BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(93), 2020, Pages 76–87
ISSN 1024–7696

Laurent-Padé approximation for locating singularities of meromorphic functions with values given on simple closed contours

Capcelea Maria, Capcelea Titu

Abstract. In the present paper the Padé approximation with Laurent polynomials is examined for a meromorphic function on a finite domain of the complex plane. Values of the function are given at the points of a simple closed contour from this domain. Based on this approximation, an efficient numerical algorithm for locating singular points of the function is proposed.

Mathematics subject classification: 65E05, 41A21.
Keywords and phrases: Padé approximation, meromorphic function, simple closed contour, localization of singular points .

1 Introduction and problem formulation

Let consider a meromorphic function $f(z)$ defined on a finite domain $\Omega \subset \mathbb{C}$ that contains a simple closed contour Γ. The domain within the contour Γ is denoted by Ω^+, whilst the complementary domain to $\Omega^+ \cup \Gamma$ is denoted by $\Omega^- := \bar{\mathbb{C}} \setminus \{\Omega^+ \cup \Gamma\}$, where $\bar{\mathbb{C}}$ is the complex plane extended with the point at infinity. We consider that the point $z = 0 \in \Omega^+$.

According to the Riemann mapping theorem there exists a conformal map $z = \psi(w)$ of the domain $D^- := \{w \in \mathbb{C} : \ |w| > 1\}$ onto Ω^- such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. The function $\psi(w)$ transforms the unit circle $\Gamma_0 := \{w \in \mathbb{C} : \ |w| = 1\}$ onto the contour Γ. Next, we consider that the points of the contour Γ are defined by means of the Riemann function $\psi(w)$.

The function $f(z)$ admits a finite number of singular points of polar type on the domain Ω, but their number and locations are not known. Also, on the contour Γ the function $f(z)$ can have both poles and jump discontinuity points (see Figure 1). Considering that the finite values $f_j := f(t_j)$ of the function $f(z)$ are known at the points $t_j \in \Gamma$ and that these values form a dense set on Γ, we aim to determine the locations of the singular points on Ω (including those on the contour Γ) for the function $f(z)$.

The approach applied here for singular points determination on Ω is based on the fact that the Padé approximation of a meromorphic function $f(z)$, defined on the domain Ω, allows to locate the poles of $f(z)$ on Ω. According to Montessus de Ballore's theorem [1,2], the poles of the sequence of Padé approximations to $f(z)$ form convergent sequences to the corresponding poles of the function $f(z)$ on Ω. But it is necessary to keep in mind that although the mentioned theorem theoretically ensures pointwise convergence, numerically this convergence does not take place due to the influence of rounding errors.

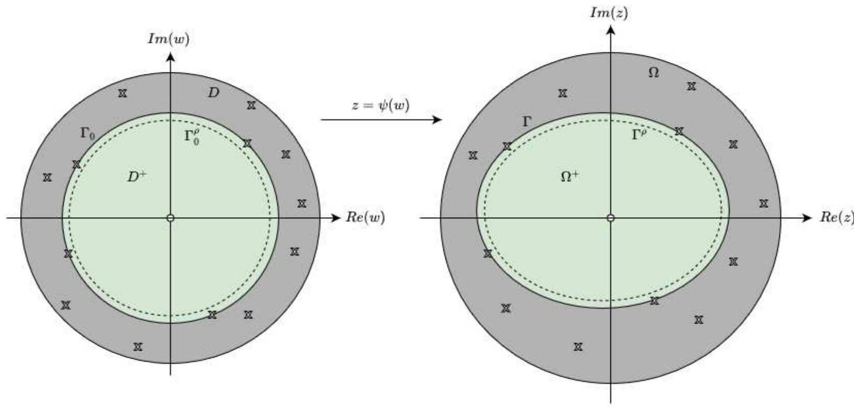

Figure 1: Data representation for the considered problem

Since jump discontinuities can be considered as apparent singularities (or "zero" order poles), they can be detected according to the same approach of searching for the poles of $f(z)$.

In Section 2 we set out the theoretical basis for the algorithm of locating the singular points of the function, based on the Laurent-Padé approximation of $f(z)$. In Section 3 we examine a formula for numerical approximation of the Laurent coefficients from the Padé approximation, and in Section 4 an algorithm for evaluating the number of poles of f on $\Omega^+ \cup \Gamma$ and $\Omega^- \cup \Gamma$. In Section 5 we consider some numerical examples that confirm the efficiency of the proposed algorithm and simultaneously show some of its numerical difficulties.

