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d^m-convex functions in the complex of multi-ary relations

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Abstract

In the present work the notions of d^m-convexity and d^m-convex function are defined. Some properties of these functions are mentioned. We study the complexes of multi-ary relations for which the median function is d^m-convex.

Keywords: complex of multi-ary relations, m-dimensional chain, d^m-convex set, d^m-convex function.

1 Introduction

Solving of many optimization problems consists in examination of some functions on discrete structures. In this context, convex functions have a special role. The functions property of being convex guarantees elaboration of efficient methods that determine the optimal solution of problem. For these reasons, the determination of conditions that ensure the convexity of special functions in a complex of multi-ary relations is important. Such functions frequently occur in process of studying location problems and are known as median-functions.

Let $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ be a complex of multi-ary relations, determined by a finite set of elements $X = \{x_1, x_2, ..., x_p\}$. The complex of relations $\Re^{n+1}$ was thoroughly defined and studied in the work [1]. According to the definition, R^m, $1 \leq m \leq n+1$ represents a subset of the Cartesian product of rank m of the set X and is not empty.

We choose two elements $r^k \in R^k$, $r^q \in R^q$, $1 \leq k, q < n+1$. We generalize the concept of chain in $\Re^{n+1}$ used in works [2], [3].

The sequence of the elements $r^m_{t_1}$,$r^m_{t_2}$,...,$r^m_{t_s} \in R^m$, $max\{k, q\} < m \leq n + 1$, with following properties:

a) $r^k \subset r^m_{t_1}$;

b) $r^q \subset r^m_{t_s}$;

c) $r^m_{t_p} \cap r^m_{t_{p+1}} \in R^l$, $1 \leq l < m$, for any p, $1 \leq p \leq s - 1$,

is called m-dimensional chain with the extremities in $r^k \in R^k$, $r^q \in R^q$ and is denoted by $L^m(r^k, r^q) = [r^m_{t_1}, r^m_{t_2}, ..., r^m_{t_s}]$.

The number s is called the length of the chain $L^m(r^k, r^q)$. The minimal length of m-dimensional chains that connect elements $r^k \in R^k$ and $r^q \in R^q$ is called m-distance between these elements and is denoted by $d^m(r^k, r^q)$. If between two elements $r^k \in R^k$ and $r^q \in R^q$ there does not exist m-dimensional chain, then it is considered that $d^m(r^k, r^q) = +\infty$.

2 d^m-convex functions

We mention that m-distance is defined on the elements of the set $R^1 \cup R^2 \cup ... \cup R^{m-1}$, $m \geq 2$, and possesses metric properties:

a) $d^m(r^k, r^q) \geq 0$, for any two elements $r^k \in R^k$ and $r^q \in R^q$, and $d^m(r^k, r^q) = 0$ if and only if $r^k \in r^q$;

b) $d^m(r^k, r^q) = d^m(r^q, r^k)$, for any two elements $r^k \in R^k$, $r^q \in R^q$;

c) $d^m(r^k, r^q) \leq d^m(r^k, r^l) + d^m(r^l, r^q)$, for any three elements $r^k \in R^k$, $r^l \in R^l$, $r^q \in R^q$ $(k, l, q < m)$.

Definition 1. *A complex of multi-ary relations $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ is m-conex if for any two elements $r^k \in R^k$ and $r^q \in R^q$, $1 \leq k, q < m$ an m-dimensional chain $L^m(r^k, r^q)$ exists.*

Theorem 1. *If $\Re^{n+1}$ is an m-dimensional complex of multi-ary relations, then it is also h-dimensional for any h, $2 \leq h < m$.*

Let $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$ be a complex of multi-ary relations and $\Re^{m-1} = (R^1, R^2, ..., R^{m-1})$, $2 \leq m \leq n + 1$ be a subcomplex from $\Re^{n+1}$. On elements of $\Re^{n+1}$ we define a function with values in the set of real numbers $f : \Re^{n+1} \to R$.

Similarly to those mentioned in the paper [4], this function is d^m-convex if for any three elements r^k, r^q and r^h from subcomplex $\Re^{m-1}$ with the property $d^m(r^k, r^q) = d^m(r^k, r^h) + d^m(r^h, r^q)$ in $\Re^{n+1}$, the following inequality holds: $f^m(r^h) \leq \frac{d^m(r^h, r^q)}{d^m(r^k, r^q)} f^m(r^k) + \frac{d^m(r^k, r^h)}{d^m(r^k, r^q)} f^m(r^q)$. Of course, the function is defined if the right side of the inequality exists.

Theorem 2. *If f^m is a d^m-convex function defined on the complex of multi-ary relations $\Re^{n+1}$, then every local extremum of this function coincides with the global.*

Theorem 3. *The sum of two d^m-convex functions defined on the complex of multi-ary relations $\Re^{n+1}$ is a d^m-convex function in $\Re^{n+1}$.*

Definition 2. *The set $A \subset R^1 \cup R^2 \cup ... \cup R^{m-1}$ is called m-convex in $\Re^{n+1}$ if every m-dimensional chain that connects two elements of A contains only elements from A.*

Theorem 4. *If f is a d^m-convex function in $\Re^{n+1}$ and α is a real number, then the set $\{r \in A = R^1 \cup R^2 \cup ... \cup R^{m-1} : f(r) \leq \alpha\}$ is d^m-convex in the complex of multi-ary relations $\Re^{n+1} = (R^1, R^2, ..., R^{n+1})$.*

3 Median function in the complex of multi-ary relations

Median functions are used to solve services centre location problems. A location problem on a complex of multi-ary relations consists in minimisation of the function $F : A = R^1 \cup R^2 \cup ... \cup R^{m-1} \rightarrow R$ of the following type: $F^m(r) = \sum\limits_{z \in A} d^m(r, z)$.

The solving of such problems is quite complicated, because, in general case, the function $F^m(r)$ does not have any properties that would facilitate the determinations of the extremum. However, for some special complexes, the situation becomes quite favourable.

If $r^k = r^q$ we say that the chain $L^m(r^k, r^q)$ is a m-dimensional cycle.

Theorem 5. *If the complex of multi-ary relations $\Re^{n+1}$ does not contain h-dimensional cycles, $h \leq m$, then the median function $F^m(r)$ is d^m-convex.*

Theorem 6. *If $F^m(r)$ is a d^m-convex median function, then all extremal points of the function $F^m(r)$ generate a convex subcomplex in the complex of multi-ary relations.*

4 Conclusion

In this paper there are presented the results about the study of convex functions properties, based on the generalization of the notion of metric convexity, known from graphs theory. The obtained results contribute to the development of the convexity theory in complexes of multi-ary relations.

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