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Algorithm for the localization of singularities of functions defined on closed contours

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Abstract

A numerical algorithm for locating polar singularities of functions defined on a discrete set of points of a simple closed contour in the complex plane is examined. The algorithm uses the Faber-Padé approximation of the function and the fact that the zeros of its denominator give us approximations of the poles of function. The numerical performance of the algorithm is being analyzed on test issues.

Keywords: Padé algorithm, singular points, closed contour.

1 Problem Formulation

Methods for solving differential and integral equations whose solutions are meromorphic functions usually start from the premise that locations of the singularities of functions are known. In this paper we examine the following problem of locating singularities (discontinuity points and poles, not the essential singularities) of functions defined on contours in complex plane.

Let $\Omega^+ \subset \mathbb{C}$ be a simply connected domain bounded by a piecewise smooth closed curve Γ. We consider that the point $z = 0 \in \Omega^+$. By the Riemann mapping theorem there exists a conformal map $z = \psi(w)$ of $D^- := \{w \in \mathbb{C} \mid |w| > 1\}$ onto $\Omega^- := \overline{\mathbb{C}} \setminus \{\Omega^+ \cup \Gamma\}$ such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. The function $\psi(w)$ transforms the circle $\Gamma_0 := \{w \in \mathbb{C} \mid |w| = 1\}$ onto Γ.

Let $f(z)$ be a meromorphic function in the finite domain $\Omega \supset \Omega^+ \cup \Gamma$ that is analytic in Ω^+ (we will denote this fact by $f \in H(\Omega^+)$). Considering that the function $f(z)$ is defined by a finite set of values $f_k = f(z_k)$ in the points $\{z_k\}$, $z_k = \psi(w_k) \in \Gamma$, $w_k \in \Gamma_0$, we aim to find the singularities of f on Γ.

2 The theory underlying the algorithm

According to the theory of analytic continuation of functions of a complex variable, the properties of a function analytic at a point are contained in its Taylor series expansion at that point. We know that the Padé approximants perform an analytic continuation of the series outside its domain of convergence and can be used effectively in determining information about the singularity structure of a function from its Taylor series coefficients [1].

The problem of recovering the meromorphic function F in the disk $D_m(F)$, where F has m poles taking into account their multiplicities, is solved on the basis of the theorem of de Montessus de Ballore [1]. According to Montessus's theorem, the poles of the sequence of Padé approximants to the function F converge to the poles of F on $D_m(F)$.

In the formulated problem in order to apply the classical result of Montessus's theorem we used the properties of the Faber transform [2,3] and of the conformal map $z = \psi(w)$. Since the function f is analytic on simply connected domain Ω^+ with boundary Γ, the Faber series expansion [2,3] is used to represent the function f instead of the Taylor series that is defined on the disk.

The Faber transform T associates to $F \in H(D^+)$ the Faber series expansion of the function $f \in H(\Omega^+)$, $f(z) = \sum_{k=0}^{\infty} c_k F_k(z)$, $z \in \Omega^+$, where $F_k(z)$ is the Faber polynomial of degree k [2,3]. The coefficients of the expansion are defined by the formula $c_k = \frac{1}{2\pi i} \int_{\Gamma_0} f(\psi(w))\, w^{-(k+1)} dw$. For $f \in H(\Omega^+)$ there exists an $F \in H(D^+)$ such that $f = T(F)$.

An important property of the Faber operator is that it induces a bijective correspondence between the set of rational functions with poles on $D^{-}\cup\Gamma_0$ and the set of rational functions with poles on $\Omega^{-}\cup\Gamma$. Moreover, this transformation keeps intact the number of poles and their multiplicities [3]. The poles of $T(R)$ are obtained as images under the Riemann function ψ of the poles of rational function R [3].

Based on the Montessus's theorem [1] it can be shown for the meromorphic function f with M poles on $\Omega^{-}\cup\Gamma$ that for sufficiently large N, the Padé approximants $r_{(N,M)}$ to f have M poles. As $N\to\infty$ the sequence $r_{(N,M)}$ converges to f uniformly inside the domain Ω' obtained from Ω by deleting the poles of f and the poles of the sequence $r_{(N,M)}$ tend to the poles of f. Each pole of f attracts a number of poles of $r_{(N,M)}$ equal to its multiplicity.

3 An algorithm for the localization of singular points on Γ

For the classical Padé approximant $R_{(N,M)}(w)$ to the function $F(w)$ we determine the poles that belong to Γ_0. Next, by using the properties of the Faber transform, the singular points on Γ of the Faber-Padé approximation are located. We perform the following steps:

1. We compute the coefficients q_j, $j=1,...,M$ of the polynomial $Q_M(w)$ from the Padé approximation $R_{(N,M)}(w)=P_N(w)/Q_M(w)$ to $F(w)$. If we have $P_N(w)=\sum_{k=0}^{N}p_kw^k$, $Q_M(w)=\sum_{j=0}^{M}q_jw^j$, $q_0=1$, then the coefficients q_j, $j=1,...,M$ are determined as a solution of the system of linear equations (abbreviated SLE):

$$\sum_{j=1}^{M}c_{k-j}q_j=-c_k,\ k=N+1,...,N+M,$$

where c_j are the Faber coefficients that coincide with the Taylor coefficients for the function $F(w)=\sum_{j=0}^{\infty}c_jw^j$.

Taking into account that the function f is defined by its values on the boundary Γ, we will compute approximations to c_j based on m - point trapezoidal rule for the contour integral. To avoid the situation when the singularities of f on Γ affect the accuracy of the approximation, we will approximate the integral on Γ_0 that defines c_k by the integral on the perturbed circle $\Gamma_0^\rho := \{ w \in \mathrm{C} \mid \ |w| = \rho \}$. Here we have $\rho = 1 - \varepsilon > 0$ and $\varepsilon > 0$ is given arbitrarily small positive number, for example, $\varepsilon = 0.01$. If $\theta_k \in [0, 2\pi]$, $k = 0, 1, ..., m$ are the polar angles corresponding to the points z_k, $k = 0, 1, ..., m$ on Γ, then according to m - point trapezoidal rule we have

$$c_j \approx \tilde{c}_j^{(m)} := \frac{1}{4\pi\rho^j} \sum_{k=1}^{m} (\theta_k - \theta_{k-1}) \left(f_{k-1} e^{-ij\theta_{k-1}} + f_k e^{-ij\theta_k} \right),$$

where the values $f_k = f\left(\psi\left(\rho e^{i\theta_k}\right)\right)$, $k = 0, ..., m-1$ are initially given.

2. We find the zeros of the polynomial $Q_M(w) = \sum_{j=0}^{M} q_j w^j$, $w \in \Gamma_0$, where q_j, $j = 1, ..., M$ is the solution of SLE. Next we find the zeros of the polynomial $q_M(z)$, $z \in \Gamma$, as images under ψ of the zeros of $Q_M(w)$. The obtained values are the candidates for the desired singularities.

The numerical performance of the algorithm is analyzed. Also, we discuss how to eliminate the spurious poles of the Padé approximants.

References

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