Computer Science Journal of Moldova, vol.**26**, no.1(76), 2018

On Nontrivial Covers and Partitions of Graphs by Convex Sets

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Abstract

In this paper we prove that it is NP-complete to decide whether a graph can be partitioned into nontrivial convex sets. We show that it can be verified in polynomial time whether a graph can be covered by nontrivial convex sets. Also, we propose a recursive formula that establishes the maximum nontrivial convex cover number of a tree.

Keywords: Convexity, complexity, nontrivial convex cover, nontrivial convex partition, tree.

MSC 2010: 68R10, 68Q25, 05C35, 05C05.

1 Introduction

We denote by $G = (X; U)$ a simple undirected connected graph with vertex set X, $|X| = n$, and edge set U, $|U| = m$. We also specify the vertex set of G by $X(G)$. The *neighborhood* of $x \in X$ is the set of all vertices $y \in X$ such that y is adjacent to x, and it is denoted by $\Gamma(x)$. The *distance* $d(x, y)$ between two vertices $x, y \in X$ is the length of the shortest path between x and y. The *diameter* of G, denoted by $diam(G)$, is the distance between two farthest vertices of G. A set $S \subset X$ is called *nontrivial* if $3 \leq |S| \leq |X| - 1$.

We remind some notions defined in [1]. A set $S \subseteq X$ is called *convex* if $\{z \in X | d(x, z) + d(z, y) = d(x, y)\} \subseteq S$ for all vertices $x, y \in S$. The *convex hull* of $S \subseteq X$, denoted by $d - conv(S)$, is the smallest convex set containing S.

The concept of *convex p-cover* of a graph is introduced by Artigas et al. in [6] and is studied in a series of papers [6] – [13]. We defined a *nontrivial convex cover* $\mathcal{P}(G)$ of a graph G in [10] as a family of sets that satisfies the following conditions:

1) every set of $\mathcal{P}(G)$ is nontrivial and convex in G;
2) $X = \bigcup_{Y \in \mathcal{P}(G)} Y$;
3) $Y \not\subseteq \bigcup_{Z \in \mathcal{P}(G), Z \neq Y} Z$ for every $Y \in \mathcal{P}(G)$.

If $|\mathcal{P}(G)| = p$, then we say that this family is a *nontrivial convex p-cover* of G and write $\mathcal{P}_p(G)$. A nontrivial convex cover is a *nontrivial convex partition* if the sets of the cover are pairwise disjoint. Correspondingly, a nontrivial convex p-cover is said to be a *nontrivial convex p-partition* if it is a nontrivial convex partition.

As it can be seen, in $\mathcal{P}(G)$ for each set $S \in \mathcal{P}(G)$ there exists a vertex x_S such that $x_S \in S$ and $x_S \neq S'$ for any $S' \in \mathcal{P}(G)$, $S' \neq S$. We call such uniquely covered vertices *resident vertices* [10].

The largest $p \geq 2$ for which a graph G admits a nontrivial convex cover with p sets is called the *maximum nontrivial convex cover number* of G and is denoted by $\varphi_{cn}^{max}(G)$. Respectively, the *maximum nontrivial convex cover* $\mathcal{P}_{\varphi_{cn}^{max}}(G)$ is the nontrivial convex p-cover of G such that $p = \varphi_{cn}^{max}(G)$ [11]. Indeed, it is natural under conditions 1) – 3) to maximize the number of nontrivial convex sets in the cover. Moreover, it is applicable for determination of nontrivial convex p-cover of graphs. We know that it is NP-complete to decide whether a graph has a nontrivial convex p-cover or p-partition for a fixed $p \geq 2$ [10]. Some consistent results are obtained for a tree [11]. Among these results the most important are that a tree T on $n \geq 4$ vertices has a nontrivial convex p-cover for every p, $2 \leq p \leq \varphi_{cn}^{max}(T)$, and it can be decided in polynomial time whether T on $n \geq 6$ vertices has a nontrivial convex p-partition for a fixed p, $2 \leq p \leq \lfloor \frac{n}{3} \rfloor$. There exist graphs for which there are no nontrivial convex covers or partitions. For instance, if any proper nonempty convex set of a graph is a vertex or an edge we obtain the so-called convex simple graph [3]. Obviously, this kind of graph has no any nontrivial convex cover. Further, it is of interest to determine the complexity of decision whether a graph can be covered or partitioned into nontrivial convex sets. In the present paper we study this problem and continue our research on nontrivial convex p-cover problem of a tree.

