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Preface

This spring (2014) the Institute of Mathematics and Computer Science (IMCS) of the Academy of Sciences of Moldova has celebrated its 50 years anniversary. Today we have all the reasons to assert that IMCS is a center for advanced research in the domains of pure mathematics, the applied one and computer science.

Academician V. Andrunachievici was the founder of the Institute. He was a talented mathematician and organizer who managed to form an advanced team.

Throughout the years, the researchers of the Institute have obtained remarcable results in the fields of algebra, functional analysis, differential equations, mathematical logics, computing mathematics, mathematical modeling and computer science. Towards the end of 70-th (XXth century), famous scientific schools in the theory of algebraic rings (the founder is acad. V. Andrunachievici), in the qualitative theory of differential equations (acad. C. Sibirschi), in the theory of quasigroups (cor.m. of Academy of Pedagogical Sciences of the former Soviet Union, Professor V. Belousov), in functional analysis (cor.m. I. Gohberg) and in mathematical logic (Dr. A. Kuzneov) has been formed.

Currently, most mentioned schools continue their successful work and enjoy international authority. Within the framework of these schools, over 3000 of scientific papers and about 150 monographs in different domains of mathematics and computer science have been published. Also 127 volumes from series “Mathematical research” (in Russian “Matematicheskie issledovania”) and 15 volumes “Applied mathematics and programming” (in Russian “Prikladnaia matematika i programmirovanie”) had appeared.

In the Institute, 30 doctors in habilitation and about 400 PhDs were trained through doctoral study or with contribution of Specialized Scientific Councils. Starting from 1989, the Institute publishes the journal "Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica" (to this day 73 volumes were out). Also, starting from 1993, the journal "Computer Science Journal of Moldova" (64 volumes) is published. In collaboration with colleagues from Poland the journal "Quasigroups and related systems" (21 volumes) is published.

Over a period of last 20 years the Institute took part in executing 50 international projects with partners from advanced research centers in different countries of Europe, Asia, America.

The country's highest distinctions in research – State Award and National Award – were conferred to 11 researchers, 19 respective awards were conferred to young researchers. Ten persons (researchers) from IMCS became laureates of "Academician Constantin Sibirschi" award for achievements in mathematics and computer science.

The **Third Conference of Mathematical Society of the Republic of Moldova dedicated to the 50th anniversary of the foundation of Institute of Mathematics and Computer Science "IMCS-50"** is a continuation of the number of scientific events, which started with the International Conference on Intelligent Information Systems – IIS 2013 and the 14th International Conference on Membrane Computing – CMC14, organized by IMCS in August 2013 and dedicated to this anniversary.

We are confident that these Proceedings, which comprise the last results of moldovan scientists and their foreign colleagues, will contribute considerably to the development and promotion of research in mathematics and computer science not only in Republic of Moldova, but abroad as well.

Mitrofan Choban
Svetlana Cojocaru

Section 1

Algebra, Geometry and Topology

Proceedings of the Third Conference of Mathematical Society of Moldova
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Functional subordination in general spaces

Loriana Andrei, Mitrofan Choban

Abstract

In the present paper we study the subordination equations for certain analytic functions in the open unit disk. These results are obtained by investigating classes of admissible functions.

Keywords: analytic functions, subordination, differential subordination.

1 Introduction

Every space is considered to be a completely regular Hausdorff space. Let X, Y, Z be non-empty spaces and $\varphi, \psi : X \times Z \longrightarrow Y$ be continuous mappings. Consider the expression $\varphi(\omega, z) = \psi(x, z)$ (1). A solution of the equation (1) is a function $\omega : X \times Z \longrightarrow X$ such that $\varphi(x, z) = \psi(\omega(x, z), z)$ for all $x, z \in X \times Z$. If the solution of the equation (1) exists, then we put: $\varphi(x, z) \prec\prec \psi(x, z)$ (2). The expression (2) is called a subordination. A strong solution of the equation (1) is a function $\omega : X \longrightarrow X$ such that $\varphi(x, z) = \psi(\omega(x), z)$ for all $x, z \in X \times Z$. Any strong solution of the equation (1) is a solution of the equation (1) too. From the Axiom of Choice it follows that there exist solutions of the equation (1) if and only if $G_{\varphi\psi}(x, z) = \psi^{-1}(\varphi(x, z)) \neq \emptyset$ for all $x \in X$ and $z \in Z$. Obviously, $G_{\varphi\psi}(x, z) \neq \emptyset$ for all $x \in X$ and $z \in Z$ if and only if $\varphi(X \times \{z\}) \subseteq \psi(X \times \{z\}$ for each $z \in Z$. The equation (1) is called Z-simple if $G_{\varphi\psi}(x, z) = G_{\varphi\psi}(x, w)$ for all $x \in X$ and $z, w \in Z$. In this case any solution $\omega : X \times Z \longrightarrow X$ of the equation (1) and any point $z \in Z$ generates the strong solution $\omega_z : X \longrightarrow X$, where $\omega_z(x) = \omega(x, z)$ for each $x \in X$, of the equation (1). Let Z

be a singleton space. In this case we may assume, without loss of generality, that $X \times Z = X$ and the mappings $\varphi, \psi : X \longrightarrow Y$ generate the following equation of the subordination $\varphi(\omega(x)) = \psi(x)$ (3). Any solution of the equation (3) is a strong solution. If $\varphi : X \times Z \longrightarrow Y$ and $\psi : X \longrightarrow Y$ are continuous mappings, then we obtain the intermediary equation of the subordonation $\varphi(\omega(x, z), z) = \psi(x)$ (4). The equations of the form (4) were examined in [3].

