\Gamma\right) \subset ... \subset NE\left({}_n^J\Gamma\right) \subset ... \subset NE({}_n\Gamma)$$

holds.

If some players have information about the chosen strategies of other players then their sets of strategies will be sets of functions defined on the product of the strategies sets of those players whose strategies are known; for the rest of players (which do not know the chosen strategies of other players) the sets of strategies will be the same sets from the initial game Γ. Thus, we can define many informational extended games for which the outcomes will contain the strategies $x_j \in X_j$ (for the players $j \in J \subset I$ which do not know the chosen strategies of other players), and the strategies $\varphi_k \in \overline{X}_k$ (for the players $k \in I \backslash J$ which know the chosen strategies of other players).

Thus, if some players are informed about the chosen strategies of other players, then the Nash equilibria sets for any informational extended game will be more than for the initial game.

So if more players are informed, then the informational extended game will have many Nash equilibria and it is possible that for some Nash equilibria some players will have the best payoff.

Thus, our aim is to indicate the importance of the information for all make-decision problems and for conflict problem solving. These models can be used in several situations in various social domains, i.e. economy, management and political theory.

References

[1] L. Novac, Informational extended games. *Second conference of the Mathematical Society of the Republic of Moldova (dedicated to the 40 anniversary of the foundation of the Institute of Mathematics and Computer Science of ASM). Communications* / Chişinău, August, 2004, p. 232-234.

[2] B. Hâncu, L. Novac, Informational aspects in the Game Theory. *Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity.* **Vol. 3**, Cluj-Napoca, 2005, Mediamira Science Publisher, p. 25-34.

[3] L. Novac, Nash equilibria in the two players noncooperative informational extended games. *Studia Universitatis, Seria Ştiinţe Exacte şi Economice*, **nr.3 (13)**, 2008, USM, Chişinău p. 34-38.

Prezentat la 20.09.2008