2 Nontrivial convex covers of graphs

This section is dedicated to studying the complexity of decision whether a graph can be covered or partitioned into nontrivial convex sets. The first problem can be formulated as follows:

Problem: Nontrivial Convex Cover (NCC).
Instance: A graph G.
Question: Is there a nontrivial convex cover of G?

Equivalently, it can be defined the Nontrivial Convex Partition problem (NCP). For this purpose, we only change the question of NCC problem like this: Is there a nontrivial convex partition of G?

Note that if for every vertex of G there exists at least one nontrivial convex set that contains it, then there is a nontrivial convex cover of G. The converse affirmation is also true. Based on these statements, we propose a polynomial algorithm, represented below, that solves the NCC problem.

Algorithm 1.
Input: A graph $G = (X; U)$.
Output: Nontrivial convex cover $\mathcal{P}(G)$ or nothing.

$\mathcal{P}(G) \leftarrow \varnothing$
$M \leftarrow \varnothing$
for $x \in X$ **do**
    if $x \notin M$ **then**
        $flag \leftarrow 0$
        for $y, z \in X \backslash \{x\}$, $y \neq z$ **do**
            $S \leftarrow d - conv(\{x, y, z\})$
            if $S \neq X$ **then**
                $\mathcal{P}(G) \leftarrow \mathcal{P}(G) \cup \{S\}$
                $M \leftarrow M \cup S$
                $flag \leftarrow 1$
                break
        if $flag = 0$ **then**
            stop: there does not exist any nontrivial convex set containing x

15: **for** $Y \in \boldsymbol{\mathcal{P}}(G)$ **do**
16: **if** $Y \subseteq \bigcup_{Z \in \boldsymbol{\mathcal{P}}(G), Z \neq Y} Z$
17: $\boldsymbol{\mathcal{P}}(G) \leftarrow \boldsymbol{\mathcal{P}}(G) \setminus \{Y\}$
18: **return** $\boldsymbol{\mathcal{P}}(G)$

Theorem 1. *Algorithm 1 decides in time $O(n^4 m)$ whether a graph G can be covered by nontrivial convex sets.*

Proof. Steps 1), 2) and 18) run in constant time. Steps 3) – 14) determine whether, for every vertex $x \in X$, there exists a nontrivial convex set $S \subset X$, $x \in S$. If there is such a set S, then there are at least two different vertices $y, z \in X$, $y \neq x$, $z \neq x$, for which $d-conv(\{x, y, z\}) \subseteq S$. Consequently, it is sufficient to build convex hull for all sets of three vertices, one of which is x. A convex hull of $S \subset X$ can be constructed in time $O(|d - conv(S)|m)$ [4]. Since $|d - conv(S)|$ can reach up to n vertices, steps 3) – 14) run in time $O(n^4 m)$.

Steps 15) – 17) exclude from $\boldsymbol{\mathcal{P}}(G)$ all sets contained in union of other sets of $\boldsymbol{\mathcal{P}}(G)$. So, we obtain a nontrivial convex cover of G. The resulting family $\boldsymbol{\mathcal{P}}(G)$ has at most $n - 2$ sets and every set of $\boldsymbol{\mathcal{P}}(G)$ contains no more than $n - 1$ vertices. Further, steps 15) – 17) run in time $O(n^3)$ and the complexity of the whole algorithm is $O(n^4 m)$. □

In the sequel, we show that NCP problem is NP-complete by reducing to NCP the well-known NP-complete problem Partition Into Triangles [2] that is formulated as follows:

Problem: Partition Into Triangles (PIT).
Instance: A graph $G = (X; U)$ with $|X| = 3q$, where $q \in N$.
Question: Is there a partition of X into q disjoint subsets $X_1, X_2, \ldots, X_q$ of size 3 such that each X_i, $1 \leq i \leq q$, induces a triangle in G?

Note that the PIT problem remains NP-complete even if input graph G is tripartite [5] (a graph is tripartite iff it can be partitioned in 3 independent sets). Also notice that every tripartite graph has no 4-cliques.

