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Topological mixing and specification in weakly contracting relations

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Abstract

We establish topological mixing and Specification property of the dynamics of a weakly contracting compact-valued function on its attractor. These properties strengthen our earlier results, obtained for weakly contracting relations, namely, the topological transitivity and Shadowing property.

Keywords: Set-valued maps, weak contractions, attractor, shadowing, specification property, topological mixing.

1 Introduction

In [1] the authors have stated the existence of the compact global attractor for a relation, which is contracting with respect to the Hausdorff-Pompeiu metrics. Moreover, some characteristics of set-valued dynamics of these relations restricted to their attractors, have been stated, as, e.g., "asymptotic phase property", topological transitivity, denseness of periodic points, minimality with respect to "big orbits", and Shadowing property. In [2] some of these properties, including "asymptotic phase property", topological transitivity and Shadowing have been generalized for weakly contracting multi-functions. In this article we strengthen the last two properties up to topological mixing and Specification property, respectively.

2 Multi-valued weak contractions

Let (X, d) be a complete metric space and let $\mathcal{P}(X)$ denote the family of all non-empty compact subsets, endowed with Hausdorff-Pompeiu metrics H. We are concerned with the dynamics, generated by compositions of upper-semi-continuous multi-functions $f : X \to \mathcal{P}(X)$, called also as *relations*. In this context, a finite or infinite sequence $\{x_n\} \subset X$ is called a *chain* for the multi-function f, if $x_{n+1} \in f(x_n)$ for all n. Similarly, given $\delta > 0$, the sequence $\{x_n\}$ is called a δ-*chain*, if $\varrho(x_{n+1}, f(x_n)) \leq \delta$ for all n (here $\varrho(a, B) := \inf\limits_{b \in B} d(a, b)$).

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* [3], if φ is monotonically increasing and $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t \geq 0$.

Following [3], we will say that $f : X \to \mathcal{P}(X)$ is a *weak contraction*, if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $H(f(x), f(y)) \leq \varphi(d(x, y))$ $(\forall\, x, y \in X)$.

A nonempty closed subset $A \subset X$ is called *attractor* for f, if $f[A] \supset A$ and there is a closed neighborhood $\overline{V(A, \delta)}$ of A, where $V(A, \delta) := \{x \in X \mid \varrho(x, A) < \delta\}$, such that $\bigcap\limits_{n \geq 0} f^n[\overline{V(A, \delta)}] \subset A$.

One says that the relation $f : X \to \mathcal{P}(X)$ has the *Shadowing property* on the compact invariant subset $A \subset X$ if, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any δ-chain $\{x_n\}_{n \in \mathbb{N}} \subset V(A, \delta)$ there exists a chain $\{y_n\}_{n \in \mathbb{N}} \subset A$ satisfying $d(x_n, y_n) \leq \varepsilon$ for all $n \in \mathbb{N}$.

Theorem 1. [2] *Any weakly contracting compact valued mapping has a nonempty compact attractor and this attractor is unique. If, in addition, f is weakly contracting with respect to a right-continuous comparison function, then the multi-function f has the Shadowing property on the attractor.*

3 Topological mixing and specification

Recall (see, e.g. [1]) that a multi-function $f : X \to \mathcal{P}(X)$ is called *transitive* on the compact invariant subset A, if there is a dense chain, or equivalently, if for any two open subsets $U, V \subset A$ and any $x \in U$ there is a chain $(x_k)_{k=1}^n \subset A$ such that $x_1 = x$ and $x_k \in V$. If, in addition, there is a chain $(x_n)_{n\in\mathbb{N}} \subset A$, which starts in U and which remains in V for all large enough n, then one speaks about *topological mixing*. Topological transitivity of a contracting relation on its attractor has been stated in [2]. In this article we establish stronger analogous of transitivity and Shadowing properties, namely topological mixing and specification.

Theorem 2. *Every multi-function, which is weakly contracting with respect to a right-continuous comparison function, is topologically mixing on its attractor.*

The Specification property for diffeomorphisms was introduced by R. Bowen [4]. It says that any finite collection of consecutive pieces of orbits of $f : X \to X$ can be shadowed by an individual orbit, provided that the time-lag between the specified orbit segments is large enough.

The following definition represents a generalization of the Specification property, given in [5] for homeomorphisms (see also [6, 7]). More precisely, given the multi-function $f : X \to \mathcal{P}(X)$, we call *specification* for f the pair $S = (\tau, P)$, consisting of a finite family of time-segments $\tau = \{I_1, I_2, \dots, I_m\}, I_j \subset \mathbb{N}$, and a mapping $P : \bigcup\limits_{j=1}^{m} I_j \to X$, such that $P(t+1) \in f(P(t))$ for all $t \in I \in \tau$, provided that $t+1 \in I$. We say that the specification S is *ε-shadowed* by the chain $(x_n)_{n\in\mathbb{N}}$, if $d(x_n, P(n)) < \varepsilon$ for all $n \in \bigcup\limits_{j=1}^{m} I_j$. Given the natural number M, the specification is called *$M-$spaced*, if the gap between two consecutive time-segments is at least M.

One says that the multi-function f has *the Specification property* on the invariant compact subset $A \subset X$ if, given $\varepsilon > 0$, there is a natural M such that each $M-$spaced specification is ε-shadowed by a chain.

Theorem 3. *Every multi-function, which is weakly contracting with respect to a right-continuous comparison function, has the Specification property on its attractor.*

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References

[1] V. Glavan, V. Guţu. *On the Dynamics of Contracting Relations.* Analysis and Optimization of Differential Systems. Edited by V. Barbu, I. Lasiecka, D. Tiba and C. Varsan, Kluwer Acad. Publ. 2003. pp. 179–188.

[2] V. Glavan, V. Guţu. *Attractors and fixed points of weakly contracting relations*, Fixed Point Theory, 5, no. 2 (2004), pp. 265–284.

[3] I.A. Rus. *Generalized Contractions and Applications.* Cluj University Press, Cluj-Napoca, 2001.

[4] R. Bowen. *Periodic points and measures for Axiom A diffeomorphisms*, Trans. Amer. Math. Soc., 154, (1971), pp. 377–397.

[5] A. Katok, B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, 1995.

[6] W. Cordeiro, M. Denker, M. Yuri. *A note on specification for iterated function systems*, Discrete and Continuous Dynamical Systems, Series B, Issue 10, (2015), pp. 3475–3485.

[7] B.E. Raines, T. Tennant. *The specification property on a set-valued map and its inverse limit*, arXiv:1509.08415.

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