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On the Computational Complexity of Optimization Convex Covering Problems of Graphs

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Abstract

In this paper we present further studies of convex covers and convex partitions of graphs. Let G be a finite simple graph. A set of vertices S of G is convex if all vertices lying on a shortest path between any pair of vertices of S are in S. If $3 \leq |S| \leq |X| - 1$, then S is a nontrivial set. We prove that determining the minimum number of convex sets and the minimum number of nontrivial convex sets, which cover or partition a graph, is in general NP-hard. We also prove that it is NP-hard to determine the maximum number of nontrivial convex sets, which cover or partition a graph.

Keywords: NP-hardness, convex cover, convex partition, graph.

MSC 2010: 05A18, 05C35; 68Q25; 68R10

1 Introduction

We denote by $G = (X; U)$ a simple undirected graph with vertex set X, $X(G)$, and edge set U, $U(G)$. The *neighborhood* of a vertex $x \in X$ is the set of all vertices $y \in X$ such that $y \sim x$ (i.e., adjacent to x), and it is denoted by $\Gamma(x)$. Let S be a subset of X. If every two vertices of S are adjacent in G, then it is called a *clique*. If every vertex of $X \backslash S$ is adjacent to at least one vertex of S, then S is a dominating set of G. If $3 \leq |S| \leq |X| - 1$, then S is a *nontrivial* set. The distance $d(x, y)$ between two vertices $x, y \in X$ is the length of the shortest path between x and y. The diameter of G, denoted by $diam(G)$, is the distance between two farthest vertices of G. We denote by $G[S]$ the subgraph of G induced by S.

We remind some notions defined in [1]. A set $S \subseteq X$ is called *convex* if the inclusion $\{z \in X : d(x,z) + d(z,y) = d(x,y)\} \subseteq S$ holds for any two vertices $x, y \in S$. The *convex hull* of $S \subseteq X$, denoted by $d - conv(S)$, is the smallest convex set containing S.

By [4], the family of sets $\boldsymbol{\mathcal{P}}(G)$ is called the *convex cover* of a graph $G = (X;\, U)$ if the following statements hold:

1) each set of $\boldsymbol{\mathcal{P}}(G)$ is convex in G;

2) $X = \bigcup_{S \in \boldsymbol{\mathcal{P}}(G)} S$;

3) $S \not\subseteq \bigcup_{C \in \boldsymbol{\mathcal{P}}(G), C \neq S} C$ for each $S \in \boldsymbol{\mathcal{P}}(G)$.

If $|\boldsymbol{\mathcal{P}}(G)| = p$, then we say that $\boldsymbol{\mathcal{P}}(G)$ is a *convex p-cover* of G. If any two sets of $\boldsymbol{\mathcal{P}}(G)$ are disjoint, then this family is called a *convex partition* of G. The family $\boldsymbol{\mathcal{P}}(G)$ is said to be the *nontrivial convex cover* of G if each set of $\boldsymbol{\mathcal{P}}(G)$ is nontrivial and convex. A vertex $x \in X$ is called *resident* in $\boldsymbol{\mathcal{P}}(G)$ if x belongs to only one set of $\boldsymbol{\mathcal{P}}(G)$.

We know from [4], [5] and [9] that it is NP-complete to decide whether a graph has a convex p-cover or a convex p-partition for a fixed $p \geq 2$. If the nontrivial sets are considered as elements of convex p-covers or convex p-partitions of a graph, the problems also remain NP-complete for a fixed $p \geq 2$.

Since the general convex p-cover problem is NP-complete, several classes of graphs for which there exist polynomial algorithms for deciding whether a graph can be covered or partitioned by a fixed number $p \geq 2$ of convex sets were identified [4], [5], [8], [11].

Note that there exist graphs for which there are no nontrivial convex covers or nontrivial convex partitions or both. For example, a convex simple graph (a graph that does not contain any nontrivial convex sets [3]) can not be covered by nontrivial convex sets. The problem of determining whether a graph G can be partitioned into an arbitrary number of nontrivial convex sets is NP-complete, but it can be established in polynomial time whether G can be covered by an arbitrary number of nontrivial convex sets [12].

