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Pseudo-automorphisms of middle Bol loops

Ion Grecu

Abstract

The set of Moufang elements in a middle Bol loop is considered in the present work. We prove that every inner mapping of the Moufang part (which is a subloop) of a middle Bol loop $(Q,\cdot)$ extends to a right pseudo-automorphism of $(Q,\cdot)$.

Keywords: loop, multiplication group, inner mapping, middle Bol loop, pseudo-automorphism.

A grupoid $(Q,\cdot)$ is called a quasigroup if the equations $a\cdot x=b$ and $y\cdot a=b$ have unique solutions, for $\forall a,b\in Q$. A loop is a quasigroup with a neutral element. Two quasigroups $(Q,\cdot)$ and $(Q,*)$ are isotopic, if there exist $\alpha,\beta,\gamma\in S_Q$, such that $x*y=\gamma^{-1}(\alpha(x)\cdot\beta(y))$, $\forall x,y\in Q$. If (Q,A) is a quasigroup and $\sigma\in S_3$, then the operation ${}^{\sigma}A$, defined by the equivalence ${}^{\sigma}A(x_{\sigma(1)},x_{\sigma(2)})=x_{\sigma(3)}\Leftrightarrow A(x_1,x_2)=x_3$, is called a σ-parastrophe of the operation A. The product of an isotopy and a parastrophy, in any order, of a quasigroup $(Q,\cdot)$ is called an isostrophy of $(Q,\cdot)$.

A loop $(Q,\cdot)$ is called a middle Bol loop if it satisfies the identity: $x(yz\backslash x)=(x/z)(y\backslash x)$. It is proved in [4] that middle Bol loops are isostrophes of left (resp. right) Bol loops. Namely, a loop $(Q,\circ)$ is middle Bol if and only if there exists a right (left) Bol loop $(Q,\cdot)$, such that, $\forall x,y\in Q$:

$$x\circ y=y^{-1}\backslash x,\quad (resp.\ \ x\circ y=x/y^{-1}). \tag{1}$$

Let $(Q,\cdot)$ be a loop. We consider the sets:

$$M_l^{(\cdot)}=\{a\in Q\mid a(y\cdot az)=(ay\cdot a)z,\forall y,z\in Q\},$$

$$M_r^{(\cdot)} = \{a \in Q \mid (za \cdot y)a = z(a \cdot ya), \forall y, z \in Q\},$$

$$M^{(\cdot)} = \{a \in Q \mid ay \cdot za = a(yz \cdot a), \forall y, z \in Q\}.$$

Lemma 1. [3] *If $(Q, \cdot)$ is a middle Bol loop, then $M_l^{(\cdot)} = M_r^{(\cdot)} = M^{(\cdot)}$ and form a subbloop in $(Q, \cdot)$.*

Definition. *Let $(Q, \cdot)$ be a middle Bol loop. The subbloop $M^{(\cdot)}$ is called the Moufang part of $(Q, \cdot)$.*

Let $(Q, \cdot)$ be an arbitrary loop, $\varphi \in S_Q$ and $c \in Q$. Recall that:
a) φ is called a left (resp. right) pseudo-automorphism of $(Q, \cdot)$, with the companion c, if the equality

$$c \cdot \varphi(x \cdot y) = [c \cdot \varphi(x)] \cdot \varphi(y),$$

respectively,

$$\varphi(x \cdot y) \cdot c = \varphi(x) \cdot [\varphi(y) \cdot c],$$

holds, for every $x, y \in Q$.

Pseudo-automorphisms (left, right) have been introduced by Bruck [1] and were studied by many authors (see, for example, [1-3,5]). Bruck proved in [1] that every inner mapping of a Moufang loop is a pseudo-automorphism of this loop. Recall that a mapping α of the multiplication group $M(Q, \cdot) =< L_x^{(\cdot)}, R_y^{(\cdot)} | x, y \in Q >$ of a loop $(Q, \cdot)$ is called an inner mapping of $(Q, \cdot)$ if $\alpha(e) = e$, where e is the neutral element of this loop.

