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Levitan Almost Periodic Solutions of Infinite-dimensional Linear Differential Equations

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Abstract. The known Levitan's Theorem states that the finite-dimensional linear differential equation

$$x' = A(t)x + f(t) \tag{1}$$

with Bohr almost periodic coefficients $A(t)$ and $f(t)$ admits at least one Levitan almost periodic solution if it has a bounded solution. The main assumption in this theorem is the separation among bounded solutions of homogeneous equations

$$x' = A(t)x \ . \tag{2}$$

In this paper we prove that infinite-dimensional linear differential equation (3) with Levitan almost periodic coefficients has a Levitan almost periodic solution if it has at least one relatively compact solution and the trivial solution of equation (2) is Lyapunov stable. We study the problem of existence of Bohr/Levitan almost periodic solutions for infinite-dimensional equation (3) in the framework of general nonautonomous dynamical systems (cocycles).

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1 Introduction

This paper is dedicated to studying the problem of Levitan almost periodicity of solutions for infinite-dimensional linear differential equation

$$x'(t) = A(t)x(t) + f(t) \tag{3}$$

with Levitan almost periodic in time coefficients $A(t)$ and $f(t)$. We prove that if the coefficients of equation (3) are Levitan almost periodic and equation (3) has a relatively compact on semi-axis $\mathbb{R}_+$ solution, then equation (3) admits at least one Levitan almost periodic solution if the trivial solution of equation $x'(t) = A(t)x(t)$ is Lyapunov stable.

Let (X, ρ) be a complete metric space. Denote by $C(\mathbb{R}, X)$ the space of all continuous functions $\varphi : \mathbb{R} \to X$ equipped with the distance

$$d(\varphi, \psi) := \sup_{L>0} \min\{\max_{|t|\le L} \rho(\varphi(t), \psi(t)), L^{-1}\}.$$

The space $(C(\mathbb{R}, X), d)$ is a complete metric space.

Let $h \in \mathbb{R}$ and $\varphi \in C(\mathbb{R}, X)$. Denote by φ^h the h-translation of function φ, i.e., $\varphi^h(t) := \varphi(t+h)$ for any $t \in \mathbb{R}$ and by $\mathfrak{N}_\varphi := \{\{h_k\} : \ \varphi^{h_k} \to \varphi\}$. Note that the convergence $\varphi^{h_k} \to \varphi$ as $k \to \infty$ means the convergence uniform on every compact $[-l, l] \subset \mathbb{R}$ $(l > 0)$.

Definition 1. Let $\varepsilon > 0$. A number $\tau \in \mathbb{R}$ is called ε-*almost period* of the function φ if

$$\rho(\varphi(t+\tau), \varphi(t)) < \varepsilon$$

or all $t \in \mathbb{R}$. Denote by $\mathcal{T}(\varphi, \varepsilon)$ the set of ε-almost periods of φ.

Definition 2. A function $\varphi \in C(\mathbb{R}, X)$ is said to be *Bohr almost periodic* if the set of ε-almost periods of φ is *relatively dense* for each $\varepsilon > 0$, i.e., for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that $\mathcal{T}(\varphi, \varepsilon) \cap [a, a+l] \neq \emptyset$ for all $a \in \mathbb{R}$.

Definition 3. Let $\varphi \in C(\mathbb{R}, X)$ and $\psi \in C(\mathbb{R}, Y)$. A function $\varphi \in C(\mathbb{R}, X)$ is called *Levitan almost periodic* if there exists a Bohr almost periodic function $\psi \in C(\mathbb{R}, Y)$ such that $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$, where Y is some metric space (generally speaking $Y \neq X$).

Remark 1. The function $\varphi \in C(\mathbb{R}, \mathbb{R})$ defined by equality

$$\varphi(t) = \frac{1}{2 + \cos t + \cos \sqrt{2}t}$$

is Levitan almost periodic, but it is not Bohr almost periodic (because it is not bounded).

B. M. Levitan [18] studied the problem of existence of Levitan almost periodic solutions of equation

$$x' = A(t)x + f(t) \quad (x \in \mathbb{R}^n) \tag{4}$$

with the matrix $A(t)$ and vector-function $f(t)$ Levitan almost periodic.

Along with equation (4), consider the homogeneous equation

$$x' = A(t)x \ . \tag{5}$$

Theorem 1. *(Levitan's theorem [18]-[21]) Linear differential equation (4) with Bohr almost periodic coefficients admits at least one Levitan almost periodic solution if it has a bounded solution and each bounded on $\mathbb{R}$ solution $\varphi(t)$ of equation (5) is separated from zero, i.e.*

$$\inf_{t \in \mathbb{R}} |\varphi(t)| > 0.$$

Denote by

$$H(A, f) := \overline{\{(A^h, f^h) | \ h \in \mathbb{R}\}},$$

where by bar we denoted the closure in the space $C(\mathbb{R}, [\mathbb{R}^n]) \times C(\mathbb{R}, \mathbb{R}^n)$, where $[\mathbb{R}^n]$ is the space of all linear operators acting on the space $\mathbb{R}^n$.

Theorem 2. *(Zhikov's theorem [30]) Linear differential equation (4) with Bohr almost periodic coefficients admits at least one Levitan almost periodic "limiting" solution if it has a bounded solution, i.e., there exists a limiting equation*

$$x'(t) = B(t)x(t) + g(t), \tag{6}$$

where $(B, g) \in H(A, f)$.

Denote by $\Omega := \{(B, g) \in H(A, f) |$ such that equation (6) has a Levitan almost periodic solution $\}$. Zhikov proved that the set Ω has a second category of Baire.

Open problem (V. V. Zhikov [30]). Is equality $\Omega = H(A, f)$ true? In other words, can we state that every equation (4) admits at least one Levitan almost periodic solution if (4) has a bounded on $\mathbb{R}$ solution?

From our result[11] it follows the positive answer to this question. In this paper we generalize this result for infinite-dimensional equations (4). In particular we prove the following statement.

Theorem 3. *Let* $\mathfrak{B}$ *a uniform convex Banach space and* $[\mathfrak{B}]$ *be a Banach space of all linear bounded operators acting on the space* $\mathfrak{B}$. *Suppose that the following conditions are fulfilled:*

1. *the operator-function* $A \in C(\mathbb{R}, [\mathfrak{B}])$ *and function* $f(t)$ *are Levitan almost periodic;*
2. $\mathfrak{B}^s_A$ *is a subspace of the Banach space* $\mathfrak{B}$, *where* $\mathfrak{B}^s_A := \{u \in \mathfrak{B} | \sup\limits_{t \geq 0} |\varphi(t, u, A)| < \infty\}$ *and* $\varphi(t, u, A)$ *is a unique solution* $x(t)$ *of equation*
$$x'(t) = A(t)x(t) \tag{7}$$
with the initial condition $x(0) = u$;
3. *equation*
$$x'(t) = A(t)x(t) + f(t) \quad (x \in \mathfrak{B}) \tag{8}$$
has a relatively compact on $\mathbb{R}_+$ *solution.*

Then equation (8) has at least one Levitan almost periodic solution.

Corollary 1. *Under the conditions of theorem if the coefficients* $A(t)$ *and* $f(t)$ *are Bohr almost periodic, then equation (8) admits at least one Levitan almost periodic solution.*

Remark 2. Note that B^s_A is closed if the trivial solution of equation (7) is Lyapunov stable because in this case $\mathfrak{B}^s_A = \mathfrak{B}$ (see Lemma 4).

This paper is organized as follow.

In Section 2 we collect some notions and facts from the theory of dynamical systems (some classes of Poisson stable motions, comparability by character of recurrence of Poisson stable motions, nonautonomous dynamical systems, conditionally compactness).

Section 3 is dedicated to the study of the problem of existence of a common fixed point for noncommutative affine semigroup of mappings (Theorems 6).

In Section 4 we study the problem of existence of at least one compatible motion/solution of linear nonhomogeneous dynamical system (Theorem 7).