2 Detection of singularities based on Laurent-Padé approximation

Let consider that the function $f(z)$ is analytic in the annulus $A := \{z \in \mathbb{C} : r < |z| < R\}$, $r > 0$, $R < \infty$. Then the Laurent series of the function $f(z)$, $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$, can be represented in the form

$$f(z) = f^+(z) + f^-(z), \; z \in A, \tag{1}$$

where the function $f^+(z) = c_0/2 + \sum_{k=1}^{\infty} c_k z^k$ is analytic on the set $\{z \in \mathbb{C} : \; |z| < R\}$ and $f^-(z) = c_0/2 + \sum_{k=-\infty}^{-1} c_k z^k$ is analytic on $\{z \in \mathbb{C} : \; |z| > r\}$.

Next, for $M, N \in \mathrm{N}$ such that $N \geq M$, we consider the Laurent-Padé approximation of order (N, M) for $f(z)$ [1], that is defined as the sum of two classical Padé approximations [2] for $f^+(z)$ at $z = 0$ and, respectively, for $f^-(z)$ at $z = \infty$. Thus, two series are approximated: one is a power series in z and the other is a power series in $1/z$. Then the obtained approximations are summed.

Let $R^{+}_{(N_1,M_1)}(z) = P^{+}_{N_1}(z) \Big/ Q^{+}_{M_1}(z)$ be the Padé approximation of order (N_1, M_1), $N_1 \geq M_1$ for the function $f^{+}(z)$ at $z = 0$ that satisfies the following condition

$$Q^{+}_{M_1}(z)\, f^{+}(z) - P^{+}_{N_1}(z) = \mathcal{O}\left(z^{M_1+N_1+1}\right). \tag{2}$$

The polynomials $P^{+}_{N_1}(z)$ and $Q^{+}_{M_1}(z)$ are of the form $P^{+}_{N_1}(z) = \sum_{k=0}^{N_1} p^{+}_k z^k$, $Q^{+}_{M_1}(z) = \sum_{j=0}^{M_1} q^{+}_j z^j$ and we consider that they do not have common zeros. In order to avoid division by zero in the relation for $R^{+}_{(N_1,M_1)}(z)$, the normalization condition $q^{+}_0 = 1$ is imposed.

Analogously we define the (N_2, M_2) - order Padé approximation for the function $f^{-}(z)$ at $z = \infty$, $R^{-}_{(N_2,M_2)}(z) = P^{-}_{N_2}(1/z) \Big/ Q^{-}_{M_2}(1/z)$, where $P^{-}_{N_2}(1/z) = \sum_{k=0}^{N_2} p^{-}_k (1/z)^k$, $Q^{-}_{M_2}(1/z) = \sum_{j=0}^{M_2} q^{-}_j (1/z)^j$, $q^{-}_0 = 1$, and the coefficients of the approximation are determined from the following condition

$$Q^{-}_{M_2}(1/z)\, f^{-}(z) - P^{-}_{N_2}(1/z) = \mathcal{O}\left(z^{-(M_2+N_2+1)}\right). \tag{3}$$

The Laurent-Padé approximation of order (N, M) for $f(z)$ is defined as follows:

$$R_{(N,M)}(z) := R^{+}_{(N_1,M_1)}(z) + R^{-}_{(N_2,M_2)}(z) = P_{N_1,N_2}(z) / Q_{M_1,M_2}(z),$$

where $P_{N_1,N_2}(z) := P^{+}_{N_1}(z)\, Q^{-}_{M_2}(1/z) + P^{-}_{N_2}(1/z)\, Q^{+}_{M_1}(z)$ and $Q_{M_1,M_2}(z) := Q^{+}_{M_1}(z)\, Q^{-}_{M_2}(1/z)$ are Laurent polynomials of the form $\sum_{j=-N_2}^{N_1} p_j z^j$ and, respectively, $\sum_{j=-M_2}^{M_1} q_j z^j$, and $N = \max(N_1, N_2)$, $M = \max(M_1, M_2)$. Taking into account the relation for $R_{(N,M)}(z)$ we can deduce that the zeros of $Q^{+}_{M_1}(z)$ and $Q^{-}_{M_2}(1/z)$ are approximations of the singular points of the function $f(z)$.

Let the function $f(z)$ allow a meromorphic extension on the annulus $\tilde{A} := \left\{z \in \mathbb{C} : \ \tilde{r} < |z| < \tilde{R}\right\}$ so that the poles of $f(z)$ belong to the set $\tilde{A}$, more precisely to the annulus $A_1 := \left\{z \in \mathbb{C} : \ R \leq |z| < \tilde{R}\right\}$ and $A_2 := \{z \in \mathbb{C} : \ \tilde{r} < |z| \leq r\}$ (see Figure 2). Taken into account the relation (1), we can conclude that the function $f^{+}(z)$ admits a meromorphic extension on the set A_1 because $f^{-}(z)$ is analytic on $\{z \in \mathbb{C} : \ |z| > r\}$. Analogously it can be shown that the function $f^{-}(z)$ admits a meromorphic extension on the set A_2.

