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Maximum nontrivial convex cover of a tree

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Abstract

The nontrivial convex p-cover problem of a tree is studied. We propose the recursive formula that determines the maximum nontrivial convex cover number of a tree.

Keywords: convexity, nontrivial convex cover, tree.

1 Introduction

A vertex set S of a graph G is called *convex* if all vertices of every shortest path between two of its vertices are in S [1]. Generally, the concept of *convex p-cover* of a graph is introduced in [2] and is examined in [2, 3]. We defined a nontrivial convex p-cover of a graph as a special case of general convex p-cover in [3]. A family of sets $\mathcal{P}_p(G)$ is called a *nontrivial convex p-cover* of a graph G if the following conditions hold:

1) every set of $\mathcal{P}_p(G)$ is convex in G;
2) every set S of $\mathcal{P}_p(G)$ satisfies inequalities: $3 \leq |S| \leq |X(G)| - 1$;
3) $X(G) = \bigcup_{Y \in \mathcal{P}_p(G)} Y$;
4) $Y \not\subseteq \bigcup_{\substack{Z \in \mathcal{P}_p(G) \\ Z \neq Y}} Z$ for every $Y \in \mathcal{P}_p(G)$;
5) $|\mathcal{P}_p(G)| = p$.

Particularly, we showed that it is NP-complete to decide whether a graph has a nontrivial convex p-cover for a fixed $p \geq 2$ [3]. Nontrivial convex p-covers for some classes of graphs are studied in [7, 8]. The most consistent results are obtained for trees [4, 5, 6]. In the present paper we continue our research on nontrivial convex p-cover problem of a tree.

2 Main Results

Recall that a vertex $x \in X(G)$ is called *resident* in $\mathcal{P}_p(G)$ if x belongs to only one set of $\mathcal{P}_p(G)$ [3]. The greatest $p \geq 2$ for which a graph G has a nontrivial convex p-cover is said to be the *maximum nontrivial convex cover number* $\varphi_{cn}^{max}(G)$ [4].

Let T be a tree on n vertices and let $C(T)$ be a set of terminal vertices of T, $p = |C(T)|$. An important result is given by the following lemma.

Lemma 1. *If $n \geq 4$, then there exists a maximum nontrivial convex cover $\mathcal{P}_{\varphi_{cn}^{max}}(T)$ such that every set $S \in \mathcal{P}_{\varphi_{cn}^{max}}(T)$ contains a path $L = [x, y, z]$, where x is a resident vertex in $\mathcal{P}_{\varphi_{cn}^{max}}(T)$.*

Suppose that $diam(T) \geq 4$, then we define the set:

$$N(T) = X(T) \setminus \left(C(T) \cup \bigcup_{y \in C(T)} \Gamma(y) \right).$$

The set $N(T)$ is empty if and only if every nonterminal vertex of T is adjacent to at least one terminal vertex of T, but in this case, according to [4], we get $\varphi_{cn}^{max}(T) = p$. Let x be a vertex of $N(T)$. Since x is an articulation vertex, through the elimination of x from T we obtain $|\Gamma(x)|$ connected components T_x^y, $y \in \Gamma(x)$. For every vertex $y \in \Gamma(x)$ we get the family of subtrees:

$$\mathcal{V}_x^y(T) = {}^*T_x^y \cup \bigcup_{z \in \Gamma(x) \setminus y} T_x^z,$$

where ${}^*T_x^y$ is a subtree of T obtained by adding x to T_x^y such that x is adjacent to y.

Finally, we get the family of subfamilies of subtrees:

$$\mathcal{V}_x(T) = \bigcup_{y \in \Gamma(x)} \mathcal{V}_x^y(T).$$

For the sake of estimation of the number $\varphi_{cn}^{max}(T)$, we consider that if $0 \leq n \leq 2$, then $\varphi_{cn}^{max}(T) = 0$, and if $n = 3$, then $\varphi_{cn}^{max}(T) = 1$. Combining Lemma 1 with results from [4, 5, 6], we obtain the recursive formula, reflected in Theorems 1 and 2, that determines the maximum nontrivial convex cover number $\varphi_{cn}^{max}(T)$.

Theorem 1. *If $diam(T) \leq 5$ or $diam(T) \geq 6$ and $N(T) = \varnothing$, then the following relation holds:*

$$\varphi_{cn}^{max}(T) = \begin{cases} p, & \text{if } 3 \leq diam(T) \leq 5 \text{ or} \\ & diam(T) \geq 6 \text{ and } N(T) = \varnothing; \\ p\text{-}1, & \text{if } diam(T) = 2; \\ 0, & \text{if } 0 \leq diam(T) \leq 1. \end{cases}$$

Theorem 2. *If $diam(T) \geq 6$ and $N(T) \neq \varnothing$, then the following relation holds:*

$$\varphi_{cn}^{max}(T) = max\left\{p, max_{x\in N(T)}\left\{max_{y\in\Gamma(x)}\left\{\sum_{H\in\mathcal{V}_x^y(T)} \varphi_{cn}^{max}(H)\right\}\right\}\right\}.$$

3 Conclusion

In this paper we propose the recursive formula that establishes the maximum nontrivial convex cover number of a tree, based on which an efficient algorithm that determines whether a tree has a nontrivial convex p-cover for a fixed $p \geq 2$ can be developed. Taking into account our previous results [4, 5, 6] together with these new findings we arrive to the conclusion that the nontrivial convex p-cover problem of a tree is almost completely solved.

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