Theorem 2. *Problem NCP is NP-complete.*

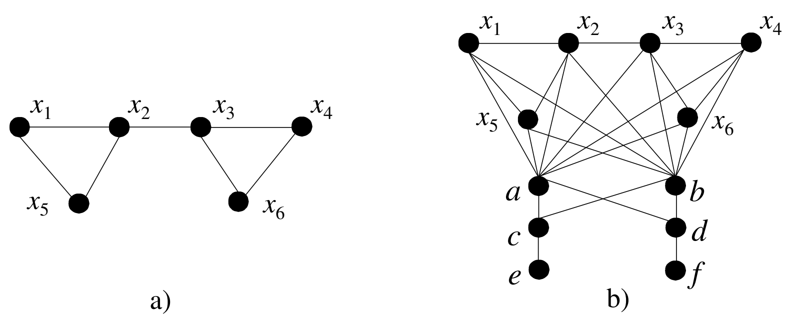

Figure 1. The graph G' (case b) that admits a nontrivial convex partition is obtained from the graph G (case a) that admits a partition into triangles.

Proof. First notice that NCP is in NP, because any nontrivial convex cover of G contains a linear number of sets and verifying if a family of sets with a linear number of sets is a nontrivial convex cover can be done in polynomial time [4]. We reduce the NP-complete problem PIT for tripartite graphs to NCP. Let $G = (X; U)$ be a tripartite instance of PIT, $|X| = 3q$, $q \in N$. From G we will derive an instance $G' = (X'; U')$ of NCP in the following way:

1) $X' = X \cup \{a, b, c, d, e, f\}$;
2) $U' = U \cup \{\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, e\}, \{d, f\}\} \cup$
$\cup \{\{a, x\}, \{b, x\} | x \in X\}$.

In Figure 1 the graph G' (case b) that corresponds to a particular instance of PIT problem G (case a) is represented.

We need to show that G admits a partition into q triangles if and only if there exists a nontrivial convex cover of G'.

Let $\mathcal{P}_q(G) = \{X_1, X_2, \ldots, X_q\}$ be a family of triangles that partitions G. Since every triangle is a clique in G, it follows that each X_i, $1 \leq i \leq q$, is nontrivial and convex in G' and the set $\{a, b, c, d, e, f\}$ remains uncovered in G'. Observe that $d - conv_{G'}(\{a, c, e\}) = \{a, c, e\}$ and $d - conv_{G'}(\{b, d, f\}) = \{b, d, f\}$. For this reason, the family of sets $\mathcal{P}_q(G) \cup \{\{a, c, e\}, \{b, d, f\}\}$ generates a partition of G' into $q + 2$ nontrivial convex sets.

Let $\boldsymbol{\mathcal{P}}(G')$ be a partition of G' into nontrivial convex sets and let S be a set of $\boldsymbol{\mathcal{P}}(G')$. We distinguish some properties of S:

1) $\{a,b\} \not\subset S$. Assuming the contrary, namely that $\{a,b\} \subset S$, we see that $d - conv_{G'}(\{a,b\}) = X \cup \{a,b,c,d\}$ and further obtain $X' \backslash d{-}conv_{G'}(\{a,b\}) = \{e,f\}$. Note that the set $\{e,f\}$ is not nontrivial and convex. Hence, $\boldsymbol{\mathcal{P}}(G')$ can not partition G' into nontrivial convex sets. We get the required contradiction.

2) $\{c,d\} \not\subset S$. Assuming the converse, it can easily be checked that $\{a,b\} \subset d - conv_{G'}(\{c,d\})$. Therefore, property 1) is not satisfied and we obtain a contradiction.

3) $\{e,f\} \not\subset S$. Conversely, we have $\{a,b,c,d\} \subset d - conv_{G'}(\{e,f\})$ and consequently properties 1) and 2) are not satisfied. This implies a contradiction.

4) $\{x,y\} \not\subset S$ for every vertex $x \in X$ and $y \in \{c,d\}$. Assuming the converse, there exist $x \in X$ and $y \in \{c,d\}$ such that $\{x,y\} \subset S$. Since $\{a,b\} \subset d - conv_{G'}(\{x,y\})$, we get a contradiction.