Based on the Montessus de Ballore theorem it can be shown that the condition $\lim\limits_{N_1 \to \infty} R^{+}_{(N_1,M_1)}(z) = f^{+}(z)$ is satisfied, that means that the Padé approximation $R^{+}_{(N_1,M_1)}(z)$ converges to $f^{+}(z)$ for a sufficiently large and fixed value M_1 and $N_1 \to \infty$. Convergence is locally uniform over the set $\left\{z \in \mathbb{C} : \ |z| < \tilde{R}\right\} \setminus \left\{z^{+}_j\right\}_{j=1}^{k^{+}}$, where z^{+}_j, $j = 1, ..., k^{+}$, are the poles of the function $f^{+}(z)$ of multiplicities l^{+}_j correspondingly. Moreover, the condition $\lim\limits_{N_1 \to \infty} Q^{+}_{M_1}(z) = \prod_{j=1}^{k^{+}} \left(1 - z \big/ z^{+}_j\right)^{l^{+}_j}$ is satisfied. Analogously it can be shown that $\lim\limits_{N_2 \to \infty} R^{-}_{(N_2,M_2)}(z) = f^{-}(z)$ and

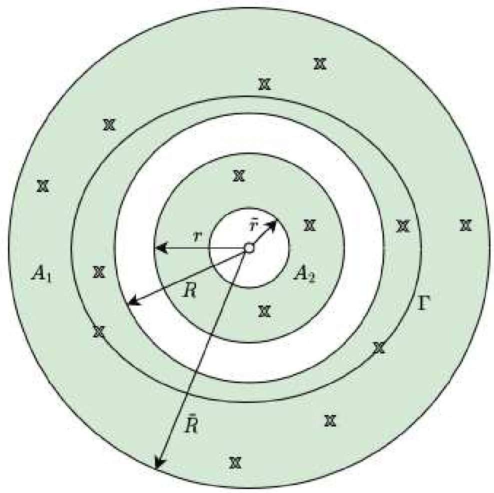

Figure 2: Regions of meromorphic extension for the function f

$\lim\limits_{N_2\to\infty} Q^{-}_{M_2}(1/z) = \prod_{j=1}^{k^-}\left(1 - z_j^- \big/ z\right)^{l_j^-}$, where z_j^-, $j = 1, ..., k^-$, are the poles of the function $f^-(z)$ of multiplicities l_j^- correspondingly. The zeros of polynomials $Q^+_{M_1}(z)$ and $Q^-_{M_2}(1/z)$ are approximations of the poles of the function $f(z)$. Thus, the zeros of the polynomial $Q^+_{M_1}(z)$ approximate the poles from the domain $\Omega^- \cup \Gamma$ and the zeros of $Q^-_{M_2}(1/z)$ - the poles from $\Omega^+ \cup \Gamma$.

The relations (2) and (3) can be used to determine the coefficients p_k^+, q_j^+, $k = 0, 1, ..., N_1$, $j = 0, 1, ..., M_1$, and p_r^-, q_m^-, $r = 0, 1, ..., N_2$, $m = 0, 1, ..., M_2$, of the polynomials $P^+_{N_1}(z), Q^+_{M_1}(z)$ and $P^-_{N_2}(1/z), Q^-_{M_2}(1/z)$, correspondingly. Based on the relation (2) we obtain the system of linear algebraic equations

$$\sum_{j=0}^{k-1} c_{k-j}q_j^+ + 0.5c_0q_k^+ - p_k^+ = 0,\ k = 0, 1, ..., M_1,$$

$$\sum_{j=0}^{M_1} c_{k-j}q_j^+ - p_k^+ = 0,\ k = M_1 + 1, ..., N_1,$$

$$\sum_{j=0}^{M_1} c_{k-j}q_j^+ = 0,\ k = N_1 + 1, ..., N_1 + M_1.$$

The coefficients q_j^+, $j = 1, ..., M_1$, of the polynomial $Q^+_{M_1}(z)$ are determined using only the relations $\sum_{j=0}^{M_1} c_{k-j}q_j^+ = 0$, $k = N_1 + 1, ..., N_1 + M_1$, that define a system of M_1 linear algebraic equations with $M_1 + 1$ unknowns and a Toeplitz matrix as a coefficient matrix of the system. Taking into account the normalization condition

$q_0^+ = 1$, the relation that defines the system is written as

$$\sum_{j=1}^{M_1} c_{k-j} q_j^+ = -c_k,\ k = N_1+1, ..., N_1+M_1. \tag{4}$$

The system (4) has a unique solution.

Analogously from condition (3) we obtain the system of linear algebraic equations

$$\sum_{j=1}^{M_2} c_{j-k} q_j^- = -c_{-k},\ k = N_2+1, ..., N_2+M_2. \tag{5}$$

The components q_j^-, $j = 1, ..., M_2$, of the solution of the system (5) (together with $q_0^- = 1$) are the coefficients of the polynomial $Q_{M_2}^-(1/z)$. The zeros of $Q_{M_2}^-(1/z)$ are obtained by inverting the zeros of the polynomial $Q_{M_2}^-(t) = \sum_{j=0}^{M_2} q_j^- t^j$.