In our previous works, we have studied six different invariants that consistently help to determine the existence of convex covers and partitions of graphs. The least $p \geq 2$ for which a graph G has a convex p-cover is said to be the *minimum convex cover number* $\varphi_c^{min}(G)$. Similarly, the least $p \geq 2$ for which G has a convex p-partition is said to be the *minimum convex partition number* $\theta_c^{min}(G)$. In the same way, *minimum nontrivial convex cover number* $\varphi_{cn}^{min}(G)$, *minimum nontrivial convex partition number* $\theta_{cn}^{min}(G)$, *maximum nontrivial convex cover number* $\varphi_{cn}^{max}(G)$ and *maximum nontrivial convex partition number* $\theta_{cn}^{max}(G)$ are defined in the case when the nontrivial convex sets are considered. For supplementary information about estimation of these invariants the papers [9], [10], [11] and [12] can be consulted.

It is obvious that for any graph G we have $\varphi_c^{min}(G) \leq \theta_c^{min}(G)$. As before, if G can be partitioned into nontrivial convex set, then $\theta_c^{min}(G) \leq \theta_{cn}^{min}(G)$ and:

$$\varphi_{cn}^{min}(G) \leq \theta_{cn}^{min}(G) \leq \theta_{cn}^{max}(G) \leq \varphi_{cn}^{max}(G).$$

Anyway, if graph G can be covered by nontrivial convex sets, then $\varphi_c^{min}(G) \leq \varphi_{cn}^{min}(G)$.

2 NP-hardness

In this section, we show that it is NP-hard to determine the values of the invariants $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$ and $\theta_{cn}^{max}(G)$ for a graph G. For each problem, firstly, we formulate the corresponding decision problem.

Determination of minimum convex cover number $\varphi_c^{min}(G)$ has the following decision problem:

PROBLEM: Minimum convex cover (MinCC).
INSTANCE: Graph $G = (X; U)$, integer p, $2 \leq p \leq |X|$.
QUESTION: Is there a convex q-cover of G such that $2 \leq q \leq p$?

As for invariants $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, their decision problems are defined in the same manner and only the appropriate specification of the type of convex cover (convex q-partition, nontrivial convex q-cover, nontrivial convex q-partition) in the questions is required.

The clique partitioning problem is defined as follows:

PROBLEM: Clique partition (CP).
INSTANCE: Graph $G = (X;\, U)$, integer p, $3 \leq p \leq |X|$.
QUESTION: Is there a partition of X into p disjoint sets $X_1, \ldots, X_p$, such that the subgraph induced by X_i is a complete graph for each i, $1 \leq i \leq p$?

In the sequel, we show that MinCC, minimum convex partition (MinCP), minimum nontrivial convex cover (MinNCC) and minimum nontrivial convex partition (MinNCP) problems are NP-complete. In order to achieve this goal, we reduce the CP that is a well-known NP-complete problem [2] to the problems of interest.

Theorem 1. *The MinCC problem is NP-complete.*

Proof. Verifying whether a set of vertices is convex can be done in polynomial time [6]. Hence, MinCC problem is in NP.

Let $G = (X;\, U)$ be a generic graph of CP problem and p be an integer, $3 \leq p \leq |X|$. Without loss of generality, it can be assumed that X is not a clique. We obtain a particular graph $G' = (X';\, U')$ of MinCC problem from G by adding auxiliary sets $Y = \{y_1, y_2, \ldots, y_p\}$ and $Z = \{z_1, z_2, \ldots, z_p\}$ to X such that $X' = X \cup Y \cup Z$, where $\Gamma(y_i) = X \cup \{z_i\}$ and $\Gamma(z_i) = X \cup \{y_i\}$ for each i, $1 \leq i \leq p$. Obviously, this construction of G' can be done in polynomial time.

The MinCC instance is defined by the graph G' and the number p. In Figure 1 it is shown how a particular graph G' of MinCC problem is obtained from a graph G of CP problem.

It can easily be checked that for every two nonadjacent vertices $a, b \in X'$ we get $X' \subseteq d - conv(\{a, b\})$. In consequence, a set $S \subset X'$ is convex in G' if and only if S is a clique, and further G' cannot be covered by k, $k < p$, convex sets.

If G can be partitioned into p disjoint cliques $X_1, X_2, \ldots, X_p$, then we obtain a convex p-cover $\boldsymbol{\mathcal{P}}(G') = \{X'_1, X'_2, \ldots, X'_p\}$, where $X'_i = X_i \cup \{y_i, z_i\}$ for each i, $1 \leq i \leq p$.