2 Conditions of the existence of solutions

Proposition 2.1. *Let $\varphi, \psi : X \times Z \longrightarrow Y$ be continuous mappings with the following property: for each $z \in Z$ the mapping $\psi_z : X \longmapsto Y$, where $\psi_z(x) = \psi(x, z)$ for each $x \in X$, is an open mapping of X into Y. Then:*

- the mapping ψ is open;

- if the equation (1) is Z-simple, then the mapping $G_{\varphi\psi}$ is lower semicontinuous.

Proposition 2.2. *Let $\varphi, \psi : X \times Z \longrightarrow Y$ be continuous mappings with the following property: for each $z \in Z$ the mapping $\psi_z : X \longmapsto Y$, where $\psi_z(x) = \psi(x, z)$ for each $x \in X$, is an injection and $\varphi(X \times \{z\}) \subseteq \psi_z(X)$. Then $\omega = G_{\varphi\psi}$ is the unique single-valued solution of the equation (1). Moreover, if the mapping $G_{\varphi\psi}$ is lower (upper) semicontinuous, then the mapping ω is continuous.*

Proposition 2.3. *Let $\varphi, \psi : X \times Z \longrightarrow Y$ be continuous mappings, X be a complete metric space, $X \times Z$ be a paracompact space and $dim(X \times Z) = 0$. If the mapping $G_{\varphi\psi}$ is lower semicontinuous and $\varphi(X \times \{z\}) \subseteq \psi(X \times \{z\})$ for each $z \in Z$, then there exists a continuous solution of the equation (1).*

3 Almost locally homeomorphisms

A point $x \in X$ is a point of a locally homeomorphism of a mapping $f : X \longrightarrow Y$ if there exists an open subset V of X such that $x \in V$, the set $W = f(V)$ is open in Y and $f|V$ is a homeomorphism of V onto W.

A mapping $f : X \longrightarrow Y$ is called an almost locally homeomorphism if f is an open continuous mapping and if $x \in X$ is not a point of locally homeomorphism of the mapping f, then there exists an open subset V of X such that $x \in V$ and any point $y \in V \setminus \{x\}$ is a point of locally homeomorphism of the mapping f.

Theorem 3.1. *Let X, Y be topological spaces, $\varphi : X \longrightarrow Y$ be a continuous mapping, $\psi : X \longrightarrow Y$ be an almost locally homeomorphism and $\varphi(X) \subseteq \psi(X)$. Then there exist an open subset D of X and a continuous mapping $h : D \longrightarrow X$ such that: $\varphi(x) = \psi(h(x))$ for each $x \in D$; the set D is dense in X; if the mapping φ is open, then the mapping h is open too, and for each point $x \in D$ there exist two open subsets V and W of X such that $x \in V$, $h(x) \in W$, $\varphi(V) = \psi(W)$, and $\psi|W$ is a homeomorphism of W onto $\psi(W)$; if the mapping φ is an almost locally homeomorphism, then the mapping h is open and for each point $x \in D$ there exist two open subsets V such that $h|V$ is a homeomorphism of V onto $h(V)$.*

4 Differential subordinations

Let Y be a commutative Banach algebra over the field of complex numbers $\mathbb{C}$. Denote by $\mathbf{1}$ the unity in Y. If we identify $z \in \mathbb{C}$ with $z \cdot \mathbf{1}$, then we obtain that $\mathbb{C}$ is a subalgebra of Y. Denote by $U = \{z \in \mathbb{C} : \ |z| < 1\}$ the unit disc of the complex plane $\mathbb{C}$ and by $\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}$ the closed unit disc of the complex plane. Consider the analytic functions $\varphi_n : \mathbb{C}^{n+1} \times U \times \overline{U} \longrightarrow Y$, and $\psi : U \longrightarrow Y$, where $n \geq 2$. In this case we may write the differential subordination of the order n: $\varphi_n(p(z), zp'(z), ..., z^n p^{(n)}(z), z, \xi) \prec\prec \psi(z)$ (5).

Let $\mathcal{H}(a, n)$ be the class of analytic functions $f : U \to \mathbb{C}$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$, $\mathcal{A}_1 = \mathcal{H}(1, 1)$, Q_a be the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $E(q) = \{\zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty\}$, for which $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$ and $q(0) = a$. Fix $\Omega \subseteq \mathbb{C}$ and $q(z) \in Q_1 \cap \mathcal{H}(q(0), 1)$. The class of admissible functions $\Phi[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\phi\,(u, v, w; z) \notin \Omega$, whenever $u = q(\zeta)$, $v =$

$q(\zeta)+\lambda\frac{k\zeta q'(\zeta)}{q(\zeta)}$, $q(\zeta)\neq 0$, $\lambda\geq 0$, $Re(\frac{v(w-v)}{\lambda(v-u)}+\frac{v-2u}{\lambda})\geq kRe(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1)$, $z\in U$, $\zeta\in\partial U\backslash E(q)$, $k\geq 1$.

Let $\lambda\geq 0$ and $n,m\in\mathbb{N}$. Denote by $DR_\lambda^{m,n}:\mathcal{A}\to\mathcal{A}$ the operator given by the Hadamard product of the generalized Sălăgean operator D_λ^m and the Ruscheweyh operator R^n (see [1, 2, 3]).