Section 5 is dedicated to the application of our general results obtained in Section 4 to the linear ordinary differential (Theorem 8 and Corollary 3) and linear partial differential (Theorem 9 and Corollary 4) equations.

2 Some notions and facts from the theory of autonomous and nonautonomous dynamical systems

2.1 Poisson stable motions

Let (X, ρ) be a complete metric space with metric ρ, $\mathbb{R}$ be a group of real numbers, $\mathbb{R}_+$ be a semigroup of the nonnegative real numbers, $\mathbb{T} \subseteq \mathbb{R}$ ($\mathbb{R}_+ \subseteq \mathbb{T}$) be a sub-semigroup of additive group $\mathbb{R}$ and $\mathbb{T}_+ := \{t \in \mathbb{T} |\ t \geq 0\}$.

Let $(X, \mathbb{T}, \pi)$ be a dynamical system. Let us recall the classes of Poisson stable motions we study in this paper, see [12, 23, 24, 26, 28] for details.

Definition 4. A point $x \in X$ is called *stationary* (respectively, *τ-periodic*) if $\pi(t, x) = x$ (respectively, $\pi(t + \tau, x) = \pi(t, x)$) for all $t \in \mathbb{T}$.

Definition 5. For given $\varepsilon > 0$, a number $\tau \in \mathbb{T}$ is called a *ε-shift of x* (respectively, *ε-almost period of x*) if $\rho(\pi(\tau, x), x) < \varepsilon$ (respectively, $\rho(\pi(\tau + t, x), \pi(t, x)) < \varepsilon$ for all $t \in \mathbb{T}$).

Definition 6. A point $x \in X$ is called *almost recurrent* (respectively, *Bohr almost periodic*) if for any $\varepsilon > 0$ there exists a positive number l such that any segment of length l contains a ε-shift (respectively, ε-almost period) of x.

Definition 7. If a point $x \in X$ is almost recurrent and its trajectory $\Sigma_x := \{\pi(t, x) : t \in \mathbb{T}\}$ is precompact, then x is called *(Birkhoff) recurrent*.

Definition 8. A point $x \in X$ is called *Levitan almost periodic* [21] (see also [3, 9, 12, 20]) if there exists a dynamical system $(Y, \mathbb{T}, \sigma)$ and a Bohr almost periodic point $y \in Y$ such that $\mathfrak{N}_y \subseteq \mathfrak{N}_x$.

Definition 9. A point $x \in X$ is called *almost automorphic* if

1. it is stable in the sense of Lagrange, i.e., its trajectory $\Sigma_x := \{\pi(t, x) |\ t \in \mathbb{T}\}$ is relatively compact and

2. x is Levitan almost periodic.

2.2 Comparability of motions by the character of recurrence

Following B. A. Shcherbakov [25, 26] (see also [8],[10, ChI]) we introduce the notion of comparability of motions of dynamical system by the character of their recurrence. While studying stable in the sense of Poisson motions, this notion plays the very important role (see, for example,[24, 26]).

Let $(X, \mathbb{T}, \pi)$ and $(Y, \mathbb{T}, \sigma)$ be dynamical systems, $x \in X$ and $y \in Y$. Denote $\mathfrak{M}_x := \{\{t_n\} : \text{such that } \{\pi(t_n, x)\} \text{ converges as } n \to \infty\}$, $\mathfrak{N}_x := \{\{t_n\} : \text{such that } \pi(t_n, x) \to x \text{ as } n \to \infty\}$ and $\mathfrak{N}_x^{+\infty} := \{\{t_n\} \in \mathfrak{N}_x : \text{such that } t_n \to +\infty \text{ as } n \to \infty\}$.

Definition 10. A point $x_0 \in X$ is called comparable by the character of recurrence with $y_0 \in Y$ if there exists a continuous mapping $h : \Sigma_{y_0} \mapsto \Sigma_{x_0}$ satisfying the condition

$$h(\sigma(t, y_0)) = \pi(t, x_0) \quad \text{for any } t \in \mathbb{R}.$$

Definition 11. Let $(X, \mathbb{T}_1, \pi)$ and $(Y, \mathbb{T}_2, \sigma)$ be two dynamical systems, $\mathbb{T}_1 \subseteq \mathbb{T}_2$ and $h : X \mapsto Y$ be a homomorphism of $(X, \mathbb{T}_1, \pi)$ on $(Y, \mathbb{T}_2, \sigma)$. A triplet $\langle (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h \rangle$ is said to be a nonautonomous dynamical system [12, ChI].

Definition 12. The point $y \in Y$ is called (see, for example,[26] and [28]) positively stable in the sense of Poisson if there exists a sequence $t_n \to +\infty$ such that $\sigma^{t_n} y \to y$.

Theorem 4. *[7, 25] Let $y \in Y$ be Poisson stable in the positive direction, then the following statement are equivalent:*

1. *the point $x \in X$ is comparable with $y \in Y$ by the character of recurrence;*
2. $\mathfrak{N}_y^{+\infty} \subseteq \mathfrak{N}_y^{+\infty}$*;*
3. *for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $d(\sigma(\tau, y), y) < \delta$ implies $\rho(\pi(\tau, x), x) < \varepsilon$, where d (respectively, ρ) is the distance on the space Y (respectively, on the space X).*

Theorem 5. *[25] Suppose that the point x is comparable with $y \in Y$ by the character of recurrence. If the point y is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable), the point x is so.*

2.3 Some general fact about nonautonomous dynamical systems

Definition 13. (Conditional compactness). Let (X, h, Y) be a fibre space, i.e., X and Ya be two metric spaces and $h : X \to Y$ be a homomorphism from X into Y. The subset $M \subseteq X$ is said to be conditionally relatively compact if the pre-image $h^{-1}(Y') \bigcap M$ of every relatively compact subset $Y' \subseteq Y$ is a relatively compact subset of X, in particular $M_y := h^{-1}(y) \bigcap M$ is relatively compact for every y. The set M is called conditionally compact if it is closed and conditionally relatively compact.

Example 1. Let K be a compact space, $X := K \times Y$, $h = pr_2 : X \to \Omega$, then the triplet (X, h, Y) is a fibre space, the space X is conditionally compact, but not compact.

Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a nonautonomous dynamical system and $y \in Y$ be a positively Poisson stable point. Denote by

$$\mathcal{E}_y^+ := \{\xi | \quad \exists \{t_n\} \in \mathfrak{N}_y^{+\infty} \quad \text{such that} \quad \pi^{t_n}|_{X_y} \to \xi\},$$

where $X_y := \{x \in X | \quad h(x) = y\}$ and $\to$ means the pointwise convergence.

Lemma 1. *[12] Let $y \in Y$ be a positively Poisson stable point, $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h \rangle$ be a nonautonomous dynamical system and X be a conditionally compact space, then $\mathcal{E}_y^+$ is a nonempty compact subsemigroup of the semigroup $X_y^{X_y}$ (w.r.t. composition of mappings).*

3 Markov-Kakutani's fixed point theorem for noncommutative affine semigroup of mappings

Let (X, ρ) be a metric space. Denote by $B[a, r] := \{x \in X | \ \rho(x, a) \le r\}$, where $a \in X$ and $r \ge 0$. For any $x_1, x_2 \in X$ and $\alpha \in [0, 1]$ denote by $S(\alpha, x_1, x_2)$ the intersection of $B[x_1, \alpha r]$ and $B[x_2, (1 - \alpha) r]$, where $r = \rho(x_1, x_2)$.