Let $\boldsymbol{\mathcal{P}}(G')$ be a convex cover of G' such that $|\boldsymbol{\mathcal{P}}(G')| \leq p$. For the above reason, $|\boldsymbol{\mathcal{P}}(G')| = p$. We define a family of convex sets $\boldsymbol{\mathcal{P}} = \varnothing$.

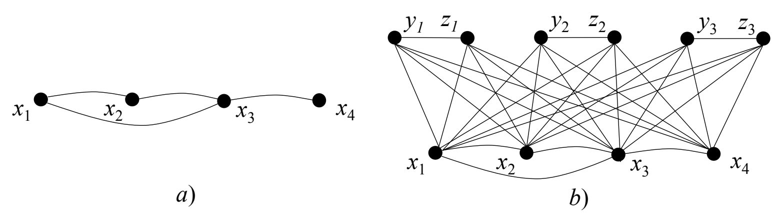

Figure 1. a) Graph G of CP problem with $p = 3$ b) Particular graph G' of MinCC problem obtained from G.

For each set $S \in \boldsymbol{\mathcal{P}}(G')$, if $S \backslash (Y \cup Z) \neq \varnothing$, we add set $S \backslash (Y \cup Z)$ to $\boldsymbol{\mathcal{P}}$. Then, by removing from $\boldsymbol{\mathcal{P}}$ all sets contained in the union of other sets of the family $\boldsymbol{\mathcal{P}}$ we obtain a convex k-cover $\boldsymbol{\mathcal{P}}'(G)$ of G, $2 \leq k \leq |\boldsymbol{\mathcal{P}}| \leq |\boldsymbol{\mathcal{P}}(G')|$. Note that if any graph H can be covered by k cliques and there exists a set S of this cover such that $|S| \geq 2$, then by removing every element of S from other cliques and by splitting S into two cliques, we obtain a cover of H by $k+1$ cliques. Thus, G can be covered by p cliques. It stands to reason that G can be partitioned into p cliques.

So, G can be partitioned into p disjoint cliques if and only if there exists a cover of G' by at most p convex sets. Thus, it is proved that the MinCC problem is NP-complete. □

In view of demonstration of the Theorem 1, we obtain the correctness of Corollaries 1, 2 and 3.

Corollary 1. *The MinCP problem is NP-complete.*

Corollary 2. *The MinNCC problem is NP-complete.*

Corollary 3. *The MinNCP problem is NP-complete.*

Determination of maximum nontrivial convex partition number $\theta_{cn}^{max}(G)$ has the following decision problem:

PROBLEM: Maximum nontrivial convex partition (MaxNCP)
INSTANCE: Graph $G = (X;\, U)$, integer p, $2 \leq p \leq |X|$.

QUESTION: Is there a nontrivial convex q-partition of G such that $q \geq p$?

The partition into triangles problem is defined as follows:

PROBLEM: Partition into triangles (PIT).
INSTANCE: A graph $G = (X;\, U)$ with $|X| = 3k$, where $k \in N$.
QUESTION: Is there a partition of X into k disjoint subsets $X_1, X_2, \dots, X_k$ of three vertices each such that the three possible edges between vertices of every X_i, $1 \leq i \leq k$, are in U?

We reduce PIT, the well known NP-complete problem [2], to MaxNCP problem. Therefore, we prove that MaxNCP problem is NP-complete too.

Theorem 2. *The MaxNCP problem is NP-complete.*

Proof. Notice that MaxNCP problem is in NP because verifying whether a set of vertices is convex can be done in polynomial time [6].

Let $G = (X;\, U)$ be an instance of PIT problem, $|X| = 3k$, $k \in N$. Firstly, we determine the structure of a particular graph $G' = (X';\, U')$ of MaxNCP problem that corresponds to G. We know that the PIT problem remains NP-complete even if the input graph G is tripartite [7]. Note also that every tripartite graph has no cliques with $r \geq 4$ vertices. Here and in the sequel we consider that G has no cliques with $r \geq 4$ vertices.

We construct the graph $G' = (X';\, U')$ as follows:

1) $X' = X \cup \{a, b, c, d, e, f\}$;

2) $U' = U \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, e\}, \{d, f\}\}$
$\cup \{\{a, x\}, \{b, x\} : x \in X\}$.

The MaxNCP instance is defined by the graph G' and a number $p = k + 2$. It is easy to see that G' can be constructed in polynomial time. We exhibit in Figure 2 how a particular graph G' of MaxNCP problem is obtained from a graph G of PIT problem.