In order to find the solutions of the systems (4) and (5) it is necessary firstly to determine the Laurent coefficients $c_{N_1-M_1+1}, ..., c_{N_1+M_1}$ and $c_{-(N_2+M_2)}, ..., c_{-(N_2-M_2+1)}$.

3 Numerical approximation of Laurent coefficients

In this section, we examine a formula for numerical approximation of Laurent coefficients $c_k = \frac{1}{2\pi i}\int_\Gamma f(z)\, z^{-k-1} dz$, where Γ is a simple closed contour that satisfies the conditions mentioned in Section 1. The values f_j of the function $f(z)$ on the contour Γ are used for the calculation of the Laurent coefficients that define the coefficient matrices of the systems (4) and (5).

Since the function $f(z)$ can have a finite number of poles on the contour Γ, we insignificantly disturb the contour and consider that the values f_j of the continuous function $f(z)$ on the disturbed contour Γ^ρ are known. Since the contour Γ is defined by the Riemann function $\psi(w)$, that transforms the unit circle $\Gamma_0 = \{w \in \mathbb{C} : |w| = 1\}$ onto the contour Γ, we define the perturbed contour Γ^ρ as follows:

$$\Gamma^\rho := \{t \in \mathbb{C} :\ t = \psi(w),\, w \in \Gamma_0^\rho\},$$

where $\Gamma_0^\rho := \{w \in \mathbb{C} :\ |w| = \rho,\, \rho = 1 \pm \varepsilon,\, \varepsilon > 0\}$. We consider small values for ε, for example, $\varepsilon = 0.001$.

The Laurent coefficient c_k is approximated as follows:

$$c_k \approx \frac{1}{2\pi i}\int_{\Gamma^\rho} f(z)\, z^{-k-1} dz =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_0^\rho} (\psi(w))^{-k-1} f(\psi(w))\, \psi'(w)\, dw = \frac{1}{2\pi i}\int_{\Gamma_0^\rho} g(w)\, dw,$$

where $g(w) := (\psi(w))^{-k-1} f(\psi(w))\, \psi'(w),\ w \in \Gamma_0^\rho$.

We approximate the function $g(w)$ with Lagrange interpolation polynomial

$$(L_n g)(w) = \sum_{s=-n}^{n} \Lambda_s w^s, \ \Lambda_s = \frac{1}{2n+1} \sum_{j=0}^{2n} g(w_j) w_j^{-s}, \ s = -n, -n+1, ..., n,$$

defined on the uniform set of nodes on Γ_0^ρ, $w_j = \rho e^{(2\pi j/(2n+1))i}$, $i^2 = -1$, $j = 0, 1, ..., 2n$. Then, taking into account the relation $\frac{1}{2\pi i} \int_{\Gamma_0^\rho} w^s dw = \begin{cases} 0, & for\ s \neq -1 \\ 1, & for\ s = -1 \end{cases}$, we can write the approximation for the coefficient c_k as follows:

$$c_k \approx \frac{1}{2\pi i} \int_{\Gamma_0^\rho} g(w)\, dw \approx c_k^{(n)} := \frac{1}{2\pi i} \int_{\Gamma_0^\rho} \sum_{s=-n}^{n} \Lambda_s w^s dw =$$

$$= \sum_{s=-n}^{n} \Lambda_s \frac{1}{2\pi i} \int_{\Gamma_0^\rho} w^s dw = \Lambda_{-1} = \frac{1}{2n+1} \sum_{j=0}^{2n} g(w_j)\, w_j.$$

If we apply the approach presented in [3] for estimating the error of approximation of the coefficient c_k by $c_k^{(n)}$, then it can be shown that when $n \to \infty$ the sequence of approximations $c_k^{(n)}$ converges to c_k with rate of a geometric progression.

4 Evaluating the number of poles

The solutions q_j^+, $j = 0, 1, ..., M_1$, and q_j^-, $j = 0, 1, ..., M_2$, of the systems of equations (4) and, respectively, (5), are the coefficients of the polynomials $Q_{M_1}^+(z)$ and, respectively, $Q_{M_2}^-(1/z)$, whose zeros are considered as approximations of the poles of the function $f(z)$. Thus, the algorithm based on the Laurent-Padé approximation generates $M_1 + M_2$ approximations of the poles of the function $f(z)$.

The zeros of the polynomial $Q_{M_1}^+(z)$ are approximations of the poles that belong to the domain $\Omega^+ \cup \Gamma$ and the zeros of $Q_{M_2}^-(1/z)$ are approximations of the poles that belong to $\Omega^- \cup \Gamma$. In order the Laurent-Padé approximation algorithm to be applied efficiently, it is necessary to know in advance the number of poles of $f(z)$ belonging to $\Omega^+ \cup \Gamma$ and $\Omega^- \cup \Gamma$, respectively. We denote these values by m_1 and m_2, respectively. Next, we present an approach that allows evaluation of values m_1 and m_2.