Theorem 4.1. *Let $\Omega\subset\mathbb{C}$ and $\phi\in\Phi\left[\Omega,q\right]$, or, in particular, q be univalent in U, $q\left(0\right)=1$ and $\phi\in\Phi\left[\Omega,q_\rho\right]$, for some $\rho\in(0,1)$, where $q_\rho\left(z\right)=q\left(\rho z\right)$. If $f\in\mathcal{A}$ satisfies*

$$\{\phi\left(\frac{DR_\lambda^{m+1,n}f(z)}{DR_\lambda^{m,n}f(z)},\frac{DR_\lambda^{m+2,n}f(z}{DR_\lambda^{m+1,n}f(z)},\frac{DR_\lambda^{m+3,n}f(z)}{DR_\lambda^{m+2,n}f(z)};z\right):z\in U\}\subset\Omega,$$

then $(DR_\lambda^{m+1,n}f(z)):(DR_\lambda^{m,n}f(z))\prec q(z)$.

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Properties of Main Lattices of Group Topologies

V.I. Arnautov, G.N. Ermakova

Abstract

This paper is a survey of results about properties of various lattices of group topologies.

Keywords: Hausdorff topology, countable group, group topology, lattice of group topologies, basis of the filter of neighbourhoods, number of group topologies, metrizable group topology.

This paper is a survey of results about the properties of various lattices of group topologies.

A.A.Markov started study of the lattice of group topologies in an article of 1945. He proposed a method of constructing of some Hausdorff group topologies on any countable group.

Later we proposed a more general method of specifying such group topologies on countable groups. This method allows to get any metrizable group topology on any countable group.

Notation 1. For any group G the following sets of subsets are considered:

– the set Δ of all normal subgroups of the group G;

– the set Ω_0 of all group topologies on the group G;

– the set Ω_1 of all group topologies on the group G, for each of which the topological group is a subgroup of a complete group;

– the set Ω_2 of all group topologies on the group G, for each of which the left and right uniform structures match;

– the set Ω_3 of all group topologies on the group G, for each of which the topological group has a basis of the filter of neighborhoods of unity, which consists of subgroups;

– the set Ω_4 of all group topologies on the group G, for each of which the topological group has a basis of the filter of neighborhoods of unity, which consists of normal subgroups;

– the set Ω_5 of all group topologies on the group G, for each of which the topological group has a finite basis of the filter of neighborhoods of unity;

– the set Ω_6 of all group topologies on the group G, for each of which the topological group is a subgroup of a compact group.

Theorem 2. *For any group each of the above sets, with the procedure, which is defined by the inclusion, is a complete lattice, and the following conditions are true:*

- the lattices Ω_5 and Δ are anti-isomorphic;

- the lattices Δ, Ω_2, Ω_4, Ω_5 and Ω_6 are modular lattices.

Theorem 3. *Let τ_1 and τ_2 be group topologies on the group G, and let B_1 and B_2 be filters of bases of neighborhoods of the unity in topological groups (G, τ_1) and (G, τ_2), respectively. Then the following assertions are true:*

- if $\Omega \in \{\Omega_0, \Omega_1, \Omega_2, \Omega_{3,}, \ \Omega_4, \ \Omega_5, \Omega_6\}$ and $\tau_1, \tau_2 \in \Omega$, then $\Omega = \{V \cap U | V \in \mathrm{B}_1, U \in \mathrm{B}_2\}$ is a basis of the filter of neighborhoods of unity in the topological group $(G, \sup\{\tau_1, \tau_2\})$;

- if $\Omega \in \{\Omega_2, \ \Omega_4, \ \Omega_5, \Omega_6\}$ and $\tau_1, \tau_2 \in \Omega$, then $\Omega = \{V \cdot U | V \in \mathrm{B}_1, U \in \mathrm{B}_2\}$ is a basis of the filter of neighborhoods of unity in the topological group $(G, \inf\{\tau_1, \tau_2\})$.

The methods for specifying group topologies, which are given above, allow to get the following criterion (see below Theorem 5) of the existence of non-discrete, Hausdorff group topologies on a countable group.

Definition 4. The equality $g_1 \cdot x^{\pm 1} \cdot g_2 \cdot x^{\pm 1} \cdot \ldots \cdot g_n \cdot x^{\pm 1} g_{n+1} = g_0$, where $g_i \in G$ for any natural number $0 \leq i \leq n+1$, will be called an equation over a group G with variable x.

Theorem 5. *Any countable group G admits a non-discrete, Hausdorff, group topology iff for any finite number of equations over the group G such that the identity element of the group G is not a root of each of these equations, there is an element such that it is not equal to the identity element and it is not a root of each of these equations.*

Theorem 6. *Every infinite Abelian group admits non-discrete Hausdorff group topology (A.Kertesz, T.Szele 1953).*

Remark 7. A.Yu. Ol'shansky constructed an example of countable group, which does not allow non-discrete Hausdorff group topologies in 1980.

Theorem 8. *If a countable group G admits group topology in which topological group does not have a finite basis of the filter of neighborhoods of unity, then the following statements are true:*

- the lattice of all group topologies on the group G contains a sublattice isomorphic to the lattice of real numbers with the usual order;

- the lattice of all group topologies on the group G contains two to the power continuum of pairwise incomparable topologies.

Theorem 9. *If a countable group G admits a non-discrete group topology in which topological space (G, τ) is Hausdorff, then the lattice of all group topologies on the group G contains two to the power of the continuum coatoms.*

Definitions 10. As usual, we say that:

- an element a_2 covers an element a_1 in partially ordered set $(A, <)$, if $a_1 < a_2$ and there is no element $b \in A$, such that $a_1 < b < a_2$. In this case we write $a_1 \prec a_2$.