We have to show that there exists a partition of X into triangles if and only if there exists a nontrivial convex partition of G' of size at least $k + 2$.

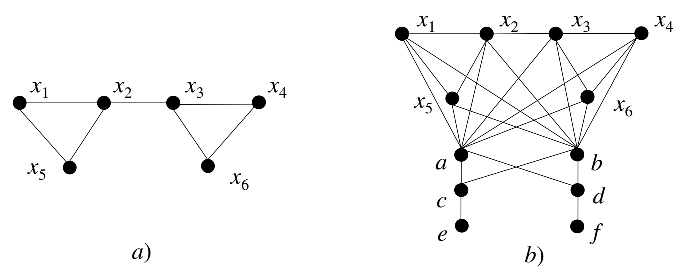

Figure 2. a) Graph G of PIT problem b) Particular graph G' of MaxNCP problem obtained from G.

Let $\mathcal{P}(G) = \{X_1, X_2, \ldots, X_k\}$ be a family of triangles that partitions G. Since every triangle is a clique in G, it follows that each X_i, $1 \leq i \leq k$, is nontrivial and convex in G' and the set $\{a, b, c, d, e, f\}$ remains uncovered in G'. Observe that $d-conv(\{a, c, e\}) = \{a, c, e\}$ and $d-conv(\{b, d, f\}) = \{b, d, f\}$. For this reason, the family of sets $\mathcal{P}(G) \cup \{\{a, c, e\}, \{b, d, f\}\}$ generates a partition of G' into $k+2$ nontrivial convex sets.

Let $\mathcal{P}(G')$ be a partition of G' into nontrivial convex sets and let S be a set of $\mathcal{P}(G')$. We distinguish some properties of S:

1) $\{a, b\} \not\subset S$. Assuming the contrary, namely that $\{a, b\} \subset S$, we see that $d-conv(\{a, b\}) = X \cup \{a, b, c, d\}$ and further we obtain $X' \backslash d-conv(\{a, b\}) = \{e, f\}$. Note that the set $\{e, f\}$ is not nontrivial and convex. Hence, $\mathcal{P}(G')$ cannot partition G' into nontrivial convex sets. We get the required contradiction.

2) $\{c, d\} \not\subset S$. Assuming the converse, $\{a, b\} \subset d-conv(\{c, d\})$. Therefore, the property 1) is not satisfied and we obtain a contradiction.

3) $\{e, f\} \not\subset S$. Conversely, we have $\{a, b, c, d\} \subset d-conv(\{e, f\})$ and consequently the properties 1) and 2) are not satisfied. This implies a contradiction.

4) $\{x, y\} \not\subset S$ for every vertex $x \in X$ and $y \in \{c, d\}$. Assuming the converse, there exist $x \in X$ and $y \in \{c, d\}$ such that $\{x, y\} \subset S$. Considering that vertices a and b belong to $d-conv(\{x, y\})$, we obtain

a contradiction.

5) $\{x,y\} \not\subset S$ for every two nonadjacent vertices $x,y \in X$. In the converse case, there are two nonadjacent vertices x and y of X for which $\{x,y\} \subset S$. And it follows that $\{a,b\} \subset d-conv(\{x,y\})$. Have a contradiction.

Let $S_1 = \{a,c,e\}$, $S_2 = \{b,d,f\}$, $S_3 = \{b,c,e\}$ and $S_4 = \{a,d,f\}$. Taking into account the properties 1) – 5) and the fact that each vertex of X' belongs exactly to one set of $\mathcal{P}(G')$, it is seen that $\mathcal{P}(G')$ contains strictly a pair of sets of the following two: S_1, S_2 or S_3, S_4. Each pair of sets covers vertices a,b,c,d,e and f. Hence, vertices of $X' \backslash \{a,b,c,d,e,f\}$ remain to be partitioned into nontrivial convex sets. By property 5), all of these sets are cliques. As mentioned above, G has no cliques with $r \geq 4$ vertices. Further, all of these sets are triangles and by elimination of a pair of sets S_1, S_2 or S_3, S_4 from $\mathcal{P}(G')$ we obtain a family of triangles $\mathcal{P}(G)$ that partitions G. $\mathcal{P}(G')$ contains exactly $k+2$ sets and thus if G has a nontrivial convex partition $\mathcal{P}(G')$ of size at leas $k+2$, then there exists a partition of X into triangles. $\square$

The decision problem for maximum nontrivial convex cover number $\varphi_{cn}^{max}(G)$ is formulated similarly to MaxNCP, but with different question: Is there a nontrivial convex q-cover of G such that $q \geq p$?