Definition 14. A metric space (X, ρ) is called:

1. a metric space with a convex structure [29] if there exists a mapping $W : [0, 1] \times X \times X \to X$ satisfying

$$\rho(u, W(\alpha, x_1, x_2)) \le \alpha \rho(u, x_1) + (1 - \alpha) \rho(u, x_2);$$

2. strictly convex [29] if for any $x, y \in X$ and $\alpha \in [0, 1]$ there exists a unique element x ($x = S(\alpha, x_1, x_2)$) such that $\rho(x, x_1) = \alpha \rho(x_1, x_2)$ and $\rho(x, x_2) = (1 - \alpha) \rho(x_1, x_2)$;

3. strongly convex [6, 16] (or strictly convex space with convex round balls) if (X, ρ) is a strictly convex metric space and for any $x_1, x_2, x_3 \in X$ ($x_2 \neq x_3$) and $\alpha \in (0, 1)$ the inequality $\rho(x_1, S(\alpha, x_2, x_3)) < \max\{\rho(x_1, x_2), \rho(x_1, x_3)\}$ holds.

Definition 15. Let X be a metric space with a convex structure (respectively, strictly convex or strongly convex). A subset M of X is said to be convex (respectively, strictly convex or strongly convex) if $S(\alpha, x_1, x_2) \in M$ for any $\alpha \in (0, 1)$ and $x_1, x_2 \in M$.

Remark 3. 1. Closed balls may be not convex sets and intersection of convex sets may be non convex set [14].

2. Intersection of convex sets is a convex set in strictly convex metric space [14].

3. There exist strictly convex metric spaces in which closed balls are not convex [5].

4. The closed ball $B[c,r]$ for every $r>0$ and every $c\in X$ is a convex set in the strongly convex metric space (X,ρ) [6].

Definition 16. A Banach space X is said to be:

1. uniformly convex if the inequality $|p_1-p_2|\ge \delta\max\{|p_1|,|p_2|\}$ implies $|\frac{1}{2}(p_1+p_2)|\le (1-\varphi(\delta))\max\{|p_1|,|p_2|\}$ ($\varphi(\delta)>0$ for any $0<\delta\le 2$);
2. strictly convex if for any $x,y\in X$ with $|x|=|y|=1$ and $x\neq y$, $|\lambda x+(1-\lambda)y|<1$ for any $\lambda\in(0,1)$.

Remark 4. 1. Uniformly convex Banach spaces are strictly convex, but the converse is not true.

2. If $(X,|\cdot|)$ is a strictly convex Banach space, then the metric space (X,ρ) ($\rho(x_1,x_2):=|x_1-x_2|$) is strictly convex (see, for example,[6, 14]).

3. If M is a convex subset of strictly convex Banach space $(X,|\cdot|)$, then the metric space (M,ρ) ($\rho(x_1,x_2):=|x_1-x_2|$) is strictly convex.

4. Every convex closed subset X of the Hilbert space H equipped with metric $\rho(x_1,x_2)=|x_1-x_2|$ is a strongly convex metric space.

Lemma 2. *[11] If $(X,|\cdot|)$ is a uniformly convex Banach space, then the metric space (X,ρ) ($\rho(x_1,x_2):=|x_1-x_2|$) is strongly convex.*

For any subset C of X we denote by coC (respectively, $\overline{co}C$) the convex envelope (respectively, closed convex envelope) of C, i.e., $\overline{co}C$ (respectively, $\overline{co}C$) is the intersection of all metric-convex (respectively, closed, metric-convex) sets containing C.

Definition 17. A mapping $f:M\to M$ of compact strictly metric-convex space (M,ρ) is said to be:

1. segment preserving if $f([x_1,x_2])\subseteq[f(x_1),f(x_2)]$, where $[x_1,x_2]:=\{S(\alpha,\ x_1,\ x_2)|\ \ 0\le\alpha\le 1\}$, for any $x_1,x_2\in M$;
2. quasi-affine [27] if $f(coA)\subseteq cof(A)$ for any subset A of M;
3. strongly quasi-affine if $f(\overline{co}A)\subseteq\overline{co}f(A)$ for any subset A of M;
4. affine if $f(S(\alpha,x_1,x_2))=S(\alpha;f(x_1),f(x_2))$ for any $x_1,x_2\in M$ and $\alpha\in[0,1]$.

Remark 5. 1. If the mapping $f:M\to M$ is quasi-affine, then it is segment preserving because $[x_1,x_2]=co(\{x_1,x_2\})$.

2. If $M=[0,1]$ and $f:[0,1]\to[0,1]$ is a continuous and strongly monotone, then it is quasi-affine [27].

3. If the mapping $f:M\to M$ is affine, then it is quasi-affine.

Theorem 6. *[11] Let (M,ρ) be a compact strongly convex metric space. Suppose that the following conditions are fulfilled:*

1. *E is a compact sub-semigroup of the semigroup M^M;*

2. *every $\xi \in E$ is continuous and quasi-affine.*

Then there exists a common fixed point $\bar{x} \in M$ of E, i.e., $\xi(\bar{x}) = \bar{x}$ for any $\xi \in E$.

4 Favard's theory for infinite-dimensional systems

Let $(\mathfrak{B}, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $\mathbb{T} \supseteq \mathbb{R}_+$ be a subsemigroup of group $\mathbb{R}$ and $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ (or shortly φ) be a linear cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$, i.e., φ is a continuous mapping from $\mathbb{T} \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ satisfying the following conditions:

1. $\varphi(0, u, y) = u$ for any $u \in \mathfrak{B}$ and $y \in Y$;

2. $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for any $t, \tau \in \mathbb{T}$, $u \in \mathfrak{B}$ and $y \in Y$;

3. for any $(t, y) \in \mathbb{T} \times Y$ the mapping $\varphi(t, \cdot, y) : \mathfrak{B} \mapsto \mathfrak{B}$ is linear.

Denote by $[\mathfrak{B}]$ the Banach space of any linear bounded operators A acting on the space $\mathfrak{B}$ equipped with the operator norm $||A|| := \sup\limits_{|x| \leq 1} |Ax|$.

Example 2. Let Y be a complete metric space and $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y. Consider the following linear differential equation

$$x' = A(\sigma(t, y))x, \quad (y \in Y) \tag{9}$$

where $A \in C(Y, [\mathfrak{B}])$. Note that the following conditions are fulfilled for equation (9):

a. for any $u \in \mathfrak{B}$ and $y \in Y$ equation (9) has exactly one solution that is defined on $\mathbb{R}$ and satisfies the condition $\varphi(0, u, y) = u$;

b. the mapping $\varphi : (t, u, y) \to \varphi(t, u, y)$ is continuous in the topology of $\mathbb{R} \times \mathfrak{B} \times Y$.

Under the above assumptions equation (9) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$.

Example 3. Consider differential equation

$$x' = A(t)x, \tag{10}$$

where $A \in C(\mathbb{R}, [\mathfrak{B}])$. Along this equation (10) consider its H-class, i.e., the following family of equations

$$x' = B(t)x, \tag{11}$$

where $B \in H(A)$. Note that the following conditions are fulfilled for equation (10) and its H-class (25):

a. for any $u \in \mathfrak{B}$ and $B \in H(A)$ equation (11) has exactly one solution $\varphi(t, u, B)$ satisfying the condition $\varphi(0, u, B) = v$;

b. the mapping $\varphi : (t, u, B) \to \varphi(t, u, B)$ is continuous in the topology of $\mathbb{R} \times \mathfrak{B} \times C(\mathbb{R}; [\mathfrak{B}])$.

Denote by $(H(A), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A)$. Under the above assumptions equation (10) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ over dynamical system $(H(A), \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$.

