On invariance of recursive differentiability under the isotopy of left Bol loops

Inga Larionova-Cojocaru, Parascovia Syrbu

Abstract

We prove that the recursive derivatives of order 1 of isotopic left Bol loops are isotopic and that every loop, isotopic to a recursively 1-differentiable left Bol loop, is recursively 1-differentiable. The recursive differentiability of di-associative loops is also considered.

Keywords: Recursively differentiable quasigroup, recursive derivative, isostrophe, core, LIP-loop, left Bol loop

1 Introduction

Recursively s-differentiable quasigroups ($s \geq 1$) have been defined in [1], were they appear as check functions of complete recursive codes. Let $(Q, \cdot)$ be a quasigroup and let i be a natural number. The operation $(\overset{i}{\cdot})$, defined recursively on Q as follows:

$$x \overset{0}{\cdot} y = x \cdot y,\ x \overset{1}{\cdot} y = y \cdot (x \cdot y),\ x \overset{i}{\cdot} y = (x \overset{i-2}{\cdot} y) \cdot (x \overset{i-1}{\cdot} y),$$

for $\forall x, y \in Q$, is called the recursive derivative of order i of $(Q, \cdot)$. A quasigroup $(Q, \cdot)$ is called recursively s-differentiable if its recursive derivatives $(Q, \overset{i}{\cdot})$ are quasigroups, for all $i = 0, 1, ..., s$. If $(Q, \cdot)$ is a loop then the grupoid $(Q, +)$, where $x + y = x \cdot (y \setminus x), \forall x, y \in Q$, is called the core of $(Q, \cdot)$. The notion of core of a loop was introduced by R. Bruck [2] for Moufung loops and studied by V. Belousov in left Bol loops [3]. It is shown in [4] that in LIP-loops the core is isostrophic to the recursive derivative of order 1.

We study the invariance of recursive differentiability under the isotopy of LIP-loops in the present work. It is proved that the recursive derivatives of order 1 of isotopic left Bol loops are isotopic and that every loop, isotopic to a recursively 1-differentiable left Bol loop, is recursively 1-differentiable. Also we show that the recursive derivative of order 1 of a recursively 1-differentiable di-associative loop is an RIP-quasigroup.

2 Recursive derivatives and cores

Let $(Q, \cdot)$ be an LIP-loop. We'll denote below by $I^{(\cdot)}$ the inversion in $(Q, \cdot) : I^{(\cdot)}(x) = x^{-1}$, $\forall x \in Q$.

Theorem 1. *If the cores of two LIP-loops are isotopic then their recursive derivatives of order one are isotopic.*

Proof. Let $(Q, \oplus)$ and $(Q, +)$ be the cores and let $(Q, \overset{1}{\circ})$ and $(Q, \overset{1}{\cdot})$ be the recursive derivatives of order 1 of two loops $(Q, \circ)$ and $(Q, \cdot)$, respectively. If $(Q, \oplus)$ and $(Q, +)$ are isotopic then $\exists \alpha, \beta, \gamma \in S_Q$: $\gamma(x \oplus y) = \alpha(x) + \beta(y) \Leftrightarrow \gamma(x \circ [I^{(\circ)}(y) \circ x]) = \alpha(x) \cdot [I^{(\cdot)}\beta(y) \cdot \alpha(x)]$, $\forall x, y \in Q$. Replacing $y \mapsto I^{(\circ)}(y)$, we obtain $\gamma[x \circ (y \circ x)] = \alpha(x) \cdot [I^{(\cdot)}\beta I^{(\circ)}(y) \cdot \alpha(x)] \Leftrightarrow \gamma(y \overset{1}{\cdot} x) = I^{(\cdot)}\beta I^{(\circ)}(y) \overset{1}{\cdot} \alpha(x)$, for $\forall x, y \in Q$, hence $(Q, \overset{1}{\circ})$ and $(Q, \overset{1}{\cdot})$ are isotopic.

V. Belousov [3] proved that the cores of isotopic left Bol loops are isomorphic. This fact and the previous theorem imply the following corollaries.

Corollary 1. *The recursive derivatives of order 1 of isotopic left Bol loops are isotopic.*

Corollary 2. *Every loop isotopic to a recursively* 1*-differentiable left Bol loop is recursively* 1*-differentiable.*

Proposition 1. *If the cores of two LIP-loops $(Q, \circ)$ and $(Q, \cdot)$ are isomorphic and φ is an isomorphism between them, then the recursive derivatives $(Q, \overset{1}{\circ})$ and $(Q, \overset{1}{\cdot})$ are isomorphic if and only if $I^{(\cdot)}\varphi I^{(\circ)} = \varphi$.*