- a chain $a_1 < a_2 < \ldots < a_n$ of elements of a partially ordered set $(A, <)$ is called an unrefinable chain if $a_k \prec a_{k+1}$ for any natural number $1 \leq k < n$.

Since for an arbitrary group $G(\cdot)$ any lattice $\Omega \in \{\Omega_2, \Omega_4, \Omega_5, \Omega_6\}$ is a modular lattice, then the following theorem is true:

Theorem 11. *Let $G(\cdot)$ be any group and let $\Omega \in \{\Omega_2, \Omega_4, \Omega_5, \Omega_6\}$. Then for any unrefinable chain $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_n$ in the lattice Ω the following statements are true*:

11.1. *If $\tau'_1 < \tau'_2 < \ldots < \tau'_k$ is a chain topologies in the lattice Ω such that $\tau_1 = \tau'_1$ and $\tau_n = \tau'_k$, then $k \leq n$. In particular, if the chain $\tau'_1 < \tau'_2 < \ldots < \tau'_k$ is a unrefinable chain, then $k = n$;*

11.2. *If $\tau \in \Omega$, then* $\sup\{\tau_i, \tau\} \prec \sup\{\tau_{i+1}, \tau\}$ *or* $\sup\{\tau_i, \tau\} = \sup\{\tau_{i+1}, \tau\}$ *for each natural number $1 \leq i \leq n-1$;*

11.3. *If $\tau \in \Omega$ and $\tau'_1 < \tau'_2 < \ldots < \tau'_k$ is a chain in the lattice Ω*

such that $\tau_1' = \inf\{\tau_1, \tau\}$ and $\tau_k' = \inf\{\tau_n, \tau\}$, then $k \le n$;

11.4. *If $\tau \in \Omega$ and $\tau_1' < \tau_2' < \ldots < \tau_k'$ is a chain in the lattice Ω such that $\tau_1' = \sup\{\tau_1, \tau\}$ and $\tau_k' = \sup\{\tau_n, \tau\}$, then $k \le n$.*

Theorem 12. *Let $G(\cdot)$ be arbitrary nilpotent group, and let $\Omega \in \{\Omega_0, \Omega_1, \Omega_2, \Omega_{3,}, \Omega_4, \Omega_5, \Omega_6\}$. Then for every unrefinable chain $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_n$ of group topologies in the lattice Ω the following statements are true:*

12.1. *If $\tau_1' < \tau_2' < \ldots < \tau_k'$ is such that $\tau_1 = \tau_1'$ and $\tau_n = \tau_k'$, then $k \le n$. In particular, if the chain $\tau_1' < \tau_2' < \ldots < \tau_k'$ is an unrefinable chain, then $k = n$;*

12.2. *If $\tau \in \Omega$, then $\sup\{\tau_i, \tau\} \prec \sup\{\tau_{i+1}, \tau\}$ or $\sup\{\tau_i, \tau\} = \sup\{\tau_{i+1}, \tau\}$ for each $1 \le i \le n-1$;*

12.3. *If $\tau_1' < \tau_2' < \ldots < \tau_k'$ is a chain such that $\tau_1' = \sup\{\tau_1, \tau\}$ and $\tau_k' = \sup\{\tau_n, \tau\}$, then $k \le n$.*

Remark 13. There exists an example of a nilpotent group G and group topologies τ_1 and τ_2 on the group G, such that τ_1 is a coatom in the lattice Ω_0 (i.e. $\tau_1 \prec \tau_d$, where τ_d is the discrete topology), and between the topologies $\inf\{\tau_1, \tau_2\}$ and $\tau_2 = \inf\{\tau_d, \tau_2\}$ there is a chain of group topologies which is infinitely increasing and infinitely decreasing.

This example shows that for nilpotent groups the lattice Ω_0 can be non modular in general case.

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Successively orthogonal systems of k-ary operations and their transformations

Galina B. Belyavskaya

Abstract

In this article systems of k-ary operations, $k \geq 2$, generalizing orthogonal sets are considered. These systems have the following property: every k successive k-ary operations of the system are orthogonal. We call these systems successively orthogonal, suggest methods of construction and consider admissible transformations of these systems.

Keywords: k-ary operation, k-ary quasigroup, orthogonal set of k-ary operations.

1 Introduction

It is known that k-ary operations, $k \geq 2$, correspond to k-dimensional hypercubes which are objects of combinatorial analysis. A binary quasigroup is an algebraic equivalent of a Latin square and a k-ary quasigroup respects to a permutation cube of the dimension k. The algebraic approach is useful for research of such combinatorial objects. All these objects and their corresponding orthogonal sets (systems) have many applications in various areas including affine and projective geometries, designs of experiments, error-correcting and error-detecting coding theory and cryptology.

2 Preliminaries

A k-ary operation A (briefly, a k-operation) on a set Q is a mapping $A : Q^k \to Q$, defined by $A(x_1^k) \to x_{k+1}$. In this case write $A(x_1^k) =$

x_{k+1}.

A k-groupoid (Q, A) is a set Q with one k-ary operation A, defined on Q.

The k-operation $E_i : E_i(x_1^k) = x_i$, $1 \leq i \leq k$, on Q is called *the i-th identity operation (or the i-th selector) of arity k.*

An i-invertible k-operation A defined on Q, is a k-operation with the following property: the equation $A(a_1^{i-1}, x, a_{i+1}^k) = a_{k+1}$ has a unique solution for each fixed k-tuple $(a_1^{i-1}, a_{i+1}^k, a_{k+1})$ of Q^k [2].