Note that equation (10) and its H-class can be written in the form (9). We put $Y := H(A)$ and denote by $\mathcal{A} \in C(Y, [\mathfrak{B}])$ defined by equality $\mathcal{A}(B) := B(0)$ for any $B \in H(A) = Y$, then $B(\tau) = \mathcal{A}(\sigma(B, \tau))$ $(\sigma(\tau, B) := B_\tau$, where $B_\tau(t) := B(t + \tau)$ for any $t \in \mathbb{R})$. Thus equation (10) with its H-class can be rewrite as follow

$$x' = \mathcal{A}(\sigma(t, B))x \ (B \in H(A)).$$

Definition 18. Let $\langle (X, \mathbb{R}_+, \pi),\ (Y, \mathbb{R}, \sigma), h\rangle$ be a linear nonautonomous (affine) dynamical system. A nonautonomous dynamical system $\langle (W,\mathbb{R}_+,\mu), (Z,\mathbb{R},\lambda),\varrho\rangle$ is said to be linear non-homogeneous, generated by linear (homogeneous) dynamical system $\langle (X, \mathbb{R}_+, \pi),\ (Y, \mathbb{R}, \sigma), h\rangle$ if the following conditions hold:

1. there exits a homomorphism q of the dynamical system $(Z, \mathbb{R}, \lambda)$ onto $(Y, \mathbb{R}, \sigma)$;

2. the space $W_y := (q \circ \rho)^{-1}(y)$ is affine for all $y \in (q \circ \varrho)(W) \subseteq Y$ and the vectorial space $X_y = h^{-1}(y)$ is an associated space to W_y ([22, p.175]). The mapping $\mu^t : W_y \to W_{\sigma^t y}$ is affine and $\pi^t : X_y \to X_{\sigma^t y}$ is its linear associated function ([22, p.179]), i.e., $X_y = \{w_1 - w_2 \mid w_1, w_2 \in W_y\}$ and $\mu^t w_1 - \mu^t w_2 = \pi^t(w_1 - w_2)$ for all $w_1, w_2 \in W_y$ and $t \in \mathbb{R}_+$.

Remark 6. The definition of linear non-homogeneous system, associated with the given linear system, is given in the work [4], but our definition is more general and sometimes more flexible.

Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be a linear cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$, $f \in C(Y, \mathbb{B})$ and ψ be a mapping from $\mathbb{R}_+ \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ defined by equality

$$\psi(t, u, y) := U(t, y)u + \int_0^t U(t - \tau, \sigma(\tau, y))f(\sigma(\tau, y))d\tau \tag{12}$$

From the definition of mapping ψ it follows that ψ possesses the following properties:

1. $\psi(0, u, y) = u$ for any $(u, y) \in \mathfrak{B} \times Y$;

2. $\psi(t + \tau, u, y) = \psi(t, \psi(\tau, u, y), \sigma(\tau, y))$ for any $t, \tau \in \mathbb{R}_+$ and $(u, y) \in \mathfrak{B} \times Y$;

3. the mapping $\psi : \mathbb{R}_+ \times \mathfrak{B} \times Y \mapsto \mathfrak{B}$ is continuous;

4. $\psi(t, u, y) - \psi(t, v, y) = \varphi(t, u - v, y)$ for any $t \in \mathbb{R}_+$, $u, v \in \mathfrak{B}$ and $y \in Y$, i.e., the mapping $\psi(t, \cdot, y) : \mathfrak{B} \mapsto \mathfrak{B}$ is affine for every $(t, y) \in \mathbb{R}_+ \times Y$.

Definition 19. A triplet $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma) \rangle$ is called an affine (nonhomogeneous) cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ if ψ is a mapping from $\mathbb{R}_+ \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ possessing the properties 1.-4.

Remark 7. If we have a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ and $f \in C(Y, \mathfrak{B})$, then equality (12) defines an affine cocycle $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma) \rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ which is called an affine (nonhomogeneous) cocycle associated with linear cocycle φ and the function $f \in C(Y, \mathfrak{B})$.

Lemma 3. *Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma) \rangle$ be a linear cocycle and*

$$\mathfrak{B}^s_y := \{x \in \mathfrak{B} | \sup_{t \geq 0} |\varphi(t, x, y)| < +\infty\}$$

be a subspace of the Banach space $\mathfrak{B}$. Then there exists a positive constant L such that

$$|\varphi(t, u, y)| \leq L|u| \tag{13}$$

for any $t \geq 0$ and $u \in \mathfrak{B}^s_y$.

Proof. Consider the family of linear bounded operators $\mathfrak{A} := \{\varphi(t, \cdot, y) | \ t \geq 0\}$ acting from $\mathfrak{B}^s_y$ into $\mathfrak{B}$. Note that for any $u \in \mathfrak{B}^s_y$ there exists a positive number $C(u)$ (for example $C(u) = \sup\limits_{t \geq 0} |\varphi(t, u, y)|$) such that

$$|Au| \leq C(u)$$

for any $A \in \mathfrak{A}$. Since $\mathfrak{B}^s_y$ is a subspace of the Banach space $\mathfrak{B}$, then by Banach-Steinhaus theorem the family of operators $\mathfrak{A}$ is bounded, i.e., there exists a positive constant L such that (13) takes place. Lemma is proved. □

Remark 8. If the Banach space $\mathfrak{B}$ is finite-dimensional, then it is evident that $\mathfrak{B}^s_y$ is a subspace of $\mathfrak{B}$.

Below we will give another important class of linear cocycles (with infinite-dimensional $\mathfrak{B}$) for which $\mathfrak{B}^s_y$ is a subspace of $\mathfrak{B}$.

Definition 20. A trivial motion of linear cocycle φ is said to be Lyapunov stable at the point $y \in Y$ if for arbitrary positive number ε there exists a positive number $\delta = \delta(\varepsilon, y)$ such that $|x| < \delta$ implies $|\varphi(t, x, y)| < \varepsilon$ for any $t \geq 0$.

Lemma 4. *The following statements are equivalent:*

1. *the trivial motion of linear cocycle φ is Lyapunov stable at the point $y \in Y$;*
2. *for any $x \in \mathfrak{B}$ we have $\sup\limits_{t \geq 0} |\varphi(t, x, y)| < \infty$, i.e., $\mathfrak{B}^s_y = \mathfrak{B}$;*
3. *there exists a positive number $L = L(y)$ such that*

$$|\varphi(t, x, y)| \leq L|x| \tag{14}$$

for any $x \in \mathfrak{B}$ and $t \geq 0$.

Proof. Let ε be an arbitrary positive number and $\delta = \delta(\varepsilon, y)$ be a positive number from the Lyapunov stability of trivial motion of cocycle φ at point $y \in Y$. Denote by $\delta_0; = \delta(1, y)$, then $|x| < \delta_0$ implies $|\varphi(t, x, y)| < 1$ for any $t \geq 0$. Let now $x \in \mathfrak{B}$ and $|x| \geq \delta_0$, then we have

$$x = \frac{2|x|}{\delta_0} x',$$

where $x' := x\delta_0/2|x|$ and, consequently,

$$|\varphi(t, x, y)| = \frac{2|x|}{\delta_0}|\varphi(t, x', y)| < \frac{2|x|}{\delta_0},$$

for any $t \geq 0$ because $|x'| < 1$. Thus the first statement implies the second one.

Let now $\mathfrak{B}^s_y = \mathfrak{B}$. Consider the family $\mathfrak{A} := \{\varphi(t, \cdot, y)| \ t \geq 0\}$ of linear bounded operators acting on the Banach space $\mathfrak{B}$. By condition 2 for any $x \in \mathfrak{B}$ there exists a positive number $C = C(x)$ such that $|Ax| \leq C|x|$ for any $A \in \mathfrak{A}$. Then by Banach-Steinhaus theorem the family of operators $\mathfrak{A}$ is bounded, i.e., there exists a positive constant L such that $||A|| \leq L$ for any $A \in \mathfrak{A}$, i.e., (14) takes place. This means that 2 implies 3.