5) $\{x,y\} \not\subset S$ for every two nonadjacent vertices $x,y \in X$. In the converse case, there are two nonadjacent vertices x and y of X for which $\{x,y\} \subset S$. It follows that $\{a,b\} \subset d - conv_{G'}(\{x,y\})$. Have a contradiction.

Let $S_1 = \{a,c,e\}$, $S_2 = \{b,d,f\}$, $S_3 = \{b,c,e\}$ and $S_4 = \{a,d,f\}$. Taking into account properties 1) – 5) and the fact that every vertex of X' belongs exactly to one set of $\boldsymbol{\mathcal{P}}(G')$, it is seen that $\boldsymbol{\mathcal{P}}(G')$ contains strictly a pair of sets of the following two: S_1, S_2 or S_3, S_4. Each pair of sets covers vertices a,b,c,d,e and f. Hence, vertices of $X' \backslash \{a,b,c,d,e,f\}$ need to be partitioned into nontrivial convex sets. By property 5), all of these sets are cliques. As mentioned above, G has no cliques with $k \geq 4$ vertices. Further, all of these sets are triangles and by elimination of a pair of sets S_1, S_2 or S_3, S_4 from $\boldsymbol{\mathcal{P}}(G')$ we obtain a family of triangles $\boldsymbol{\mathcal{P}}_q(G)$. □

3 Maximum nontrivial convex cover of a tree

We denote by T a tree and by $C(T)$, $|C(T)| = p$, a set of terminal vertices of T. Recall that a *terminal vertex* is a vertex of degree 1.

In this section we continue our research on nontrivial convex p-cover problem of a tree. Below we determine the number $\varphi_{cn}^{max}(T)$. Let us remind some results, which will be useful in the sequel.

Theorem 3. [11] *If $diam(T) \geq 3$, then there exists a maximum nontrivial convex cover $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$ such that every terminal vertex of T is resident in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$ and any two terminal vertices do not belong to the same set of $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$.*

Corollary 1. [11] *If $diam(T) \geq 3$ and every nonterminal vertex of T is adjacent to at least one terminal vertex, then $\varphi_{cn}^{max}(T) = p$.*

Corollary 2. [11] *If $3 \leq diam(T) \leq 5$, then $\varphi_{cn}^{max}(T) = p$.*

By $M(T)$, $|M(T)| = q$, we denote a set of vertices x of T, for which distance between all vertices of $C(T)$ and x is greater then or equal to 3 and there exists a vertex $c \in C(T)$, $d(x, c) = 3$.

Theorem 4. *If $diam(T) \geq 6$ and $M(T) \neq \varnothing$, then $\varphi_{cn}^{max}(T) \geq p + q$.*

Proof. We define a family of nontrivial convex sets $\boldsymbol{\mathcal{P}}(T) = \varnothing$ that will cover T. For every terminal vertex $c \in C(T)$ we select the nearest $x \in M(T)$ and the path $L = [x, x_1, x_2, \ldots, x_k, c]$, $k \geq 2$. Since the set $S_c = \{x_1, x_2, \ldots, x_k, c\}$ is nontrivial and convex, we add S_c to $\boldsymbol{\mathcal{P}}(T)$. Besides, for every $x \in M(T)$ we select a terminal vertex $c \in C(T)$, $d(x, c) = 3$, and for the obtained path L with $k = 2$ form a nontrivial convex set $S_x = \{x, x_1, x_2\}$ and add it to $\boldsymbol{\mathcal{P}}(T)$. If there remain some uncovered vertices, then we select an uncovered vertex y that is adjacent to a vertex $z \in S$, $S \in \boldsymbol{\mathcal{P}}(T)$, and further add y to S. We see that every vertex of $A = C(T) \cup M(T)$ is resident in $\boldsymbol{\mathcal{P}}(T)$ and any two vertices from A do not belong to the same set of $\boldsymbol{\mathcal{P}}(T)$. In consequence, we obtain a nontrivial convex cover $\boldsymbol{\mathcal{P}}(T)$ such that $|\boldsymbol{\mathcal{P}}(T)| = p + q$. Therefore, we have $\varphi_{cn}^{max}(T) \geq p + q$. □

An important result is given by the following theorem.