A k-tuple $< A_1^k >$ of k-operations is orthogonal if and only if the mapping $\theta = (A_1^k) : Q^k \rightarrow Q^k$, $(x_1^k) \rightarrow (A_1(x_1^k), A_2(x_1^k), ..., A_k(x_1^k)) = (A_1^k)(x_1^k)$ is a permutation on Q^k [1].

All 2-invertible binary operations, given on a set Q, form the group $(\Lambda_2; \cdot)_Q$ under the multiplication $(A \cdot B)(x, y) = A(x, B(x, y))$.

A k-ary quasigroup (or simply, a k-quasigroup) is a k-groupoid (Q, A) such that the k-operation A is i-invertible for each $i = 1, 2, \ldots, k$.

Definition 1 [1]. *A k-tuple $< A_1, A_2, ..., A_k >=< A_1^k >$ of k-operations, given on a set Q, is called orthogonal if the system $\{A_i(x_1^k) = a_i\}_{i=1}^k$ has a unique solution for all $a_1^k \in Q^k$.*

Definition 2 [1]. *A set $\{A_1, A_2, \ldots, A_t\}$, $t \geq k$, of k-operations is called orthogonal if every k-tuple of these k-operations is orthogonal.*

Definition 3 [1]. *A set $\Sigma = \{A_1^t\}$, $t \geq 1$, of k-ary operations, given on a set Q, is called strongly orthogonal if the set $\overline{\Sigma} = \{A_1^t, E_1^k\}$ is orthogonal.*

3 Successively orthogonal systems

We give the following

Definition 4. *A system $\Sigma = \{A_1^t\}$, $t \geq k$, of k-ary operations, given on a set Q, $\mid Q \mid \geq 3$, is called successively orthogonal system (briefly, a SOS) if any successive k operations are orthogonal.*

Every orthogonal set of k-operations is a successively orthogonal system.

Proposition 1. *Let* $\Sigma_1 = \{A_1, A_2, ..., A_{s_1}\}$, $\Sigma_2 = \{B_1, B_2, ..., B_{s_2}\}$ *be strongly orthogonal sets of* k*-operations. Then the system*

$$\Sigma_3 = \{E_1, E_2, ..., E_k, A_1, A_2, ..., A_{s_1}, E_1, E_2, ..., E_k, B_1, B_2, ..., B_{s_2}\}$$

is a SOS.

Let (Q, A) be a quasigroup, $A^i(x, y) = A(x, A^{i-1}(x, y))$, $i = 2, \dots$.

Theorem 1. *If* $A, A_1, A_2, ..., A_t$ *are binary quasigroups of the order* $s_0, s_1, ..., s_t$ *respectively, in the group* $(\Lambda_2; \cdot)_Q$ *of all 2-invertible binary operations, given on a set* Q*, then the system*

$$F, E, A, A^2, ..., A^{s_0-1},\ F, E, A_1, A_1^2, ..., A_1^{s_1-1},$$

$$F, E, A_2, A_2^2, ..., A_2^{s_2-1}, ...,\ F, E, A_t, A_t^2, ..., A_t^{s_t-1}$$

is a SOS.

Let A^{-1} be the inverse element to a binary 2-invertible operation A in the group $(\Lambda_2; \cdot)_Q$. Simultaneously this operation is the right inverse quasigroup for (Q, A), if (Q, A) is a quasigroup.

Proposition 2. *Let* (Q, A) *be a binary quasigroup,* s *be the order of* A *in the group* $(\Lambda_2; \cdot)_Q$, $A^{-(i)} = (A^i)^{-1}$*. Then the system*

$$\Sigma = \{A^{-(s-1)}, A^{-(s-2)}, ..., A^{-1}, E, A, A^2, ..., A^{s-1}\}$$

is a SOS of 2-invertible binary operations.

Theorem 2. *Let* A *be an 1-invertible* k*-operation on a set* Q, $\theta = (E_2, E_3, ..., E_k, A) = (E_2^k, A)$*, and* s_0 *be the order of the permutation* θ *in the group* S_{Q^k}*, then the system of* k*-operations*

$$E_1, E_2, ..., E_k, A, A\theta, A\theta^2, ..., A\theta^{k-1}, A\theta^k, ..., A\theta^{s_0-k-1},$$

$$E_1, E_2, ..., E_k, A, A\theta, A\theta^2, ..., A\theta^{k-1}, A\theta^k, ..., A\theta^{s_0-k-1}, ...$$

is successively orthogonal.

Theorem 3. *Let a permutation* (E_2^k, A) *have the order* s_0*, then a successively orthogonal system of Theorem 2 contains* s_0 *different* k*-operations. If* $s_0 = k + 1$*, then the* k*-operation* A *is a quasigroup* k*-operation. For any 1-invertible* k*-operation* $s_0 \geq k + 1$*.*

4 Transformations of successively orthogonal systems

The successively orthogonal systems admit the known transformations of orthogonal sets of k-operations. This enables to get one from another system.