Finally, we notice that 3 implies 1. In fact If ε is an arbitrary positive number and $\delta(\varepsilon, y) := \varepsilon/2L$, then evidently from $|x| < \delta$ we have $|\varphi(t, x, y)| < \varepsilon$ for any $t \geq 0$. Lemma is completely proved. □

Lemma 5. *[7] Let $\langle E, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be a cocycle and $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be the nonautonomous dynamical system generated by the cocycle φ. Assume that $x_0 := (u_0, y_0) \in X = E \times Y$ and the set $Q^+_{(u_0,y_0)} := \overline{\{\varphi(t, u_0, y_0) : t \in \mathbb{R}_+\}}$ is compact. Then the semi-hull $H^+(x_0) := \overline{\{\pi(t, x_0)| \ t \in \mathbb{R}_+\}}$ is conditionally compact.*

Lemma 6. *Let $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma)\rangle$ be an affine cocycle and $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be a nonautonomous dynamical system generated by cocycle φ ($X := \mathfrak{B} \times Y, \pi := (\varphi, \sigma)$ and $h := pr_2$). Assume that the following conditions are fulfilled:*

1. *$\mathfrak{B}^s_y$ is a subspace of the Banach space $\mathfrak{B}$;*
2. *the point $y_0 \in Y$ is Poisson stable in the positive direction;*
3. *there exits a point $u_0 \in \mathfrak{B}$ such that $\psi(\mathbb{R}_+, u_0, y_0)$ is relatively compact.*

Then the following statements hold:

1. *the set $K := \omega_{x_0} \subset X = \mathfrak{B} \times Y$ is conditionally compact, where $x_0 := (u_0, y_0)$;*
2. *$M := \overline{co} K_{y_0}$ is a compact convex subset of $X_{y_0} := \mathfrak{B} \times \{y_0\}$, where $K_{y_0} := \omega_{x_0} \bigcap X_{y_0}$;*
3. *$\mathcal{E}^+_{y_0}$ is a compact sub-semigroup of the semi-group M^M;*
4. *every $\xi \in \mathcal{E}^+_{y_0}$ is affine and continuous.*

Proof. The first statement follows from Lemma 5.

Since the set K is conditionally compact, then the set K_{y_0} is compact and, consequently, the set $M = \overline{co}K_{y_0}$ is also compact.

The third statement follows from Lemma 1.

Let m be an arbitrary natural number, $\alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}_+$ with

$$\sum_{k=1}^{m} \alpha_k = 1$$

and $x_1, x_2, \dots, x_m \in M$ ($x_i = (u_i, y_0)$ $i = 1, 2, \dots, m$). Since the maps from $\{\varphi(t, \cdot, y_0) | \ t \geq 0\}$ are affine, then we have

$$\psi(t, \sum_{k=1}^{m} \alpha_k u_k) = \sum_{k=1}^{m} \alpha_k \psi(t, u_k, y_0) \tag{15}$$

for any $t \geq 0$. Let now $\xi \in \mathcal{E}^+_{y_0}$, then there exists a sequence $\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that

$$\lim_{n\to\infty} \pi^{t_n}(x) = \xi(x) \tag{16}$$

for any $x \in M$. From (15) we get

$$\psi(t_n, \sum_{k=1}^{m} \alpha_k u_k, y_0) = \sum_{k=1}^{m} \alpha_k \psi(t_n, u_k, y_0) \tag{17}$$

for any $n \in \mathbb{N}$. Passing to the limit in (17) and taking in consideration (16) we obtain

$$\xi(\sum_{k=1}^{m} \alpha_k x_k) = \sum_{k=1}^{m} \alpha_k \xi(x_k).$$

Thus the map ξ is affine.

Let $x \in M$ and $\xi \in \mathcal{E}^+_{y_0}$, then $x = (u, y_0)$ and there exists a sequence $\{t_n\} \in \mathfrak{N}^{+\infty}_{y_0}$ such that $\xi(x) = \lim_{n\to\infty} (\psi(t_n, u, y_0), \sigma(t_n, y_0)) = (\nu(u), y_0)$, where

$$\nu(u) = \lim_{n\to\infty} \psi(t_n, u, y_0) \tag{18}$$

for any $(u, y_0) \in M$. By Lemma 4 there exists a positive constant L such that

$$|\psi(t_n, u_1, y_0) - \psi(t_n, u_2, y_0)| \leq L|u_1 - u_2| \tag{19}$$

for any $(u_1, y_0), (u_2, y_0) \in M$ and $n \in \mathbb{N}$. Passing to the limit in (19) as $n \to \infty$ and taking in consideration (18) we obtain

$$\rho(\xi(x_1), \xi(x_2)) \leq L\rho(x_1, x_2)$$

for every map $\xi \in \mathcal{E}^+_{y_0}$ because $\xi(x) = (\nu(u), y_0)$ for any $x = (u, y_0) \in M$ and $\rho(x_1, x_2) = |u_1 - u_2|$ ($x_i = (u_i, y_0), i = 1, 2$). Lemma is completely proved. $\square$

Theorem 7. *Let $\langle\mathfrak{B},\psi,(Y,\mathbb{R},\sigma)\rangle$ be an affine cocycle and $\langle(X,\mathbb{R}_+,\pi),(Y,\mathbb{R},\sigma),h\rangle$ be a nonautonomous dynamical system generated by cocycle φ ($X:=\mathfrak{B}\times Y,\pi:=(\varphi,\sigma)$ and $h:=pr_2$). Assume that the following conditions are fulfilled:*

1. *the Banach space $\mathfrak{B}$ is uniformly convex;*
2. *$\mathfrak{B}^s_y$ is a subspace of the Banach space $\mathfrak{B}$;*
3. *the point $y_0\in Y$ is Poisson stable;*
4. *there exits a point $u_0\in\mathfrak{B}$ such that $\psi(\mathbb{R}_+,u_0,y_0)$ is relatively compact.*

Then there exists at least one point $\bar{u}\in\mathfrak{B}$ such that $\mathfrak{N}^{+\infty}_{y_0}\subseteq\mathfrak{N}^{+\infty}_{\bar{x}}$, i.e., the point $\bar{x}$ is comparable with the point y_0.

Proof. Let $x_0:=(u_0,y_0)$ and $K:=\omega_{x_0}$, then by Lemma 6 we have

1. the set $K:=\omega_{x_0}\subset X=\mathfrak{B}\times Y$ is conditionally compact, where $x_0:=(u_0,y_0)$;
2. $M:=\overline{co}K_{y_0}$ is a compact convex subset of $X_{y_0}:=\mathfrak{B}\times\{y_0\}$, where $K_{y_0}:=\omega_{x_0}\bigcap X_{y_0}$;
3. $\mathcal{E}^+_{y_0}$ is a compact sub-semigroup of the semi-group M^M;
4. every $\xi\in\mathcal{E}^+_{y_0}$ is affine and continuous.

Since the Banach space $\mathfrak{B}$ is uniformly convex, then by Lemma 2 it is strongly convex and according to Theorem 6 there exists at least one point $\bar{x}=(\bar{u},y_0)\in M$ such that $\xi(\bar{x})=\bar{x}$ for any $\xi\in\mathcal{E}^+_{y_0}$. Now we show that the point $\bar{x}$ is comparable by character of recurrence with the point y_0. To this end by Theorem 4 it is sufficient to show that $\mathfrak{N}^{+\infty}_{y_0}\subseteq\mathfrak{N}^{+\infty}_{\bar{x}}$. Let $\{t_n\}\in\mathfrak{N}^{+\infty}_{y_0}$, then $\sigma(t_n,y_0)\to y_0$ as $n\to\infty$. Since $\Sigma_{\bar{x}}$ is conditionally precompact and $\{\pi(t_n,\bar{x})\}=\Sigma_{\bar{x}}\bigcap h^{-1}(\{\sigma(t_n,y_0)\})$, then $\{\pi(t_n,\bar{x})\}$ is a precompact sequence. To show that $\{t_n\}\in\mathfrak{N}^{+\infty}_{\bar{x}}$ it is sufficient to prove that the sequence $\{\pi(t_n,\bar{x})\}$ has at most one limiting point. Let p_i $(i=1,2)$ be two limiting points of $\{\pi(t_n,\bar{x})\}$, then there are $\{t_{k^i_n}\}\subseteq\{t_n\}$ such that $p_i:=\lim\limits_{n\to\infty}\pi(t_{k^i_n},x_0)$ $(i=1,2)$. Notice that the set

$$Q:=\overline{\bigcup\{\pi^{t_n}(X_y)|n\in\mathbb{N}\}}$$

is compact, because X is conditionally compact. Thus $\{\pi^{t_n}|_{X_y}\}\subseteq Q^{X_y}$ and according to Tykhonov's theorem this sequence is relatively compact and, consequently, without loss of generality we can suppose that the subsequences $\{\pi^{t_{k^i_n}}\}\subset\{\pi^{t_n}\}$ $(i=1,2)$ are convergent. Denote by $\xi^i=\lim\limits_{n\to\infty}\pi^{t_{k^i_n}}$, then $\xi^i\in\mathcal{E}^+_{y_0}$ $(i=1,2)$ and $p^i=\xi^i(\bar{x})=\bar{x}$. Thus we have $p^1=\bar{x}=p^2$. Theorem is completely proved. □