Consider the matrices of the systems of equations (4) and (5), that have the orders $M_1 \times M_1$ and $M_2 \times M_2$, respectively, i.e.

$$C_1 = \begin{pmatrix} c_{N_1} & c_{N_1-1} & \cdots & c_{N_1-(M_1-1)} \\ c_{N_1+1} & c_{N_1} & \cdots & c_{N_1-(M_1-2)} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N_1+M_1-1} & c_{N_1+M_1-2} & \cdots & c_{N_1} \end{pmatrix}$$

and

$$C_2 = \begin{pmatrix} c_{-N_2} & c_{-(N_2-1)} & \cdots & c_{-(N_2-(M_2-1))} \\ c_{-(N_2+1)} & c_{-N_2} & \cdots & c_{-(N_2-(M_2-2))} \\ \vdots & \vdots & \ddots & \vdots \\ c_{-(N_2+M_2-1)} & c_{-(N_2+M_2-2)} & \cdots & c_{-N_2} \end{pmatrix}.$$

The idea on which the mentioned approach is built is that the determinants of the matrices C_1 and C_2 are non-zero when $M_1 = m_1$ and $M_2 = m_2$, respectively, and the determinants of higher order matrices (which are obtained by adding rows and columns) have close to zero values.

Thus, we define a parameter $m > 0$ that means an estimate for the maximum value of the number of poles for $f(z)$ on $\Omega^+ \cup \Gamma$, $m_1 \le m$ (respectively, $m_2 \le m$ on $\Omega^- \cup \Gamma$). If for the initial value m we have nonzero determinant, then we consider a higher value for m. For values $m, m-1, m-2, ...$ we calculate the determinants of the Toeplitz matrices C_1 (respectively, C_2) of order $m \times m$, until we have $\det C_1 \neq 0$ (respectively, $\det C_2 \neq 0$). Obviously, a sufficiently small parameter $\delta > 0$ must be used to evaluate the last condition, for example, $\delta = 0.001$. The first value of the parameter m for which $\det C_1 \neq 0$ (respectively, $\det C_2 \neq 0$), represents an evaluation of the number of poles on $\Omega^+ \cup \Gamma$ (respectively, on $\Omega^- \cup \Gamma$).

Numerical experiments show that the accuracy of the result depends on the value of the parameters $N_1 \ge m$ $(N_2 \ge m)$ and δ as well as on the presence of multiple poles and discontinuity points.

5 Numerical examples

In this section we analyze some numerical examples that confirm the correctness of the examined localization algorithm and provide us with necessary data to analyze its efficiency. Below we examine four test functions with singularities that highlight the possibilities of the algorithm:

1. the case when $f(z)$ has only simple poles on Ω;
2. the case when $f(z)$ has multiple poles on Ω;
3. the case when $f(z)$ has points of jump discontinuity on Ω;
4. the case when $f(z)$ has both poles and discontinuity points on Ω.

In our examples we consider that the Riemann function $z = \psi(w)$ that performs the conformal transformation of the set $\{w \in \mathbb{C} \mid |w| > 1\}$ on the domain Ω^- from the outside of the contour Γ is $\psi(w) = w + 1/\left(3w^3\right)$. Thus, $\psi(w)$ transforms the circle Γ_0 onto the astroid Γ.

We consider that the values f_j of the examined function $f(z)$ are given at the points

$$z_j = \psi\left(\rho e^{i\theta_j}\right) \in \Gamma^\rho,\ \theta_j = 2\pi j/m,\ m \in \mathbb{N},\ j = 0, 1, ..., m,$$

where $\rho = 1$.

The following notations are used in the below graphical representations:
$\square$ is location of the pole;
$\diamond$ ($\times$) is the approximation of the pole determined by the Padé approximation of $f^+(z)$ ($f^-(z)$);
$\circ$ is jump discontinuity point on the contour Γ.

The following parameter values are considered in the algorithm that evaluates the number of singular points $N_1 = m+1$, $N_2 = m+1$, $\delta = 10^{-3}$, where m changes starting with the value $m^* = 10$, i.e. $m = 10, 9, 8, \ldots.$

Example 1. The function of a complex variable $f(z)$ is defined as follows:

$$f(z) = \frac{\cos\left(z^2\right)}{(z - zi_1)(z - zi_2)(z - ze_1)(z - ze_2)(z - zc_1)(z - zc_2)(z - zc_3)},$$

where

$$zc_1 = \psi\left(e^{\pi i/4}\right), \ zc_2 = \psi\left(e^{6\pi i/7}\right), \ zc_3 = \psi\left(e^{9\pi i/7}\right),$$

$$zi_1 = 0.3 + 0.1i, \ zi_2 = -0.4 - 0.2i, \ ze_1 = \psi(1.3 + 0.7i), \ ze_2 = \psi(-1.5 - 0.75i),$$

there are seven simple poles (two on Ω^-, three on the contour Γ and two on Ω^+) of $f(z)$, $i^2 = -1$.