Theorem 4. *Let a k-tuple $\varphi = (C_1, C_2, ..., C_k)$ of k-operations, $k \geq 2$, given on a set Q, be orthogonal, a system $\Sigma =< B_1, B_2, ..., B_t >$ be a SOS, then the system of k-operations $\Sigma\varphi =< B_1\varphi, B_2\varphi, ..., B_t\varphi >$, where $B_i\varphi(x_1^k) = B_i(C_1(x_1^k), C_2(x_1^k), ..., C_k(x_1^k))$, is a SOS.*

Proposition 3. *Let a system of k-operations $\Sigma = \{B_1, B_2, ..., B_t\}$, $t \geq k$, be a SOS and $\varphi = (B_{i+1}, ..., B_{i+k})$ be a permutation of successive k-operations of Σ for some $i = 0, 1, ..., t-k$. Then the system $\Sigma\varphi^{-1} = \{B_1\varphi^{-1}, B_2\varphi^{-1}, ..., B_t\varphi^{-1}\} = \{B_1\varphi^{-1}, B_2\varphi^{-1}, ..., B_i\varphi^{-1}, E_1, E_2, ..., E_k, B_{i+k+1}\varphi^{-1}, ..., B_t\varphi^{-1}\}$ is a SOS, the operation $B_i\varphi^{-1}$ is k-invertible if $i \geq 1$ and the operation $B_{i+k+1}\varphi^{-1}$ is 1-invertible if $i \leq t-k-1$.*

Theorem 5. *Any orthogonal set of k-operations can be continued to a SOS.*

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On a Method of Constructing Medial and Paramedial Quasigroups

Natalia Bobeica, Liubomir Chiriac

Abstract

In this paper was developed a new method of constructing paramedial and medial quasigroups using the special direct product of Abelian groups.

Keywords: Abelian group, special direct product, paramedial and medial quasigroups.

1 Introduction

The results established here are related to the results of M. Choban and L. Chiriac in [2] and to the research papers [1,3,4,5]. Our main goal is to prove a new method of constructing non-associative medial quasigroups and non-associative paramedial quasigroups.

2 Basic notions

A non-empty set G is said to be a *groupoid* relatively to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

A groupoid $(G, \cdot)$ is called a *quasigroup* if for every $a, b \in G$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions.

A groupoid $(G, \cdot)$ is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

If a paramedial guasigroup G contains an element e such that $e \cdot x = x (x \cdot e = x)$ for all x in G, then e is called a *left (right) identity* element of G and G is called a *left (right) paramedial loop.*

A groupoid $(G, \cdot)$ is called a *groupoid Abel-Grassmann* or *AG-groupoid* if it satisfies the left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.

A groupoid $(G, \cdot)$ is called *AD-groupoid* if it satisfies the law $a \cdot (b \cdot c) = c \cdot (b \cdot a)$ for all $a, b, c \in G$.

Example 1 *Let* $(G, +)$ *be a commutative additive group with a zero* 0*. Consider a new binary operation* $x \cdot y = y - x$*. Then* $(G, \cdot)$ *is a medial quasigroup with a* $(1, 2)$*-identity* 0*. If* $x + x \neq 0$ *for some* $x \in G$*, then* 0 *is not an identity in* $(G, \cdot)$*.*

The notion of (n, m)-identity was introduced in [2].

3 Main Results

Theorem 1 *Let* $(G, +)$ *be a commutative group. The set* $G \times G$ *with the operation*

$$(x_1, y_1) \circ (x_2, y_2) = (x_2 + y_2 - x_1, x_1 + y_1 - y_2)$$

is a paramedial, non-medial, non-associative quasigroup.

Example 2 *Let* $G = \{0, 1\}$*. We define the binary operation* ” + ”*:* $0 + 0 = 0$*,* $0 + 1 = 1$*,* $1 + 0 = 1$*,* $1 + 1 = 1$*. Then* $(G, +)$ *is a commutative group.Define a new operation* $(\circ)$ *on the set* $G \times G$ *by* $(x_1, y_1) \circ (x_2, y_2) = (x_2 + y_2 - x_1, x_1 + y_1 - y_2)$*, for all* $x_1, y_1, x_2, y_2 \in G \times G$*. If we label the elements as follows* $(0, 0) \leftrightarrow 0$*,* $(0, 1) \leftrightarrow 1$*,* $(1, 0) \leftrightarrow 2$*,* $(1, 1) \leftrightarrow 3$*, then obtain:*

$(\circ)$	*0*	*1*	*2*	*3*
0	*0*	*3*	*2*	*1*
1	*1*	*2*	*3*	*0*
2	*3*	*0*	*1*	*2*
3	*2*	*1*	*0*	*3*

Then $(G \times G, \circ)$ *is a paramedial, non-medial, non-associative quasigroup.*

Theorem 2 *Let* $(G, +)$ *be a commutative group. The set* $G \times G$ *with the operation*

$$(x_1, y_1) \circ (x_2, y_2) = (-x_1 - y_1 + y_2, -x_2 - y_2 + x_1)$$

is a non-associative, non-paramedial, medial quasigroup, AD and AG−quasigroup.