Corollary 2. *Let $\langle\mathfrak{B},\psi,(Y,\mathbb{R},\sigma)\rangle$ be an affine cocycle. Under the conditions of Theorem 7 if the point $y_0\in Y$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic,* *almost recurrent in the sense of Bebutov, Poisson stable), then there exists at least one point $\bar{x} \in X := \mathfrak{B} \times Y$ such that $\bar{x}$ has the same character of recurrence as y_0, i.e., $\bar{x}$ is τ-periodic (respectively, almost automorphic, recurrent in the sense of Birkhoff, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable).*

Proof. This statement follows from Theorems 5 and 7. ☐

Definition 21. A continuous mapping $\nu : \mathbb{R} \to \mathfrak{B}$ is said to be a full trajectory of the cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ if $\varphi(t+s, \nu(s), \sigma(s, y)) = \nu(t+s)$ for any $t \geq 0$ and $s \in \mathbb{R}$.

Remark 9. Notice that Lemma 4 and Theorem 7 remain true if we replace the set $\mathfrak{B}^s_y$ by its subset $\mathfrak{B}^0_y := \{u \in \mathfrak{B} |$ there exists a full trajectory ν of the cocycle φ such that $\nu(0) = u$ and $\nu(\mathbb{R})$ is relatively compact$\}$.

5 Applications

5.1 Ordinary linear differential equations

Example 4. Let Y be a complete metric space, $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y and $[\mathfrak{B}]$ be the space of linear bounded operators acting into Banach space $\mathfrak{B}$ equipped with the operator norm and $f \in C(Y, \mathfrak{B})$. Consider the following linear nonhomogeneous differential equation

$$x' = A(\sigma(t, y))x + f(\sigma(t, y)), \quad (y \in Y) \tag{20}$$

where $A \in C(Y, [\mathfrak{B}])$.

Equation (9) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$. According to Remark 7, by equality (12)a linear nonhomogeneous cocycle $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$ is defined . Thus every nonhomogeneous linear differential equations (20) generates a linear nonhomogeneous cocycle ψ.

Example 5. Consider a linear nonhomogeneous differential equation

$$x' = A(t)x + f(t), \tag{21}$$

where $f \in C(\mathbb{R}, \mathfrak{B})$ and $A \in C(\mathbb{R}, [\mathfrak{B}])$. Along with equation (21) consider its H-class, i.e., the following family of equations

$$x' = B(t)x + g(t), \tag{22}$$

where $(B, g) \in H(A, f)$. Notice that the following conditions are fulfilled for equation (10) and its H-class (11):

a. for any $u \in \mathfrak{B}$ and $B \in H(A)$ equation (11) has exactly one solution $\varphi(t, u, B)$ defined on $\mathbb{R}$ and the condition $\varphi(0, u, B) = v$ is fulfilled;

b. the mapping $\varphi : (t, u, B) \to \varphi(t, u, B)$ is continuous in the topology of $\mathbb{R} \times \mathfrak{B} \times C(\mathbb{R}; \mathfrak{B})$.

Denote by $(H(A, f), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A, f)$. Under the above assumptions the equation (10) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (H(A, f), \mathbb{R}, \sigma) \rangle$ over dynamical system $(H(A, f), \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$. Denote by ψ a mapping from $\mathbb{R}_+ \times \mathfrak{B} \times H(A, f)$ into $\mathfrak{B}$ defined by equality

$$\psi(t, u, (B, g)) := U(t, B)u + \int_0^t U(t - \tau, B_\tau) g(\tau) d\tau,$$

then ψ possesses the following properties:

(i) $\psi(0, u, (B, g)) = u$ for any $(u, (B, g)) \in \mathfrak{B} \times H(A, f)$;

(ii) $\psi(t + \tau, u, (B, g)) = \psi(t, \psi(\tau, u, (B, g)), (B_\tau, g_\tau))$ for any $t, \tau \in \mathbb{R}_+$ and $(u, (B, g)) \in \mathfrak{B} \times H(A, f)$;

(iii) the mapping $\psi : \mathbb{R} \times \mathfrak{B} \times H(A, f) \mapsto \mathfrak{B}$ is continuous;

(iv) $\psi(t, \lambda u + \mu v, (B, g)) = \lambda \psi(t, u, (B, g)) + \mu \psi(t, v, (B, g))$ for any $t \in \mathbb{R}_+$, $u, v \in \mathfrak{B}$, $(B, g) \in H(A, f)$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$) with the condition $\lambda + \mu = 1$, i.e., the mapping $\psi(t, \cdot, (B, g)) : \mathfrak{B} \mapsto \mathfrak{B}$ is affine for every $(t, (B, g)) \in \mathbb{R}_+ \times H(A, f)$.

Thus, every linear nonhomogeneous differential equation of the form (21) (and its H-class (22)) generates a linear nonhomogeneous cocycle $\langle \mathfrak{B}, \psi, (H(A, f), \mathbb{R}, \sigma) \rangle$ over dynamical system $(H(A, f), \mathbb{R}, \sigma)$ with the fibre $\mathfrak{B}$.

Theorem 8. *Assume that the following conditions are fulfilled:*

1. *$\mathfrak{B}$ is a uniformly convex Banach space;*
2. *$\mathfrak{B}^s_A := \{u \in \mathfrak{B} | \sup\limits_{t \geq 0} |\varphi(t, u, A)| < \infty\}$ is a subspace of the Banach space;*
3. *the function $f \in C(\mathbb{R}, \mathfrak{B})$ and operator-function $A \in C(\mathbb{R}, [\mathfrak{B}])$ are jointly Poisson stable;*
4. *there exits a relatively compact on $\mathbb{R}_+$ solution $\psi(t, u_0, A, f)$ of equation (21).*

Then there exists at least one compatible solution $\psi(t, \bar{u}, A, f)$ of equation (21), i.e., $\mathfrak{N}^{+\infty}_{(A,f)} \subseteq \mathfrak{N}^{+\infty}_{\bar{\psi}}$, where $\bar{\psi} := \psi(\cdot, \bar{u}, A, f)$.

Proof. Let $\langle \mathfrak{B}, \psi, (H(A, f), \mathbb{R}, \sigma) \rangle$ be a cocycle generated by equation (21). By Example 5 the cocycle ψ is affine. Now applying Theorem 7 to constructed cocycle ψ we complete the proof of Theorem. □

Corollary 3. *Under the conditions of Theorem 8 if the function $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, Levitan almost periodic, almost recurrent in the sense of Bebutov, recurrent in the sense of Birkhoff, Poisson stable), then equation (21) has at least one τ-periodic (respectively, almost automorphic, Levitan almost periodic, almost recurrent in the sense of Bebutov, recurrent in the sense of Birkhoff, Poisson stable) solution.*

Proof. If the function $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is τ-periodic (respectively, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable), then this statement follows from Theorem 8.