The algorithm for evaluating the number of singular points of $f(z)$ returns $m_1 = 5$ for the number of poles on $\Omega^+ \cup \Gamma$ and $m_2 = 6$ for the number of poles on $\Omega^- \cup \Gamma$.

For the function $f^+(z)$ we apply the Padé approximation of order $(N_1, M_1) = (6, 5)$ at $z = 0$, and for $f^-(z)$ - the Padé approximation of order $(N_2, M_2) = (5, 5)$ at $z = \infty$. The approximations for the poles obtained by examined algorithm are presented in Table 1 and Figure 3.

Poles of the function $f(z)$		*Approximations of the poles of $f^+(z)$*	*Approximations of the poles of $f^-(z)$*
Poles from Ω^-	1.3092 + 0.5968i	1.3100 + 0.5907i	—
	-1.5126 - 0.6805i	-1.5164 - 0.6818i	—
Poles on Γ	0.4714 + 0.4714i	0.4714 + 0.4714i	0.4714 + 0.4714i
	-0.9751 + 0.1089i	-0.9753 + 0.1091i	-0.9751 + 0.1089i
	-0.3232 - 0.6372i	-0.3232 - 0.6372i	-0.3232 - 0.6372i
Poles from Ω^+	0.3000 + 0.1000i	—	0.3000 + 0.1000i
	-0.4000 - 0.2000i	—	-0.4000 - 0.2000i

Table 1: The approximations obtained in Example 1

We can see that if the function $f(z)$ has only simple poles on Ω, then the localization algorithm converges rapidly, requiring values for N_1, M_1, N_2, M_2 close to the number of poles of $f(z)$.

Example 2. The function of a complex variable $f(z)$ is defined as follows:

$$f(z) = \frac{\cos\left(z^2\right)}{(z - zi_1)^3 (z - zi_2)(z - ze_1)^2 (z - ze_2)(z - zc_1)(z - zc_2)^2 (z - zc_3)},$$

where

$$zc_1 = \psi\left(e^{\pi i/4}\right), \ zc_2 = \psi\left(e^{6\pi i/7}\right), \ zc_3 = \psi\left(e^{9\pi i/7}\right)$$

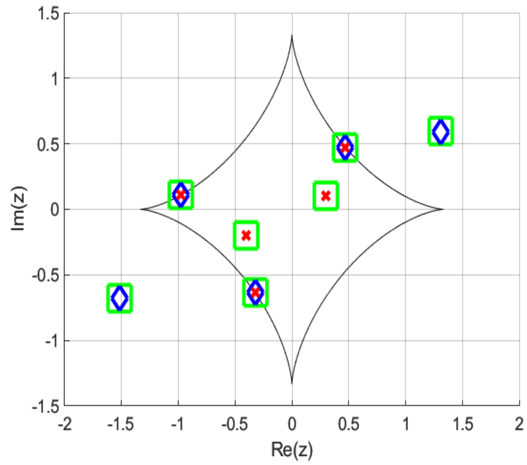

Figure 3: Approximations of the poles in Example 1

$$zi_1 = 0.3 + 0.1i,\ zi_2 = -0.4 - 0.2i,\ ze_1 = \psi\left(1.3 + 0.7i\right),\ ze_2 = \psi\left(-1.5 - 0.75i\right),$$

there are seven poles (two on Ω^-, three on the contour Γ and two on Ω^+) of $f(z)$, $i^2 = -1$. The poles ze_1 and zc_2 are of the second order, zi_1 is of the third order, and the other poles are simple.

The algorithm for evaluating the number of singular points of $f(z)$ returns $m_1 = 7$ for the number of poles on $\Omega^+ \cup \Gamma$ and $m_2 = 7$ for the number of poles on $\Omega^- \cup \Gamma$. Thus, the values m_1 and m_2 returned by the algorithm also include the multiplicities of the poles.

For the function $f^+(z)$ we apply the Padé approximation of order $(N_1, M_1) = (8, 7)$ at $z = 0$, and for $f^-(z)$ – the Padé approximation of order $(N_2, M_2) = (8, 8)$ at $z = \infty$. The approximations for the poles obtained by examined algorithm are presented in Table 2 and Figure 4.

