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On the Riemann boundary value problem in the case of a piecewise Lyapunov contour

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Abstract

The generalized Riemann boundary value-problem is investigated in the case of a piecewise Lyapunov contour. It is proved that the conditions for normal solvability depend on the coefficients of the problem, as well as on the presence of corner points on the contour of integration.

Keywords: Noetherian operator, Lyapunov contour, symbol, Riemann boundary value problem.

1 Introduction

Let Γ be a closed, oriented, piecewise Lyapunov contour which divides the complex plan into an interior domain D^+ and an exterior domain D^-. We denote by $L_p(\Gamma,\rho)$ the space L_p on Γ with weight $\rho(t)=\prod_{k=1}^{n}|t-t_k|^{\beta_k}$, where $t_1,\dots,t_n$ are distinct points of the curve Γ and $\beta_1,\dots,\beta_n$ are arbitrary real numbers satisfying the relations $-1<\beta_k<p-1$. In the space $L_p(\Gamma,\rho)$, over the field of real numbers, we consider the bounded linear operator

$$A=aP+bQ+(cP+dQ)V, \tag{1}$$

where a, b, c, d are continuous functions on Γ, $(V\phi)(t)=\bar{\phi}(t)$, and

$$(P\phi)(t)=\frac{1}{2}\phi(t)+\frac{1}{2\pi i}\int_\Gamma\frac{\phi(\tau)}{\tau-t}d\tau,\ (Q\phi)=\phi(t)-(P\phi)(t).$$

In the case when Γ is a Lyapunov contour the operator A and the Carleman integral equations with a shift and complex conjugate unknowns have been considered by many authors. We mention only [1], where a detailed bibliography can be found.

In constructing the Noether theory of the operator A a basic role is played by the fact that if at each point of the contour Γ a Lyapunov condition is satisfied, then the operator $VSV + S$ $(S = P + Q)$ is completely continuous in space $L_p(\Gamma, \rho)$ (see [1]). In this case A is Noetherian [1] if and only if the operator

$$A_V = \left\| \begin{matrix} a & c \\ \bar{d} & \bar{b} \end{matrix} \right\| P + \left\| \begin{matrix} b & d \\ \bar{c} & \bar{a} \end{matrix} \right\| Q \tag{2}$$

possesses the same property in the space $L_p^2(\Gamma, \rho) = L_p(\Gamma, \rho) \times L_p(\Gamma, \rho)$. The situation is otherwise if the contour Γ has corner points. It turns out that in this case the operator $VSV + S$ is not completely continuous in $L_p(\Gamma, \rho)$, and if A is Noetherian, so is A_V, but the converse assertion is not true. These facts disclose the essential difference between the piecewise Lyapunov contour and a Lyapunov contour.

In this note we construct the symbol of the operator (1) in the form of a matrix-valued function of the variable order. The non-degeneracy of the symbol is a necessary and sufficient condition for the operator A to be Noetherian in $L_p(\Gamma, \rho)$. Analogous results are obtained for the generalized Riemann boundary value problem.

2 Defining the symbol

Let $t_1, \ldots, t_n$ be all the corner points of the contour Γ, where $t_1 \prec t_2 \prec \cdots \prec t_n$ and the relation $t_k \prec t_{k+1}$ means that the point t_k precedes t_{k+1} on the oriented contour Γ. We denote by $\tilde{\alpha}_k,\ k = 1, 2, \ldots, n$, the non-negative angle, $0 \leq \tilde{\alpha}_k \leq 2\pi$, by which an infinitely small vector $t_k Z$ rotates when the point Z to the left of Γ and turning about t_k passes from the portion $t_k t_{k+1},\ k = 1, 2, \ldots, n,\ t_{n+1} = t_1$ of the contour Γ to the portion $t_{k-1} t_k$ $(t_0 = t_1)$. We denote by α_k $(= \alpha(t_k))$ the quantity $\alpha_k = min(\tilde{\alpha}_k,\ 2\pi - \tilde{\alpha}_k)$. In this paper we assume that $\alpha_k \neq 0$.

We now define the symbol $A(t,\xi)$, $t\in\Gamma$, $-\infty\le\xi\le\infty$ of the operator A acting in $L_p(\Gamma,\rho)$. To this end we first define the symbol of the operators aI, P and Q. The symbol aI, $a\in C(\Gamma)$, is the matrix-valued function $a(t,\xi)$, $t\in\Gamma$, $-\infty\le\xi\le\infty$ of variable order defined by the following equalities:

$$a(t,\xi)=\begin{cases} diag(a(t),\ \overline{a(t)})\ \ for\ t\neq t_k, \\ diag(a(t_k),\ \overline{a(t_k)},\ a(t_k),\ \overline{a(t_k)}), \end{cases}$$

where $diag(x_1,x_2,\dots,x_s)$ is the diagonal matrix of order s with the elements $x_1,x_2,\dots,x_s$ on the diagonal.

The symbol $P(t,\xi)$ of an operator P is the following matrix-valued function

$$P(t,\xi)=\left\|\begin{matrix}1&0\\0&0\end{matrix}\right\|\ \text{for}\ t\in\Gamma\setminus\{t_1,t_2,\dots,t_n\},$$

$$P(t,\xi)=\frac{1}{z_k^{2\pi}-1}\left\|\begin{matrix} z_k^{2\pi} & 0 & -z_k^{\alpha_k} & 0\\ 0 & -1 & 0 & z_k^{2\pi-\alpha_k}\\ z_k^{2\pi-\alpha_k} & 0 & -1 & 0\\ 0 & -z_k^{\alpha_k} & 0 & z_k^{2\pi}\end{matrix}\right\|,$$

where $z_k=\exp(\xi+i\frac{1+\beta_k}{p})$. We define the symbol $Q(t,\xi)$ of the operator Q by the formula $Q(t,\xi)=E(t)-P(t,\xi)$, where $E(t)$ is the identity matrix of second order for $t\neq t_k$ and of fourth order for $t=t_k$, $k=1,2,\dots,n$.