Suppose that $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is Bohr almost periodic (respectively, almost automorphic, recurrent in the sense of Birkhoff), then it is Levitan almost periodic. By above arguments equation (21) has at least one Levitan almost periodic solution $\bar{\psi}$. On the other hand the solution $\bar{\psi}$ is relatively compact and uniformly continuous on $\mathbb{R}$ because $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is bounded on $\mathbb{R}$. Thus the function $\bar{\psi}$ is Levitan almost periodic and stable in the sense of Lagrange and, consequently, it is almost automorphic (respectively, recurrent in the sense of Birkhoff). $\square$

5.2 Linear partial differential equations

5.2.1 Linear homogeneous differential equations

Let $(\mathfrak{B}, |\cdot|)$ be a Banach space with the norm $|\cdot|$, $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ (or shortly φ) be a linear cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$, i.e., φ is a continuous mapping from $\mathbb{R}_+ \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ satisfying the following conditions:

1. $\varphi(0, u, y) = u$ for all $u \in \mathfrak{B}$ and $y \in Y$;
2. $\varphi(t + \tau, u, y) = \varphi(t, \varphi(\tau, u, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{R}_+$, $u \in \mathfrak{B}$ and $y \in Y$;
3. for all $(t, y) \in \mathbb{R}_+ \times Y$ the mapping $\varphi(t, \cdot, y) : \mathfrak{B} \mapsto \mathfrak{B}$ is linear.

Denote by $[\mathfrak{B}]$ the Banach space of all linear bounded operators A acting on the space $\mathfrak{B}$ equipped with the operator norm $||A|| := \sup\limits_{|x| \leq 1} |Ax|$.

Example 6. Let Y be a complete metric space, $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y and Λ be some complete metric space of linear closed operators acting into Banach space $\mathfrak{B}$ (for example $\Lambda = \{A_0 + B | B \in [\mathfrak{B}]\}$, where A_0 is a closed operator that acts on $\mathfrak{B}$). Consider the following linear differential equation

$$x' = A(\sigma(t, y))x, \quad (y \in Y) \tag{23}$$

where $A \in C(Y, \Lambda)$. We assume that the following conditions are fulfilled for equation (23):

a. for any $u \in \mathfrak{B}$ and $y \in Y$ equation (23) has exactly one solution that is defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, u, y) = u$;

b. the mapping $\varphi : (t, u, y) \to \varphi(t, u, y)$ is continuous in the topology of $\mathbb{R}_+ \times \mathfrak{B} \times Y$.

Under the above assumptions the equation (23) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$.

Example 7. Let Λ be some complete metric space of linear closed operators acting into Banach space $\mathfrak{B}$. Consider the differential equation

$$x' = A(t)x, \tag{24}$$

where $A \in C(\mathbb{R}, \Lambda)$. Along with equation (24) consider its H-class, i.e., the following family of equations

$$x' = B(t)x, \tag{25}$$

where $B \in H(A)$. We assume that the following conditions are fulfilled for equation (24) and its H-class (25):

a. for any $u \in \mathfrak{B}$ and $B \in H(A)$ equation (25) has exactly one mild solution $\varphi(t, u, B)$ (i.e., $\varphi(\cdot, u, B)$ is continuous, defined on $\mathbb{R}_+$ and satisfies the equation

$$\varphi(t, v, B) = U(t, B)v + \int_0^t U(t - \tau, B^\tau)\varphi(\tau, v, B)d\tau$$

and the condition $\varphi(0, u, B) = v$);

b. the mapping $\varphi : (t, u, B) \to \varphi(t, u, B)$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}; \Lambda)$.

Denote by $(H(A), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A)$. Under the above assumptions the equation (24) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (H(A), \mathbb{R}, \sigma)\rangle$ over dynamical system $(H(A), \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$.

Note that equation (24) and its H-class can be written in the form (23). We put $Y := H(A)$ and denote by $\mathcal{A} \in C(Y, \Lambda)$ defined by equality $\mathcal{A}(B) := B(0)$ for all $B \in H(A) = Y$, then $B(\tau) = \mathcal{A}(\sigma(B, \tau))$ ($\sigma(\tau, B) := B^\tau$, where $B^\tau(t) := B(t + \tau)$ for all $t \in \mathbb{R}$). Thus the equation (24) with its H-class can be rewrite as follow

$$x' = \mathcal{A}(\sigma(t, B))x \ (B \in H(A)).$$

We will consider example of partial differential equations which satisfy the above conditions a.-b.

Example 8. *Consider the differential equation*

$$u' = (A_1 + A_2(t))u, \tag{26}$$

where A_1 is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $A_2 \in C(\mathbb{R}, [\mathfrak{B}])$. The results of [15],[21] imply that equation (26) satisfies conditions a.-b. from Example 6.

5.2.2 Linear non-homogeneous (affine) differential equations

Let $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ be a linear cocycle over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber E, $f \in C(Y, \mathbb{B})$ and ψ be a mapping from $\mathbb{T} \times \mathfrak{B} \times Y$ into $\mathfrak{B}$ defined by equality

$$\psi(t, u, y) := U(t, y)u + \int_0^t U(t-\tau, \sigma(\tau, y))f(\sigma(\tau, y))d\tau \ . \tag{27}$$

From the definition of the mapping ψ it follows that ψ possesses the following properties:

1. $\psi(0, u, y) = u$ for any $(u, y) \in \mathfrak{B} \times Y$;
2. $\psi(t+\tau, u, y) = \psi(t, \psi(\tau, u, y), \sigma(\tau, y))$ for any $t, \tau \in \mathbb{R}_+$ and $(u, y) \in \mathfrak{B} \times Y$;
3. the mapping $\psi : \mathbb{R}_+ \times \mathfrak{B} \times Y \mapsto \mathfrak{B}$ is continuous;
4. $\psi(t, \lambda u + \mu v, y) = \lambda\psi(t, u, y) + \mu\psi(t, v, y)$ for any $t \in \mathbb{R}_+$, $u, v \in \mathfrak{B}$, $y \in Y$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$) with condition $\lambda + \mu = 1$, i.e., the mapping $\psi(t, \cdot, y) : \mathfrak{B} \mapsto \mathfrak{B}$ is affine for every $(t, y) \in \mathbb{R}_+ \times Y$.

Remark 10. If we have a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$ and $f \in C(Y, \mathbb{B})$, then by equality (27) is defined an affine cocycle $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$ which is called an affine (non-homogeneous) cocycle associated by linear cocycle φ and the function $f \in C(Y, \mathfrak{B})$.

Example 9. Let Y be a complete metric space, $(Y, \mathbb{R}, \sigma)$ be a dynamical system on Y and Λ be some complete metric space of linear closed operators acting into Banach space $\mathfrak{B}$ and $f \in C(Y, \mathfrak{B})$. Consider the following linear non-homogeneous differential equation

$$x' = A(\sigma(t, y))x + f(\sigma(t, y)), \quad (y \in Y) \tag{28}$$

where $A \in C(Y, \Lambda)$. We assume that conditions a. and b. from Example 6 are fulfilled for equation (28).

Under the above assumptions equation (28) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$. According to Remark 10 by equality (27) a linear non-homogeneous cocycle $\langle \mathfrak{B}, \psi, (Y, \mathbb{R}, \sigma)\rangle$ over dynamical system $(Y, \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$ is defined. Thus every non-homogeneous linear differential equation (28), under conditions a. and b. generates a linear non-homogeneous cocycle ψ.

Example 10. Let Λ be some complete metric space of linear closed operators acting into Banach space $\mathfrak{B}$ and $f \in C(\mathbb{R}, \mathfrak{B})$. Consider a linear non-homogeneous differential equation

$$x' = A(t)x + f(t), \tag{29}$$

where $A \in C(\mathbb{R}, \Lambda)$. Along with equation (29) consider its H-class, i.e., the following family of equations

$$x' = B(t)x + g(t), \tag{30}$$

where $(B, g) \in H(A, f)$. We assume that the following conditions are fulfilled for equation (24) and its H-class (25):

a. for any $u \in \mathfrak{B}$ and $B \in H(A)$ equation (25) has exactly one mild solution $\varphi(t, u, B)$ with the condition $\varphi(0, u, B) = v$;

b. the mapping $\varphi : (t, u, B) \to \varphi(t, u, B)$ is continuous in the topology of $\mathbb{R}_+ \times \mathfrak{B} \times C(\mathbb{R}; \Lambda)$.