Poles of the function $f(z)$		*Approximations of the poles of $f^+(z)$*	*Approximations of the poles of $f^-(z)$*
Poles from Ω^-	1.3092 + 0.5968i	1.3008 + 0.6365i	—
		1.3044 + 0.5583i	—
	-1.5126 - 0.6805i	-1.5029 - 0.6807i	—
Poles on Γ	0.4714 + 0.4714i	0.4714 + 0.4714i	0.4714 + 0.4714i
	-0.9751 + 0.1089i	-0.9751 + 0.1090i	-0.9751 + 0.1089i
		-0.9314 + 0.1182i	-0.9751 + 0.1089i
	-0.3232 - 0.6372i	-0.3232 - 0.6372i	-0.3232 - 0.6372i
Poles from Ω^+	0.3000 + 0.1000i	—	0.2997 + 0.0999i
		—	0.3000 + 0.1003i
		—	0.3002 + 0.0998i
	-0.4000 - 0.2000i	—	-0.4000 - 0.2000i

Table 2: The approximations obtained in Example 2

For each multiple pole a number of approximations is generated corresponding to its order and this requires the Padé approximations to have higher orders.

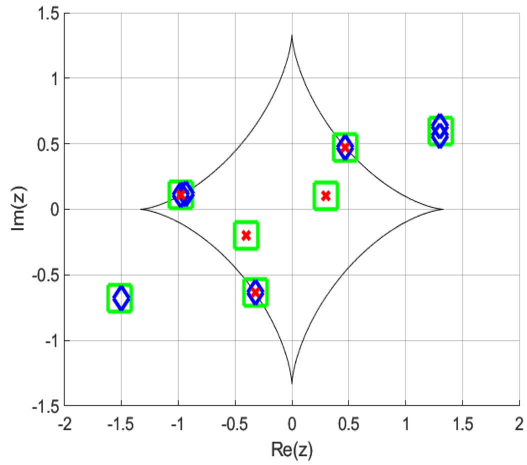

Figure 4: Approximations of the poles in Example 2

Example 3. The function of a complex variable $f(z)$ is defined as follows:

$$f(z) = \begin{cases} z^2, & for \ \ \theta \in [\zeta_2, \zeta_1] \\ \cos\left(z^2\right), & for \ \ \theta \in (\zeta_1, 2\pi + \zeta_2) \end{cases},$$

where $\zeta_1 = 7\pi/4, \zeta_2 = 7\pi/10$. We can see that $f(z)$ has two points of jump discontinuity on Γ (see Figure 5).

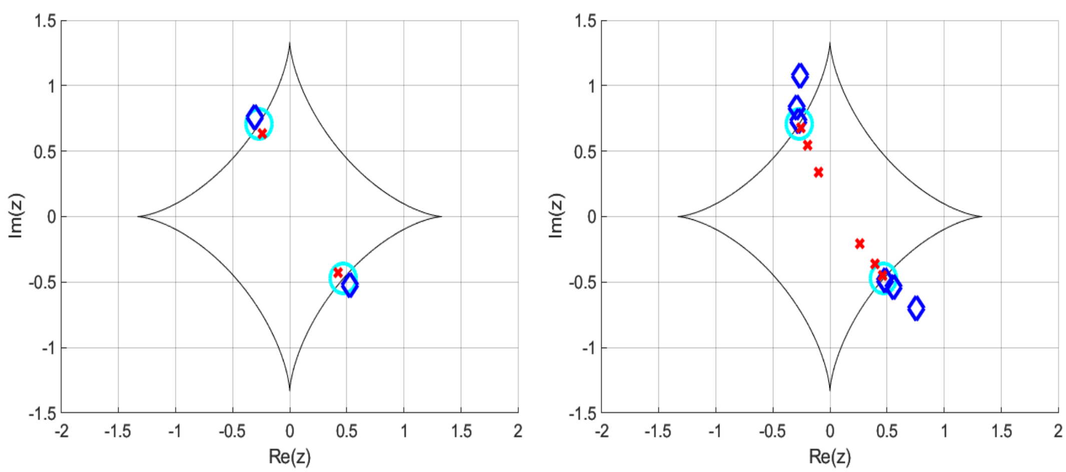

Figure 5: Approximations of the discontinuity points in Example 3

The algorithm for evaluating the number of singular points of $f(z)$ returns $m_1 = 2$ for the number of poles on $\Omega^+ \cup \Gamma$ and $m_2 = 5$ for the number of poles on $\Omega^- \cup \Gamma$. So, we observe that the algorithm gives wrong estimates in the case when the function has discontinuity points.

We apply Padé approximation of order $(N_1, M_1) = (10, 2)$ to the function $f^+(z)$ and Padé approximation of order $(N_2, M_2) = (8, 2)$ to $f^-(z)$. The approximations obtained for the discontinuity points of the function $f(z)$ on Γ are presented in Figure 5 (left).

The approximations for $(N_1, M_1) = (16, 6)$ and $(N_2, M_2) = (8, 6)$ are presented in Figure 5 (right). It can be seen that in the neighborhood of each discontinuity point a convergent sequence of approximations is generated.

