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Construction of pointwise lattices in Euclidean and Minkowski spaces with some given properties

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Abstract

In this paper concrete construction methods of the several Bravais types of pointwise lattices in Euclidean and Minkowski spaces depending on some additional conditions have been elaborated.

Keywords: pointwise lattice, Bravais type, Euclidean space, Minkowski space, crystallography.

1 Introduction

Let $\mathbb{R}_n$ (or ${}^1\mathbb{R}_n$) be the Euclidean n-dimensional pointwise space (or Minkowski space) and $\mathfrak{R}_n$ be a pointwise lattice in this space, i.e. $\mathfrak{R}_n$ is a set of points which have integer coordinates with respect to some fixed basis:

$$\mathfrak{R}_n = \{M : \overrightarrow{OM} = \textstyle\sum_{i=1}^{n} x_i\vec{a}_i, x_i \in \mathbb{Z}\}.$$

The basis $\{O, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$ is the main basis of the lattice $\mathfrak{R}_n$. The properties of the lattices depend on their transformations of symmetry. A transformation of symmetry of the pointwise lattice is a motion of space that maps this lattice onto itself. The most simple transformations of symmetry of a lattice are the parallel translations by any of its vector. The set of all the parallel translations of each lattice is a commutative group generated by the parallel translations determined by the vectors of a certain main basis. Other transformations of the symmetry of the lattice either keep invariant at least one point (these transformations are called pointwise transformations), or are compositions of pointwise transformations with parallel translations. Rotations around k-dimensional planes k=$1, 2, \ldots, n-2$ as well as reflections through hyperplanes are transformations of symmetry. The pointwise transformations keeping invariant one common point also form a group named pointwise group of the lattice. The pointwise group in certain main basis of the lattice is determined by a group of integer unimodular matrices. The set of all the transformations of symmetry of the given lattice forms its complete group of symmetry. Two lattices are related to the same Bravais type if their complete groups of symmetry are isomorphic[1].

The main problem to be solved is to build all Bravais types of the lattices in the concrete given space. It is known that in Euclidean spaces the number of the Bravais types of pointwise lattices is finite and depends only on the dimension n (there are 5 types for n=2, 14 types for n=3, 64 types for n=4). In Minkowski spaces there exists an infinite number of Bravais types of pointwise lattices. Since the problem cannot be solved at all in many cases, it is necessary to elaborate certain methods of constructing Bravais types of pointwise lattices in the given space (Euclidean or Minkowski) that possess some of the earlier properties (for example, given the characteristics of the angle of rotation around a k-dimensional space, $k = 1, 2 \ldots, n-2$, either given the characteristics of the corresponding quadratic forms, or the lattice elements of symmetry are indicated etc.).

2 Construction of the pointwise lattices having a k-dimensional plan of rotation

Let us give further some methods of constructing the Bravais types of pointwise lattices in Euclidean and Minkowski spaces depending on some additional conditions. If the pointwise lattice $\mathfrak{R}_n$ in Euclidean or Minkowski space is mapped onto itself by a rotation of the space around a k-dimensional plane $\mathcal{G}$, $k = 1, 2, \ldots, n-2$, then it is mapped onto itself by a rotation around any k-dimensional plan which is parallel to $\mathcal{G}$ and is of the same nature, but passes through a certain point of the lattice. Let us suppose that the plane of rotation $\mathcal{G}$ passes through the point O of the lattice. We construct a $(n-k)$-dimensional plane $\mathcal{P}$ that passes through the point O and is absolutely perpendicular on $\mathcal{G}$. It is knows that each of these planes contains a k-dimensional sublattice ($(n-k)$-dimensional respectively)[2]. If the plane $\mathcal{G}$ is a Euclidean or Minkowski plane, then $\mathcal{G} \cap \mathcal{P} = \{O\}$ and the lattice consists of $(n-k)$-dimensional sublattices that are situated in planes parallel to $\mathcal{P}$ intersecting $\mathcal{G}$ in the points of the sublattices in this plane. These sublattices are called fibers. Let us enumerate these fibers by integers. We suppose that the fiber in the plane $\mathcal{P}$ is the null one; let us order points of the sublattice in the plane $\mathcal{G}$ and enumerate them by integers (obviously, the set of points of any lattice is countable). In Euclidean spaces all planes are also Euclidean, and if they are absolutely perpendicular, then their intersection is a single point ($\mathcal{P} + \mathcal{G} = \mathbb{R}_n$). In Minkowski spaces the plane $\mathcal{G}$ can be semi-Euclidean, then $\mathcal{P}$ is semi-Euclidean plane too. In such a case $\mathcal{G} \cap \mathcal{P} = \ell$, ℓ being an isotropic strainght line and $\mathcal{P} + \mathcal{G} = \mathbb{R}^{(1)}_{n-1}$, $\mathbb{R}^{(1)}_{n-1}$ being a semi-Euclidean hyperplane. In this case the hyperplane $\mathbb{R}^{(1)}_{n-1}$ contains an $(n-1)$-dimensional sublattice, and the lattice $\mathfrak{R}_n$ consists of the fibers parallel to one in this hyperplane. In order to construct the corresponding lattice it is sufficient to build the fiber in the hyperplane $\mathbb{R}^{(1)}_{n-1} = \mathcal{P} + \mathcal{G}$ and to multiply it by translations that are given by the vectors $S \cdot \vec{a}$, $S \in \mathbb{Z}$, $\vec{a} \parallel \mathbb{R}^{(1)}_{n-1}$.

It is obvious that the situation between any two neighbouring enumerated fibersis common for the whole lattice. Therefore, in order to construct all Bravais types of the pointwise lattices in any n-dimensional Euclidean or Minkowski space that are mapped onto themself by a rotation around a given k-dimensional plane $\mathcal{G}$ ($k = 1, 2, \ldots, n-2$), it is necessary: 1) to determine the dimension and the nature of space in which the lattices are built, as well as the dimension and the nature of the plane (or axis) of rotation; 2) to determine the nature of the planes absolutely perpendicular to the plan of rotation; 3) to establish the Bravais types of the pointwise lattices in $(n-k)$-dimensional spaces situated in a Euclidean or Minkowski space of the same dimension; 4) to examine the possibilities of intersection of the planes $\mathcal{P}$ and $\mathcal{G}$ ($\mathcal{G}$ is fixed plane of rotation); 5) to compare the built lattices and to establish the Bravais types of these pointwise lattices in the n-dimensional space. If the groups of matrices are integer equivalent, then the lattices are related to the same Bravais types.

3 Conclusion

Based on the properties of the elements of symmetry of the pointwise lattices which are the intersection of the other elements of symmetry some methods of construction of the Brawais types of pointwise lattices from Euclidean and Minkovski n-dimensional space have been developed. The finding may be useful in construction of the lattices having a k-dimensional plane of rotation.

References

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