Proof. Let $(Q, \oplus)$ and $(Q, +)$ be the cores and $(Q, \overset{1}{\circ})$ and $(Q, \overset{1}{\cdot})$

be the recursive derivatives of order 1 of two LIP-loops $(Q,\circ)$ and $(Q,\cdot)$, respectively. If φ is an isomorphism between the cores, then $\varphi(x\oplus y)=\varphi(x)+\varphi(y)\Leftrightarrow\varphi(x\circ(I^{(\circ)}(y)\circ x))=\varphi(x)\cdot(I^{(\cdot)}\varphi(y)\cdot\varphi(x))\Leftrightarrow$ $\varphi(I^{(\circ)}(y)\overset{1}{\circ}x) = I^{(\cdot)}\varphi(y)\overset{1}{\cdot}\varphi(x)$. Replacing $I^{(\circ)}(y)\mapsto y$, we obtain $\varphi(y\overset{1}{\circ}x)=I^{(\cdot)}\varphi I^{(\circ)}(y)\overset{1}{\cdot}\varphi(x)$ for $\forall x,y\in Q$. Hence φ is an isomorfism from $(Q,\overset{1}{\circ})$ to $(Q,\overset{1}{\cdot})$ if and only if $I^{(\cdot)}\varphi I^{(\circ)}=\varphi$.

Corollary 3. *If two left Bol loops $(Q,\circ)$ and $(Q,\cdot)$ are isotopic and φ is an isomorphism between their cores then $(Q,\overset{1}{\circ})$ and $(Q,\overset{1}{\cdot})$ are isomorphic if and only if $I^{(\cdot)}\varphi I^{(\circ)}=\varphi$.*

Theorem 2. *Let $(Q,\cdot)$ be a di-associative loop. Then $(Q,\cdot)$ is recursively differentiable if and only if the mapping $x\mapsto x^2$ is a bijection. Moreover, if $(Q,\cdot)$ is recursively 1-differentiable then its recursive derivative of order 1 is an RIP-quasigroup.*

Proof. Let $(Q,\cdot)$ be a di-associative loop. Its recursive derivative of order 1, $(Q,\overset{1}{\cdot})$ is a quasigroup if the equations $a\overset{1}{\cdot}x=b$ and $y\overset{1}{\cdot}a=b$ have unique solutions in Q for all $a,b\in Q$. But $y\overset{1}{\cdot}a=b\Leftrightarrow a\cdot ya=b$ has a unique solution in Q, for all $a,b\in Q$. Hence $(Q,\cdot)$ is recursively 1-differentiable if and only if

$$a\overset{1}{\cdot}x=b\Leftrightarrow x\cdot ax=b\Leftrightarrow ax\cdot ax=ab\Leftrightarrow(ax)^2=ab$$

has a unique solution in Q, for all $a,b\in Q$, i.e. if and only if the mapping $x\mapsto x^2$ is a bijection.

If $(Q,\cdot)$ is recursively 1-differentiable, then $(Q,\overset{1}{\cdot})$ is a quasigroup, and $(y\overset{1}{\cdot}x)\overset{1}{\cdot}x^{-1}=x^{-1}\cdot(x\cdot yx)\cdot x^{-1}=y$, as $(Q,\cdot)$ is a di-associative loop. Hence $(Q,\overset{1}{\cdot})$ is an RIP-quasigroup.

Proposition 2. [5] *Let $(Q,\cdot)$ be a quasigroup and let $(Q,\circ)$ be a quasigroup with a right unit e. If $x\circ y=\varphi(y)\cdot\psi(x)$, for $\forall x,y\in Q$, and $\psi\in Aut(Q,\circ)$, then $RM(Q,\circ)\triangle LM(Q,\cdot)$.*

Corollary 4. *If $(Q,\cdot)$ is a recursively 1-differetiable left Bol loop, $(Q,+)$ is its core and $(Q,\overset{1}{\cdot})$ is its recursive derivative of order 1, then $RM(Q,\overset{1}{\cdot})\triangle LM(Q,+)$.*

Proof. Let $(Q,\cdot)$ be a recursively 1-differetiable left Bol loop, then its core $(Q,+)$ and its recursive derivative of order 1 $(Q,\overset{1}{\cdot})$ are quasigroups. It is shown in [4] that: the unit of $(Q,\cdot)$ is the right unit of $(Q,\overset{1}{\cdot})$, $x+y = I^{(\cdot)}(y)\overset{1}{\cdot}x$ and that, if $(Q,\cdot)$ is a left Bol loop, then $I^{(\cdot)}(x\overset{1}{\cdot}y) = I^{(\cdot)}(x)\overset{1}{\cdot}I^{(\cdot)}(y)$. Hence the quasigroups $(Q,+)$ and $(Q,\overset{1}{\cdot})$, where $x+y = I^{(\cdot)}(y)\overset{1}{\cdot}x$ and $I^{(\cdot)} \in Aut(Q,\overset{1}{\cdot})$, fulfill the conditions of the previous proposition, hence $RM(Q,\overset{1}{\cdot})\triangle LM(Q,+)$.

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Inga Larionova-Cojocaru[1], Parascovia Syrbu[2]

[1]Moldova State University
Email:larionovainga@yahoo.com

[2]Moldova State University
Email:syrbuviv@yahoo.com