The symbol $V(t,\xi)$ of the operator V is defined by the matrices

$$V(t,\xi)=\left\|\begin{matrix}0&1\\1&0\end{matrix}\right\|,\ \text{for}\ t\in\Gamma\setminus\{t_1,\dots,t_n\},\ V(t_k,\xi)=\left\|\begin{matrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{matrix}\right\|.$$

If the operator A has the form (1), we define its symbol $A(t,\xi)$ to be

$$A(t,\xi)=a(t,\xi)\,P(t,\xi)+b(t,\xi)\,Q(t,\xi)+$$

$$+\left[c\left(t,\xi\right)P\left(t,\xi\right)+d\left(t,\xi\right)Q\left(t,\xi\right)\right]V\left(t,\xi\right).$$

Theorem 1. *The operator $A = aP + bQ + (cP + dQ)\,V$ is Noetherian in the space $L_p\left(\Gamma,\rho\right)$ if and only if the following condition is satisfied*

$$\det A(t,\xi) \neq 0,\; t \in \Gamma,\; -\infty \leq \xi \leq \infty.$$

Corollary 1. *If the operator A is Noetherian, then the corresponding operator A_V defined by* (2) *is also Noetherian. The converse is not true, in general.*

The properties of operators of local type [2] and some results from [3] and [4] concerning singular operators with a shift along piecewise Lyapunov curves are used in the proof of Theorem 1.

Theorem 2. *The operator*

$$(VSV+S)\phi = \frac{1}{\pi i}\int_{\Gamma}\frac{\phi\left(\tau\right)}{\bar{\tau}-\bar{t}}\overline{d\tau} + \frac{1}{\pi i}\int_{\Gamma}\frac{\phi\left(\tau\right)}{\tau - t}d\tau$$

is completely continuous in space $L_p\left(\Gamma,\rho\right)$ if and only if Γ is a Lyapunov contour.

The sufficient part of this assertion was proved in [1,5]. We prove here the necessity. Suppose that $VSV + S$ is completely continuous, then the operator $R_\lambda = VSV + S - \lambda I$ is Noetherian for all $\lambda \in \mathbb{C}\setminus\{0\}$. Hence, by Theorem 1, $\det R_\lambda(t_k,\xi) \neq 0$ for all $k = 1, 2, \ldots, n$ and $-\infty \leq \xi \leq \infty$. From this we find that $\frac{z_k^{2\pi-\alpha_k} - z_k^{\alpha_k}}{z_k^{2\pi}-1} \equiv 0$, where $z_k = \exp(\xi + i\frac{1+\beta_k}{p})$.
The latter is possible only for $\alpha_k = \pi$. This means that Γ is a Lyapunov contour. The proof of the theorem is complete.

In contrast to singular operators don't containing the operator V (i.e., $A = aP + bQ$), the condition for the operator A be Noetherian essentially depends on the contour. For example, the operator $A = (1+\sqrt{2})P + (1-\sqrt{2})Q + V$ is Noetherian in all spaces $L_p\left(\Gamma,\rho\right)$, if Γ is a Lyapunov contour and is not Noetherian in $L_2\left(\Gamma\right)$, if Γ has at least one corner point with angle $\pi/2$. This follows immediately from Theorem 1.

3 Noetherian criteria

In conclusion we consider the generalized Riemann boundary value problem: find analytic functions $\Phi^+(z)$ and $\Phi^-(z)$ which can be represented by the Cauchy integral in $D^+$and D^-, with limit values on Γ which belong to $L_p(\Gamma, \rho)$, $1 < p < \infty$, and satisfy the boundary condition

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t)\overline{\Phi^-(t)} + c(t), \qquad (3)$$

where $a(t)$ and $b(t)$ are known continuous functions on Γ and $c(t) \in L_p(\Gamma, \rho)$.

The Noether theory to the problem (3) in the case of a Lyapunov contour is constructed in [1] and [6]. In particular, in these papers it was established that the inequality $|a(t)| > 0$ for all $t \in \Gamma$ is a necessary and sufficient condition for the problem to be Noetherian. In the case of a piecewise Lyapunov contour we have the following theorem.

Theorem 3. *The following conditions are necessary and sufficient for the problem* (3) *be Noetherian:*

1. $|a(t)| > 0, \; t \in \Gamma$;

2. $|a(t_k)|^2 - \dfrac{z_k^{2\pi-\alpha_k} - z_k^{\alpha_k}}{z_k^{2\pi} - 1}|b(t_k)|^2 \neq 0$, for all $k = 1, 2 \ldots, n$,

where

$$z_k = \exp(\xi + i\frac{1+\beta_k}{p}), \; -\infty \leq \xi \leq \infty.$$

Thus, in the case of a piecewise Lyapunov contour, the Noetherian property of the problem (3) depends not only by the coefficient $a(t)$, as in the case of a Lyapunov contour, but also on $b(t)$.

4 Conclusion

The Noetherian criteria for the Riemann boundary-value problems were obtained on piecewise Lyapunov curves applying the Plemelj-Sohotsky formulas in combination with the symbol of singular integral equations. The proposed method can be used in the study of boundary-value problems with discontinuous coefficients when the integration contour consists of a finite number of closed curves without common points.

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