Denote by $(H(A), \mathbb{R}, \sigma)$ the shift dynamical system on $H(A)$. Under the above assumptions the equation (24) generates a linear cocycle $\langle \mathfrak{B}, \varphi, (H(A), \mathbb{R}, \sigma) \rangle$ over dynamical system $(H(A), \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$. Denote by ψ the mapping from $\mathbb{R}_+ \times \mathfrak{B} \times H(A)$ into $\mathfrak{B}$ defined by equality

$$\psi(t, u, (B, g)) := U(t, B)u + \int_0^t U(t - \tau, B^\tau) g(\tau) d\tau,$$

then ψ possesses the following properties:

(i) $\psi(0, u, (B, g)) = u$ for any $(u, (B, g)) \in \mathfrak{B} \times H(A, f)$;

(ii) $\psi(t + \tau, u, (B, g)) = \psi(t, \psi(\tau, u, (B, g)), (B^\tau, g^\tau))$ for any $t, \tau \in \mathbb{R}_+$ and $(u, (B, g)) \in \mathfrak{B} \times H(A, f)$;

(iii) the mapping $\psi : \mathbb{R}_+ \times \mathfrak{B} \times H(A, f) \mapsto \mathfrak{B}$ is continuous;

(iv) $\psi(t, \lambda u + \mu v, (B, g)) = \lambda \psi(t, u, (B, g)) + \mu \psi(t, v, (B, g))$ for any $t \in \mathbb{R}_+$, $u, v \in \mathfrak{B}$, $(B, g) \in H(A, f)$ and $\lambda, \mu \in \mathbb{R}$ (or $\mathbb{C}$) with condition $\lambda + \mu = 1$, i.e., the mapping $\psi(t, \cdot, (B, g)) : \mathfrak{B} \mapsto \mathfrak{B}$ is affine for every $(t, (B, g)) \in \mathbb{R}_+ \times H(A, f)$.

Thus, every linear non-homogeneous differential equation of the form (29) (and its H-class (30)) generates a linear non-homogeneous cocycle $\langle \mathfrak{B}, \psi, (H(A, f), \mathbb{R}, \sigma) \rangle$ over dynamical system $(H(A, f), \mathbb{R}, \sigma)$ with the fiber $\mathfrak{B}$.

Definition 22. A closed linear operator $A : D(A) \to \mathfrak{B}$ with dense domain $D(A)$ is said [15] to be *sectorial* if one can find a $\phi \in (0, \frac{\pi}{2})$, an $M \geq 1$, and a real number a such that the sector

$$S_{a,\phi} := \{\lambda \mid |\arg\,(\lambda - a)| \leq \pi, \lambda \neq a\}$$

lies in the resolvent set $\rho(A)$ of A and $\|(\lambda I - A)^{-1}\| \leq M|\lambda - a|^{-1}$ for any $\lambda \in S_{a,\phi}$. An important class of sectorial operators is formed by elliptic operators [15],[17].

Remark 11. Consider the differential equation

$$u' = (A_1 + A_2(t))u, \tag{31}$$

where A_1 is a sectorial operator that does not depend on $t \in \mathbb{R}$, and $A_2 \in C(\mathbb{R}, [\mathfrak{B}])$.

The results of [15, 21], imply that equation (31) satisfies conditions (i)-(iii).

Note that equation (29) (and its H-class (30)) can be written in the form (23). We put $Y := H(A, g)$ and denote by $\mathcal{A} \in C(H(A, f), \Lambda)$ (respectively, $\mathcal{F} \in C(H(A, f), E)$) defined by equality $\mathcal{A}(B, g) := B(0)$ (respectively, $\mathcal{F}(B, g) = g(0)$) for any $(B, g) \in H(A, f)$, then $B(\tau) = \mathcal{A}(B^\tau, g^\tau)$ (respectively, $g(\tau) = \mathcal{F}(B^\tau, g^\tau)$), where $B^\tau(t) := B(t + \tau)$ and $g^\tau(t) := g(t + \tau)$ for any $t \in \mathbb{R}$). Thus equation (29) with its H-class can be rewrite as follow

$$x' = \mathcal{A}(\sigma(t, (B, g)))x + \mathcal{F}(\sigma(t, (B, g))) \ \big((B, g) \in H(A, f)\big).$$

5.2.3 Levitan almost periodic solutions of linear partial differential equations

Theorem 9. *Assume that the following conditions are fulfilled:*

1. $\mathfrak{B}$ *is a uniformly convex Banach space;*
2. $\mathfrak{B}^s_A := \{u \in \mathfrak{B} | \ \sup_{t \ge 0} |\varphi(t, u, A)| < \infty\}$ *is a subspace of the Banach space* $\mathfrak{B}$, *where* φ *is a linear cocycle generated by equation (24);*
3. *the function* $f \in C(\mathbb{R}, \mathfrak{B})$ *and operator-function* $A \in C(\mathbb{R}, \Lambda)$ *are jointly Poisson stable;*
4. *there exits a relatively compact on* $\mathbb{R}_+$ *solution* $\psi(t, u_0, A, f)$ *of equation (29).*

Then there exists at least one compatible solution $\psi(t, \bar{u}, A, f)$ *of equation (29), i.e.,* $\mathfrak{N}^{+\infty}_{(A,f)} \subseteq \mathfrak{N}^{+\infty}_{\bar{\psi}}$, *where* $\bar{\psi} := \psi(\cdot, \bar{u}, A, f)$.

Proof. Let $\langle \mathfrak{B}, \psi, (H(A, f), \mathbb{R}, \sigma)\rangle$ be a cocycle generated by equation (29). By Example 10 the cocycle ψ is affine. Now applying Theorem 7 to constructed cocycle ψ we complete the proof of Theorem. □

Remark 12. 1. Note that the definition of almost automorphy is equivalent to the following: the function $\varphi \in C(\mathbb{R}, X)$ is almost automorphic if and only if from every sequence $\{t'_n\} \subset \mathbb{R}$ we can extract a subsequence $\{t_n\}$ such that

$$\varphi(t + t_n) \to \psi(t) \ \text{ and } \ \psi(t - t_n) \to \varphi(t) \tag{32}$$

uniformly in t on every compact subset from $\mathbb{R}$.

2. The original definition of Bochner [2] is the following: the function $\varphi \in C(\mathbb{R}, X)$ is almost automorphic if and only if from every sequence $\{t'_n\} \subset \mathbb{R}$ we can extract a subsequence $\{t_n\}$ such that the relations in (32) take place pointwise for $t \in \mathbb{R}$.

Lemma 7. *[13] Suppose that the function $\varphi \in C(\mathbb{R}, X)$ is uniformly continuous on $\mathbb{R}$ and almost automorphic in the sense of Bochner. Then it is almost automorphic in the sense of Bohr.*

Remark 13. The function $\varphi(t) = \sin(\frac{1}{2+\cos t+\cos\sqrt{2}t})$ is

1. almost automorphic in the sense of Bochner [1, Example 3.1];

2. Levitan almost periodic, but it is not almost automorphic (in the sense of Bohr), because φ is not uniformly continuous on $\mathbb{R}$ [20, Ch.V, pp.212–213].

Corollary 4. *Under the conditions of Theorem 8 if the function $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is τ-periodic (respectively, Bohr almost periodic, almost automorphic, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable), then equation (29) has at least one τ-periodic (respectively, almost automorphic in the sense of Bochner, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable) solution.*

Proof. If the function $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is τ-periodic (respectively, Levitan almost periodic, almost recurrent in the sense of Bebutov, Poisson stable), then this statement follows from Theorem 8.

Suppose that $(A, f) \in C(\mathbb{R}, [\mathfrak{B}]) \times C(\mathbb{R}, \mathfrak{B})$ is Bohr almost periodic (respectively, almost automorphic), then it is Levitan almost periodic. By above arguments equation (29) has at least one Levitan almost periodic solution $\bar{\psi}$. On the other hand the solution $\bar{\psi}$ is relatively compact on $\mathbb{R}$ and by Remark 12 it is almost automorphic in the sense of Bochner. ☐

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