The 3-Satisfiability problem is defined as follows:

PROBLEM: 3-Satisfiability (3SAT).

INSTANCE: Given a boolean expression E in conjunctive normal form that is the conjunction of clauses, each of which is the disjunction of three distinct literals.

QUESTION: Is there a satisfying truth assignment for E?

Now we prove that MaxNCC problem is NP-complete. For this purpose, we reduce the 3SAT problem that is NP-complete [2] to MaxNCC problem.

Theorem 3. *The MaxNCC problem is NP-complete.*

Proof. The MaxNCC problem is in NP because verifying whether a set of vertices is convex can be done in polynomial time [6].

The following reduction from 3SAT to MaxNCC will establish that MaxNCC problem is NP-complete. Let E be an instance of the 3SAT problem with n variables $V_1, V_2, \ldots, V_n$ and m clauses $K_1, K_2, \ldots, K_m$. Given this instance, we construct a graph $G = (X; U)$ with $3n+m+16$ vertices. Vertices v_i, $\overline{v_i}$, y_i correspond to variable V_i, $1 \le i \le n$. One vertex k_j corresponds to clause K_j, $1 \le j \le m$. There are supplimentary vertices grouped in the four sets: $A = \{a, a_1, a_2, a_3\}$, $B = \{b, b_1, b_2, b_3\}$, $C = \{c, c_1, c_2, c_3\}$ and $\{d, e, f, h\}$. Denote by V, $\overline{V}$ and Y sets of all vertices v_i, $\overline{v_i}$ and respectively y_i, $1 \le i \le n$. By K we denote the set of all vertices k_j, $1 \le j \le m$.

The graph G has $12n + 6m + 27$ edges. The three vertices corresponding to each variable are connected by an edge (i.e., v_i, $\overline{v_i}$, y_i form a triangle). Each clause vertex is connected to its component terms (that is, the clause vertex k_l corresponding to the clause $\overline{V_i} \vee V_j \vee V_q$ is connected by edges to vertices $\overline{v_i}, v_j, v_q$). The three additional vertices a, b, c are connected to v_i, $\overline{v_i}$, y_i for all i, $1 \le i \le n$, and to k_j for all j, $1 \le j \le m$. The vertex d is connected by an edge to each vertex $r \in (A \cup B \cup \{h\}) \backslash \{a_1, b_1\}$, the vertex e is connected by an edge to each vertex $r \in (B \cup C \cup \{h\}) \backslash \{b_1, c_1\}$ and f is connected to each vertex $r \in (A \cup C \cup \{h\}) \backslash \{a_1, c_1\}$. Finally, a_1 is connected to a_2 and a_3, b_1 is connected to b_2 and b_3, c_1 is connected to c_2 and c_3.

The MaxNCC instance is defined by the graph G and a number $p = 2n + m + 3$. It should be clear that this construction of G can be done in polynomial time.

For example, consider the 3SAT instance:

$$E = (V_1 \vee V_2 \vee \overline{V_3}) \wedge (V_2 \vee V_3 \vee \overline{V_4}) \wedge (V_1 \vee \overline{V_2} \vee V_4).$$

Then the graph G corresponding to E is presented in Figure 3.

Without loss of generality, we consider that E has no clauses which contain a variable and its negation.

Let us distinguish some properties of $G = (X; U)$:

Property 1: For any two nonadjacent vertices $x, y \in V \cup \overline{V} \cup Y \cup K \cup \{a, b, c\}$, we have $d - conv(\{x, y\}) = X$.

Property 2: For any two vertices $x \in V \cup \overline{V} \cup Y \cup K$, $y \in (A \cup B \cup C \cup \{d, e, f, h\}) \backslash \{a, b, c\}$, we have $d - conv(\{x, y\}) = X$.