Example 3 *Let* $G = \{0, 1, 2\}$*. We define the binary operation " + ".*

(+)	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then $(G, +)$ *is a commutative group. Define a binary operation* $(\circ)$ *on the set* $G \times G$ *by* $(x_1, y_1) \circ (x_2, y_2) = (-x_1 - y_1 + y_2, -x_2 - y_2 + x_1)$*, for all* $x_1, y_1, x_2, y_2 \in G \times G$ *. If we label the elements as follows* $(0, 0) \leftrightarrow 0$*,* $(0, 1) \leftrightarrow 1$*,* $(0, 2) \leftrightarrow 2$*,* $(1, 0) \leftrightarrow 3$*,* $(1, 1) \leftrightarrow 4$*,* $(1, 2) \leftrightarrow 5$*,* $(2, 0) \leftrightarrow 6$*,* $(2, 1) \leftrightarrow 7$*,* $(2, 2) \leftrightarrow 8$*, then obtain:*

(∘)	0	1	2	3	4	5	6	7	8
0	0	5	7	2	4	6	1	3	8
1	6	2	4	8	1	3	7	0	5
2	3	8	1	5	7	0	4	6	2
3	7	0	5	6	2	4	8	1	3
4	4	6	2	3	8	1	5	7	0
5	1	3	8	0	5	7	2	4	6
6	5	7	0	4	6	2	3	8	1
7	2	4	6	1	3	8	0	5	7
8	8	1	3	7	0	5	6	2	4

Then $(G \times G, \circ)$ *is a non-associative, non-paramedial, medial quasigroup, AD and AG−quasigroup.*

4 Conclusion

In this article we demonstrate a method of constructing various types of medial and paramedial quasigroups using the special direct product of commutative groups.

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Semireflexive subcategories and pairs of conjugate subcategories

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Abstract

We examine some relations between semireflexive subcategories, semireflexive product, the right product of two subcategories and pairs of conjugate subcategories.

Keywords: reflector and coreflector functor, left exact functor, semireflexive spaces, semireflexive subcategories, semireflexive product of two subcategories, the right product of two subcategories.

We shall consider the category $\mathcal{C}_2\mathcal{V}$ of locally convex topological vector Hausdorff spaces, defined on the field K of real or complex numbers (see [3]). The semireflexive product and semireflexive subcategories were defined and studied by the authors in [5]. The pairs of the conjugate subcategories were defined and studied by the first author (see [4]).

We shall use the following notations: $(\mathcal{E}_u, \mathcal{M}_p)$, where $\mathcal{E}_u$ is the universal epimorphism class and $\mathcal{M}_p$ is the class of exact monomorphisms);
Π – the subcategory of complete spaces with weak topology;
$\mathcal{S}$ – the subcategory of spaces with weak topology;
Γ_0 – the subcategory of complete spaces;
$q\Gamma_0$ – the subcategory of quasicomplete spaces;
$s\mathcal{R}$ – the subcategory of semireflexive spaces;
$\widetilde{\mathcal{M}}$ – the subcategory of the spaces with Mackey topology.

A factorization structure $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R}))$, where $P''(\mathcal{R}) = \mathcal{E}_p \circ (\varepsilon\mathcal{R})$, corresponds to any nonzero reflective subcategory $\mathcal{R}$ of the category $\mathcal{C}_2\mathcal{V}$ (see [4]).

The factorization structure $(\mathcal{E}_u, \mathcal{M}_p)$ divides the lattice $\mathbb{R}$ of the nonzero reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ into three classes $\mathbb{R}(\mathcal{E}_u) = \{\mathcal{R} \in \mathbb{R},\ \mathcal{R}$ is $\mathcal{E}_u$-reflective$\}$, $\mathbb{R}(\mathcal{M}_p) = \{\mathcal{R} \in \mathbb{R},\ \mathcal{R}$ is $\mathcal{M}_p$-reflective$\}$, $\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p) = (\mathbb{R} \setminus (\mathbb{R}(\mathcal{E}_u) \cup \mathbb{R}(\mathcal{M}_p))) \cup \{\mathcal{C}_2\mathcal{V}\}$.

Definition 1. ([5]). *The subcategory $\mathcal{R}$ of the class $\mathbb{R}(\mathcal{E}_u, \mathcal{M}_p)$ is called semireflexive if there exists a subcategory $\mathcal{B} \in \mathbb{R}(\mathcal{E}_u)$ and a subcategory $\Gamma \in \mathbb{R}(\mathcal{M}_p)$ such that $\mathcal{R} = \mathcal{B} \times_{sr} \Gamma$.*

Let $(\mathcal{K}, \mathcal{R})$ be a pair of conjugate subcategories, and $\Gamma \in \mathbb{R}(\mathcal{M}_p)$. We denote by $l\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$, where $\mathcal{L} = \mathcal{R} \cap \Gamma$, the reflector functor. Since the subcategory $\mathcal{R}$ is closed under the extension, it follows that $l = g \cdot r$, where $g\colon \mathcal{C}_2\mathcal{V} \longrightarrow \Gamma$ and $r\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ are reflector functors. Moreover, according to the Theorem 4.6 [5], the subcategory $\mathcal{T} = \mathcal{R} \times_{sr} \Gamma$ is reflective. Let $t\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{T}$ and $v\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{V}$, where $\mathcal{V} = \mathcal{K} \times_d \mathcal{L}$ be reflector functors. Obviously, $\mathcal{L} \subset \mathcal{T}$.

Theorem 1. *Using the notations given above, we have:*

1. $\mathcal{R} \times_{sr} \Gamma = \mathcal{K} \times_d (\mathcal{R} \cap \Gamma)$.

2. $v \cdot r = r \cdot v = l$.

3. $v \cdot k = k \cdot l \cdot k$.

Theorem 2. *Let $k, g\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{C}_2\mathcal{V}$ be two functors. The following statements are equivalent:*

1. $k \cdot g = g \cdot k$.

2. The subcategory $\mathcal{R} \times_{sr} \Gamma$ is $\mathcal{P}''(\Gamma)$-reflective.

Theorem 3. *The following statements are equivalent:*

1. $k(\varepsilon\mathcal{L}) \subset \mathcal{M}_p$.