Theorem 1. *Let $(Q, \cdot)$ be a middle Bol loop. Each inner mapping of $M^{(\cdot)}$ extends to a pseudo-automorphism of $(Q, \cdot)$.*

Proof. Let $H =< L_x^{(\cdot)}, R_y^{(\cdot)} | x, y \in M^{(\cdot)} >$ be the multiplication group of the subloop $M^{(\cdot)}$, where $L_x^{(\cdot)}(z) = x \cdot z$ and $R_y^{(\cdot)}(z) = z \cdot y$, for all $z \in Q$. If $a \in M^{(\cdot)}$, then $L_a^{(\cdot)-1} = L_{a^{-1}}^{(\cdot)}$ and $R_a^{(\cdot)-1} = R_{a^{-1}}^{(\cdot)}$. Indeed, if $a \in M^{(\cdot)}$ then $ay \cdot za = a(yz \cdot a)$, for every $y, z \in Q$. Now, taking $z = a^{-1}$ in the last equality, we get:

$$a \cdot y = a \cdot (ya^{-1} \cdot a) \Rightarrow y = ya^{-1} \cdot a =$$
$$= R_a^{(\cdot)} R_{a^{-1}}^{(\cdot)}(y) \Rightarrow R_a^{(\cdot)-1}(y) = R_{a^{-1}}^{(\cdot)}(y),$$

$\forall y \in Q$. As $M^{(\cdot)} = M_l^{(\cdot)}$, for $a \in M^{(\cdot)}$ the equality $a(y \cdot az) = (ay \cdot a)z$ holds, for every $y, z \in Q$. Taking $y = a^{-1}$ in the last equality, we get:

$$a(a^{-1} \cdot az) = a \cdot z \Rightarrow a^{-1} \cdot az = z \Rightarrow L_{a^{-1}}^{(\cdot)} L_a^{(\cdot)}(z) = z,$$

$\forall z \in Q$, so $L_{a^{-1}}^{(\cdot)} = L_a^{(\cdot)-1}$.

If $U \in H$, then U can be expressed in the form $U = U_1 U_2 ... U_n$, where $U_i = R_{a_i}^{(\cdot)}$ or $U_i = L_{a_i}^{(\cdot)}$, for some $a_i \in M^{(\cdot)}$. Let $a \in M^{(\cdot)}$, then $ay \cdot za = a(yz \cdot a)$, for all $y, z \in Q$, so the triple

$$T_1 = (L_a^{(\cdot)}, R_a^{(\cdot)}, L_a^{(\cdot)} R_a^{(\cdot)})$$

is an autotopism of $(Q, \cdot)$. For each $a \in M^{(\cdot)} = M_r^{(\cdot)}$ we have $(za \cdot y)a = z(a \cdot ya)$, $\forall y, z \in Q$, so

$$(R_a^{(\cdot)-1}, L_a^{(\cdot)} R_a^{(\cdot)}, R_a^{(\cdot)})$$

is an autotopism of $(Q, \cdot)$ as well, hence the triple

$$T_2 = (R_a^{(\cdot)}, R_a^{(\cdot)-1} L_a^{(\cdot)-1}, R_a^{(\cdot)-1})$$

is an autotopism of $(Q, \cdot)$. As T_1 and T_2 are autotopisms of $(Q, \cdot)$ we get that, for all $U_i \in H$ there exists $V_i, W_i \in H$, such that

$$U_i(y) \cdot V_i(z) = W_i(y \cdot z),$$

$\forall y, z \in Q$. So, letting $V = V_1 V_2 ... V_n$ and $W = W_1 W_2 ... W_n$, we obtain: $W(y \cdot z) = W_1 W_2 ... W_n(y \cdot z) = W_1 W_2 ... W_{n-1}(U_n(y) \cdot V_n(z)) = ... = U_1 U_2 ... U_n(y) \cdot V_1 V_2 ... V_n(z) = U(y) \cdot V(z)$, for all $y, z \in Q$, so

$$U(y) \cdot V(z) = W(y \cdot z), \tag{2}$$

for all $y, z \in Q$. Let U be a inner mapping of $M^{(\cdot)}$, then $U(e) = e$, where e is the unit of $(Q, \cdot)$. Taking $y = e$ in (2) we obtain $V = W$, so $U(y) \cdot V(z) = V(y \cdot z)$, for all $y, z \in Q$. Taking $z = e$ in the last equality we have $U(y) \cdot V(e) = V(y)$, for all $y \in Q$. Denoting $V(e) = u$, we obtain that $V = R_u^{(\cdot)} U$, so the triple $T = (U, R_u^{(\cdot)} U, R_u^{(\cdot)} U)$ is an autotopism of $(Q, \cdot)$, which implies that U is a right pseudo-automorphism of the loop $(Q, \cdot)$, with the companion $u = V(e)$. $\square$

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Ion Grecu

Moldova State University
Email: iongrecu21@gmail.com