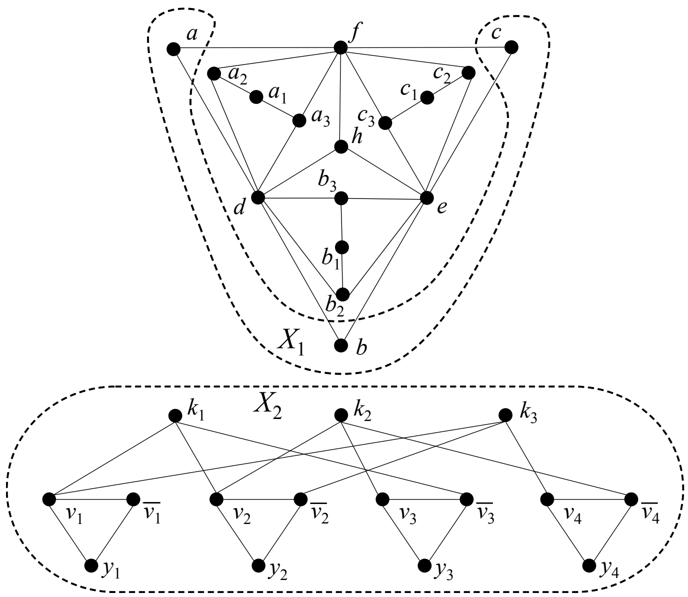

Figure 3. Graph G corresponds to the instance $E = (V_1 \vee V_2 \vee \overline{V_3}) \wedge (V_2 \vee V_3 \vee \overline{V_4}) \wedge (V_1 \vee \overline{V_2} \vee V_4)$. Every vertex of set X_1 is adjacent to every vertex of set X_2. Dashed line marks areas of sets X_1 and X_2.

Property 3: Let $\mathcal{P}(G)$ be a nontrivial convex cover or G. Then, $A^* = A \cup \{d, f, h\}$, $B^* = B \cup \{d, e, h\}$ and $C^* = C \cup \{e, f, h\}$ are in $\mathcal{P}(G)$.

Correctness of the first two properties can be easily verified. We will show that the Property 3 is correct too. Let $S \subset X$ be a nontrivial convex set that contains the vertex a_1. Since $|S| \geq 3$, the structure of G yields that S also contains vertices a_2 and a_3. Hence, we get the equality $d-conv(\{a_2, a_3\}) = A^*$. Note that for each $x \in X \backslash A^*$ we have $d - conv(A^* \cup \{x\}) = X$. Further, A^* is a unique nontrivial convex set, different from X, that contains a_1. For the same reason, B^* and C^* are unique nontrivial convex sets, different from X, which contain vertices b_1 and c_1, respectively.

We have to show that E is satisfiable if and only if there exists a nontrivial convex cover of G of size at least $2n+m+3$.

Assume E is satisfiable. Then there exists a truth assignment of variables such that all clauses evaluate to true. We will form a nontrivial convex cover of $\mathcal{P}(G)$ as follows. By the Property 3, $\mathcal{P}(G)$ includes the sets A^*, B^* and C^*. We denote by M the set of all vertices v_i for which V_i are true in the assignment, and all $\overline{v_i}$ for which V_i are false. $\mathcal{P}(G)$ includes sets $\{x, y, a\}$ for all $x \in M$, where $y \in Y$ and $y \sim x$. Also, $\mathcal{P}(G)$ includes sets $\{v_i, \overline{v_i}, a\}$ for each i, $1 \leq i \leq n$. Moreover, $\mathcal{P}(G)$ includes one set $\{x, k_j, a\}$ for each k_j, where $x \in M$, $x \sim k_j$, and the existence of such a vertex x yields from the fact that E is satisfiable. The obtained nontrivial convex cover $\mathcal{P}(G)$ contains exactly $2n+m+3$ sets.

Assume there exists a nontrivial convex cover $\mathcal{P}(G)$ of size at least $2n+m+3$. Taking into account the Property 3, $\mathcal{P}(G)$ already includes the sets A^*, B^* and C^*. In view of the Property 2, it remains to analyze a nontrivial convex cover $\mathcal{P}(G')$ resulted from $\mathcal{P}(G)$ after elimination of sets A^*, B^* and C^*, where $G' = G[V \cup \overline{V} \cup Y \cup K \cup \{a, b, c\}]$. Clearly, $|\mathcal{P}(G)| = |\mathcal{P}(G')| + 3$. It follows from the Property 1 and the structure of G that every set $S \in \mathcal{P}(G')$ is a clique, $3 \leq |S| \leq 4$, and it can be classified into one of three types:

(i) $S = \{x, y, z\}$, where $x, y \in V \cup \overline{V} \cup Y \cup K$, $x \sim y$, and $z \in \{a, b, c\}$;
(ii) $S = \{v_i, \overline{v_i}, y_i\}$ for any i, $1 \leq i \leq n$;
(iii) $S = \{v_i, \overline{v_i}, y_i, x\}$ for any i, $1 \leq i \leq n$, where $x \in \{a, b, c\}$.