Theorem 5. *If T has $n \geq 4$ vertices, then there exists a maximum nontrivial convex cover $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$ such that every set $S \in \boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$ contains a path $L = [x, y, z]$, where x is a resident vertex in $\boldsymbol{\mathcal{P}}_{\varphi_{cn}^{max}}(T)$.*

Proof. If $diam(T) = 2$, then the statement of the theorem is obvious. If $3 \leq diam(T) \leq 5$, then it follows from Corollary 2 that the theorem is true. If $diam(T) \geq 6$, then taking into account Theorem 3, there is a family $\mathcal{P}_{\varphi_{cn}^{max}}(T)$ such that for every terminal vertex $x \in C(T)$ that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(T)$ and for a set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(T)$, $x \in S$, there exists a path $L = [x, y, z]$, where $y, z \in S$.

We define a family of nontrivial convex sets $\mathcal{P}(T) = \varnothing$ that will cover T and a set of vertices $D = \varnothing$. If there is a set $A \in \mathcal{P}_{\varphi_{cn}^{max}}(T)$, containing a terminal vertex $a \in A$ that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(T)$, and there exists another set $B \in \mathcal{P}_{\varphi_{cn}^{max}}(T)$, $A \cap B \neq \varnothing$, $|A \backslash B| \geq 2$ or $|B \backslash A| \geq 2$, then we denote by B_r a set of resident vertices of B in $\mathcal{P}_{\varphi_{cn}^{max}}(T)$. Next, we select a vertex $b \in B_r$, where the distance $d(a, b)$ is maximum. Also, we denote by B_b all verices b' of B for which a path between b' and a contains b. Evidently, vertex b belongs to B_b. We now define two sets:

$$A' = A \cup (B \backslash B_b) \text{ and } B' = (A' \cup \{b\}) \backslash \{a\}.$$

It can easily be checked that A' and B' are nontrivial convex sets and there are paths $[a, c, d]$ and $[b, c', d']$, where $c, d \in A'$ and $c', d' \in B'$. Further, we replace sets A and B by A' and B' in $\mathcal{P}_{\varphi_{cn}^{max}}(T)$. If there still remains such a set $A \in \mathcal{P}_{\varphi_{cn}^{max}}(T)$ that satisfies the conditions mentioned above, then we repeat the described process. Otherwise, we define a family $\mathcal{A}$ that consists of sets from $\mathcal{P}_{\varphi_{cn}^{max}}(T)$, which contain exactly one terminal vertex that is resident in $\mathcal{P}_{\varphi_{cn}^{max}}(T)$. Next, we select a set $A \in \mathcal{A}$ and define a family $\mathcal{B}_A$, composed of sets which belong to $\mathcal{P}_{\varphi_{cn}^{max}}(T)$ and intersect A. Let $D_A = A \cup \bigcup_{B \in \mathcal{B}_A} B$. We add vertices of D_A to D, remove A and every set of $\mathcal{B}_A$ from $\mathcal{P}_{\varphi_{cn}^{max}}(T)$ and from $\mathcal{A}$, and then add them to $\mathcal{P}(T)$. If $\mathcal{A} \neq \varnothing$, then we choose another set $A \in \mathcal{A}$ and repeat the above procedure. In the contrary case, remove from T vertices of D and edges incident to them. If $X(T) = \varnothing$, then $|\mathcal{P}(T)| = |\mathcal{P}_{\varphi_{cn}^{max}}(T)|$. This implies correctness of the theorem.

If $X(T) \neq \varnothing$, then there are obtained $k \geq 1$ subtrees $T_1, T_2, \ldots, T_k$. It is clear that $X(T_i) \geq 3$, $1 \leq i \leq k$, and every set of $\mathcal{P}(T)$ does not intersect any set of $\mathcal{P}'(T)$, obtained from $\mathcal{P}_{\varphi_{cn}^{max}}(T)$ after elimination of convex sets in the actions described above. Therefore, we have:

$$|\mathcal{P}'(T)| + |\mathcal{P}(T)| = \varphi_{cn}^{max}(T).$$

If $2 \leq diam(T_i) \leq 5$, for every i, $1 \leq i \leq k$, then, considering that if $|X(T_i)| = 3$, then $|\mathcal{P}(T_i)| = 1$, by Corollary 2 a maximum nontrivial convex cover $\mathcal{P}(T_i)$ is easily obtained for every T_i such that the affirmation of the theorem is true. Conversely, for every T_i, $diam(T) \geq 6$, we define a family of sets $\mathcal{P}(T_i) = \varnothing$ that will cover T_i and recursively fill it using rationales from the demonstration. Further, we get:

$$\sum_{i=1}^{k} |\mathcal{P}(T_i)| + |\mathcal{P}(T)| = \varphi_{cn}^{max}(T).$$