Example 4. The function of a complex variable $f(z)$ is defined as follows:

$$f(z) = \begin{cases} \frac{z^2}{(z-zi_2)(z-ze_2)(z-zc_2)(z-zc_3)}, & for \ \ \theta \in [\zeta_2, \zeta_1] \\ \frac{\cos(z^2)}{(z-zi_1)(z-zc_1)(z-ze_1)}, & for \ \ \theta \in (\zeta_1, 2\pi + \zeta_2) \end{cases},$$

where

$$\zeta_1 = 7\pi/4, \zeta_2 = 7\pi/10, \ zc_1 = \psi\left(e^{\pi i/4}\right), \ zc_2 = \psi\left(e^{6\pi i/7}\right), \ zc_3 = \psi\left(e^{9\pi i/7}\right),$$

$$zi_1 = 0.3 + 0.1i, \ zi_2 = -0.4 - 0.2i, \ ze_1 = \psi(1.3 + 0.7i), \ ze_2 = \psi(-1.5 - 0.75i).$$

The function $f(z)$ has two poles inside Γ, two poles outside of Γ and three poles on the contour Γ (all simple), as well as two jump discontinuity points on Γ (see Figure 6).

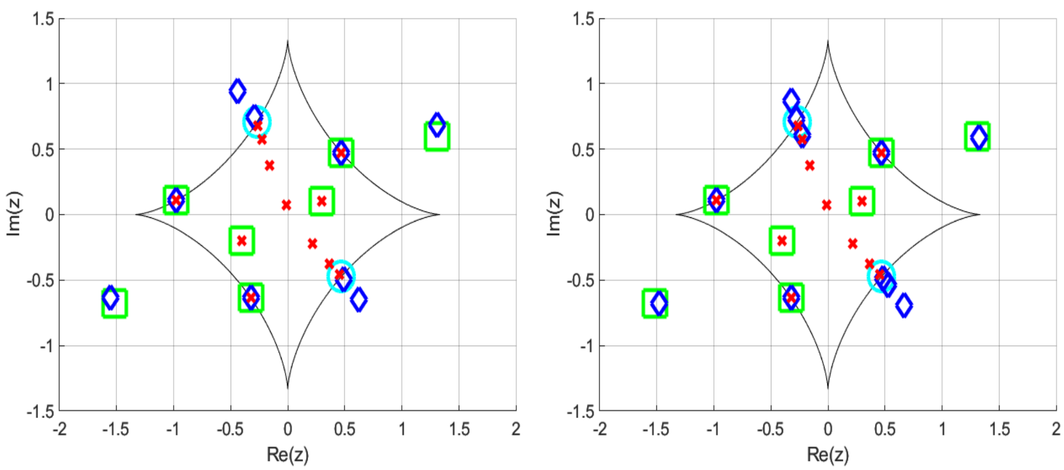

Figure 6: Approximations of the poles and discontinuity points in Example 4

The algorithm for evaluating the number of singular points of $f(z)$ returns $m_1 = 5$ for the number of poles on $\Omega^+ \cup \Gamma$ and $m_2 = 10$ for the number of poles on $\Omega^- \cup \Gamma$. The obtained numerical results confirm the idea that the examined algorithm gives a correct evaluation for the number of poles only in the case when the function has no discontinuity points.

We apply Padé approximation of order $(N_1, M_1) = (11, 9)$ to $f^+(z)$ and Padé approximation of order $(N_2, M_2) = (12, 12)$ to $f^-(z)$. The approximations obtained for the poles and discontinuity points of the function $f(z)$ are presented in Figure 6 (left).

It can be seen that in the neighborhood of each discontinuity point a convergent sequence of approximations is generated and the number of elements of the sequence increases simultaneously with the order of approximations. The approximations for $(N_1, M_1) = (17, 11)$ and $(N_2, M_2) = (12, 12)$ are presented in Figure 6 (right).

It should be noted that the presence of discontinuity points essentially reduces the convergence speed of the examined localization algorithm. This is explained by the fact that the discontinuity points "attract" the generated approximations and do not allow them to locate quickly certain poles.

As the problem of constructing the Padé approximation is generally poorly conditioned, then amplifying the order of the Padé approximations for the functions $f^+(z)$ and $f^-(z)$ usually leads to the appearance of spurious poles. Most of the spurious poles can be removed by applying the residual analysis [4].

References

[1] Bultheel A. *Laurent series and their Padé approximations.* Birkhäuser Verlag, Basel-Boston, 1987.

[2] Baker G., Graves-Morris P. *Padé approximants.* Second ed., Cambridge University Press, 1996.

[3] Lyness J. N. *Differentiation formulas for analytic functions.* Math. Comp., 1968, **22**, no. 102, 352–362.

[4] Gonnet P., Guttel S., Trefethen L. N. *Robust Padé approximation via SVD.* SIAM Review, 2013, **55**, no. 1, 101–117.

Capcelea Maria, Capcelea Titu
Moldova State University

E-mail: *mariacapcelea@yahoo.com; titu.capcelea@gmail.com*

Received May 22, 2020