Now we define a family of convex sets $\mathcal{P}(G'') = \varnothing$, obtained from $\mathcal{P}(G')$, that will cover $G'' = G'[V \cup \overline{V} \cup Y \cup K]$.

We examine each set $S \in \mathcal{P}(G')$ and consider two cases.

1) If S is of the first type (i), then we add set $S \backslash \{a, b, c\}$ to $\mathcal{P}(G'')$.

2) If S is of the second (ii) or the third (iii) type, then taking into account the fact that vertices a, b and c are already covered by sets A^*, B^* and C^*, we analyze two options. Suppose S contains only one resident vertex r in $\mathcal{P}(G')$. In this case, we add set $\{r, x\}$ to $\mathcal{P}(G'')$, where $x \in S \backslash \{a, b, c\}$, $x \neq r$. Suppose S contains at least two resident vertices r_1, r_2 in $\mathcal{P}(G')$. Then, we add sets $\{r_1, x\}$ and $\{r_2, x\}$ to $\mathcal{P}(G'')$,

where $x \in S \backslash \{a, b, c\}$, $x \neq r_1$, $x \neq r_2$.

Let us remark that for a set $S \in \mathcal{P}(G')$ of the type (ii) or (iii), any set $S' \in \mathcal{P}(G')$, $S' \neq S$, $S' \cap S \backslash \{a, b, c\} \neq \varnothing$, is of the first type (i), and thus there are no uncovered vertices in G'', i.e. $\mathcal{P}(G'')$ is a convex cover of G''. It is obvious that $|\mathcal{P}(G'')| \geq |\mathcal{P}(G')|$ and furthermore $|\mathcal{P}(G'')| \geq |\mathcal{P}(G)| - 3 = 2n + m$. We need to take a closer look at the family $\mathcal{P}(G'')$. Every set of $\mathcal{P}(G'')$ has exactly two adjacent vertices. We choose one resident vertex of each set $S \in \mathcal{P}(G'')$ and form the set W as a union of these vertices. It is clear that $D = X(G'') \backslash W$ is a dominating set of G'' such that $|\mathcal{P}(G'')| + |D| = |V \cup \overline{V} \cup Y \cup K| = 3n + m$. If we combine this with the previous inequality, we get $|D| \leq n$.

Consider the graph G''. For any i, $1 \leq i \leq n$, y_i is either in D or adjacent to a vertex of D, and y_i is connected by edges only to v_i and $\overline{v_i}$. It follows that for every i, $1 \leq i \leq n$, either v_i, $\overline{v_i}$, or y_i is in D. This already specifies n vertices, so exactly one vertex for each variable is included in D. We create a truth assignment as follows. V_i will be assigned to true if v_i is in D. Otherwise V_i will be assigned to false. Consider clause K_j. The vertex k_j is not in the dominating set. So k_j is adjacent to some v_i or $\overline{v_l}$ in the dominating set. If k_j is adjacent to v_i in D, then since V_i is set to true, it follows that k_j will be true. If k_j is adjacent to $\overline{v_l}$ in D, then v_l is not in D and V_l will be false, so K_j will be true. It follows that this assignment is a solution for E and E is satisfiable. Thus, if G has a nontrivial convex cover $\mathcal{P}(G)$ of size at least $2n + m + 3$, then E is satisfiable.

So, this completes the proof of the correctness of the reduction and we conclude that MaxNCC is NP-complete. $\square$

3 Conclusion

We have proved that MinCC, MinCP, MinNCC, MinNCP, MaxNCC and MaxNCP problems are NP-complete. This yields that the problems of determining the values of the invariants $\varphi_c^{min}(G)$, $\theta_c^{min}(G)$, $\varphi_{cn}^{min}(G)$, $\theta_{cn}^{min}(G)$, $\varphi_{cn}^{max}(G)$ and $\theta_{cn}^{max}(G)$ for a general graph G are NP-hard.

Of course, it is of interest to develop approximate algorithms, heuristics, and establish other classes of graphs for which the above mentioned invariants can be determined in polynomial time. All these are issues for further research.

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