So, now we add all sets from $\mathcal{P}(T_i)$, $1 \leq i \leq k$, to $\mathcal{P}(T)$ and see that the theorem is proved. □

Suppose that $diam(T) \geq 6$, then we define the set:

$$N(T) = X(T) \setminus \left(C(T) \cup \bigcup_{x \in C(T)} \Gamma(x) \right).$$

The set $N(T)$ is empty if and only if every nonterminal vertex of T is adjacent to at least one terminal vertex of T, but in this case, accordingly to Corollary 1, we obtain $\varphi_{cn}^{max}(T) = p$. Assume further that $N(T) \neq \varnothing$. Let x be a vertex of $N(T)$. Since x is an articulation vertex, through the elimination of x from T we obtain $|\Gamma(x)|$ connected components T_x^y, $y \in \Gamma(x)$. For every vertex $y \in \Gamma(x)$ we get the family of subtrees:

$$\mathcal{V}_x^y(T) = {}^*T_x^y \cup \bigcup_{z \in \Gamma(x) \setminus \{y\}} T_x^z,$$

where ${}^*T_x^y$ is a subtree of T obtained by adding x to T_x^y such that x is adjacent to y.

Thus, we get the family of subfamilies of subtrees:

$$\mathcal{V}_x(T) = \bigcup_{y \in \Gamma(x)} \mathcal{V}_x^y(T).$$

For the sake of estimation of the number $\varphi_{cn}^{max}(T)$, we consider that if $0 \le n \le 2$, then $\varphi_{cn}^{max}(T) = 0$, and if $n = 3$, then $\varphi_{cn}^{max}(T) = 1$. We obtain the recursive formula, reflected in Theorems 6 and 7, that determines $\varphi_{cn}^{max}(T)$.

Let us remark that for a tree T with $diam(T) = 2$, $n \ge 4$, it can easily be checked that $\varphi_{cn}^{max}(T) = p-1$. Taking into account Corollaries 1, 2 and the fact mentioned above, we get Theorem 6.

Theorem 6. *If $diam(T) \le 5$ or $diam(T) \ge 6$ and $N(T) = \varnothing$, then the following relation holds:*

$$\varphi_{cn}^{max}(T) = \begin{cases} p, & \text{if } 3 \le diam(T) \le 5 \text{ or} \\ & diam(T) \ge 6 \text{ and } N(T) = \varnothing; \\ p\text{-}1, & \text{if } diam(T) = 2; \\ 0, & \text{if } 0 \le diam(T) \le 1. \end{cases}$$

If $diam(T) \ge 6$ and $N(T) \ne \varnothing$, then by Theorem 5 we obtain Theorem 7.

Theorem 7. *If $diam(T) \ge 6$ and $N(T) \ne \varnothing$, then the following relation holds:*

$$\varphi_{cn}^{max}(T) = \max\left\{p, \max_{x \in N(T)}\left\{\max_{y \in \Gamma(x)}\left\{\sum_{H \in \boldsymbol{\mathcal{V}}_x^y(T)} \varphi_{cn}^{max}(H)\right\}\right\}\right\}.$$

4 Conclusion

In this paper we prove that it is NP-complete to decide whether a graph can be partitioned into nontrivial convex sets. We show a polynomial algorithm that determines whether a graph can be covered by nontrivial convex sets.

Also, we propose a recursive formula that establishes the maximum nontrivial convex cover number of a tree. Combining this formula with our previous results from [11], it can be easily built a recursive procedure that determines whether a tree has a nontrivial convex p-cover for a fixed $p \ge 2$. We conclude further that the nontrivial convex p-cover problem of a tree is almost completely solved.

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Received July 25, 2017
Revised November 28, 2017

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