2. The subcategory $\mathcal{R} \times_{sr} \Gamma$ is $\mathcal{M}_p$-reflective.

Definition 2. (see [4]). *Let $\mathcal{K}$ be a coreflective and $\mathcal{R}$ be a reflective, both nonzero, subcategories of the category $\mathcal{C}_2\mathcal{V}$, with the corresponding functors $k\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $r\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$. Then $(\mathcal{K}, \mathcal{R})$ is called a pair of conjugate subcategories if the following equalites hold:*

a) $k \cdot r = k$, b) $r \cdot k = r$.

In this case, $\mathcal{K}$ (respectively, $\mathcal{R}$) is called a c-coreflective, (respectively, $\mathcal{R}$ – c-reflective) subcategory.

Denote $\mu\mathcal{K} = \{m \in \mathcal{M}ono\,\mathcal{C}_2\mathcal{V} \mid k(m) \in \mathcal{I}so\}$ and $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi\,\mathcal{C}_2\mathcal{V} \mid r(e) \in \mathcal{I}so\}$.

Theorem 4. (see [4]). *For a pair $(\mathcal{K}, \mathcal{R})$ of subcategories of the category $\mathcal{C}_2\mathcal{V}$, where $\mathcal{K}$ is coreflective and $\mathcal{R}$ is reflective, the following statements are equivalent:*

1. $(\mathcal{K}, \mathcal{R})$ is a pair of conjugate subcategories.

2. $\mu\mathcal{K} = \varepsilon\mathcal{R}$.

Theorem 5. *The following statements are equivalent:*

1. $k(\varepsilon\mathcal{L}) \subset \mathcal{M}_p$.

2. The subcategory $\mathcal{R} \times_{sr} \Gamma$ is $\mathcal{M}_p$-reflective.

Proposition 1. *Let $(\mathcal{K}, \mathcal{R})$ be a pair of conjugate subcategories, Γ_1 and Γ_2 – two $\mathcal{M}_p$-reflective subcategories and $\Gamma_1 \subset \Gamma_2$. If $\mathcal{R} \times_{sr} \Gamma_1$ is a $\mathcal{M}_p$-reflective subcategory, then $\mathcal{R} \times_{sr} \Gamma_2$ is a semireflexive subcategory.*

Proposition 2. *Let $(\mathcal{K}, \mathcal{R})$ be a pair of conjugate subcategories, Γ_1 and Γ_2 – two $\mathcal{M}_p$-reflective subcategories and $\Gamma_1 \subset \Gamma_2$. If $\mathcal{R} \times_{sr} \Gamma_2$ is a semireflexive subcategory, then $\mathcal{R} \times_{sr} \Gamma_2$ is a semireflexive subcategory too.*

Proposition 3. *Let $(\mathcal{K}_1, \mathcal{R}_1)$ and $(\mathcal{K}_2, \mathcal{R}_2)$ be two pairs of conjugate subcategories, $\mathcal{R}_1 \subset \mathcal{R}_2$ and $\Gamma \in \mathbb{R}(\mathcal{M}_p)$. If $\mathcal{R}_2 \times_{sr} \Gamma$ is a semireflexive subcategory, then $\mathcal{R}_1 \times_{sr} \Gamma$ is a semireflexive subcategory too.*

Theorem 6. (see [4]). *For every nonzero reflective subcategory $\mathcal{R}$ of the category $\mathcal{C}_2\mathcal{V}$, the following statements are equivalent:*

1. $\mathcal{R}$ is c-reflective (see [2]).

2. $\mathcal{R}$ contains the subcategory of spaces with weak topology $\mathcal{S}$ and the reflector functor $r\colon \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ is left-exact (see [6]).

Theorem 7. *Let $(\mathcal{K}, \mathcal{R})$ be a pair of conjugate subcategories. If the subcategory $\mathcal{S}$ of a space with weak topology is not contained in the subcategory $\mathcal{K}$, then $\mathcal{R} \times_{sr} \Gamma_0$ is a semireflexive subcategory.*

Theorem 8. *Let $(\mathcal{K}, \mathcal{R})$ be a pair of conjugate subcategories and $\Gamma \in \mathbb{R}(\mathcal{M}_p)$. If there exists an object $X \in |\mathcal{S}|$, such that: $X \in |\mathcal{K}|$ and its Γ-replique coincides with Γ_0-replique, then $\mathcal{R} \times_{sr} \Gamma$ is a semireflexive* *subcategory.*

Example 1. *If* $(\widetilde{\mathcal{M}}, \mathcal{S})$ *is a pair of conjugate subcategories in the category* $\mathcal{C}_2\mathcal{V}$ *and* $\Pi = \mathcal{S} \cap \Gamma_0$ *then we have:*

$$\mathcal{S} \times_{sr} \Gamma_0 = \widetilde{\mathcal{M}} \times_d \Pi = \Pi.$$

Example 2. *The subcategory* $\mathcal{S}ch$ *of the Schwartz spaces is c-reflective. Let* $\mathcal{K}$ *be a coreflective subcategory of the category* $\mathcal{C}_2\mathcal{V}$*, such that* $(\mathcal{K}, \mathcal{S}ch)$ *is a pair of conjugate subcategories. Then*

$$\mathcal{S}ch \times_{sr} \Gamma_0 = i\mathcal{R} = \mathcal{K} \times_d (\mathcal{S}ch \cap \Gamma_0),$$

where $i\mathcal{R}$ *is a subcategory of an inductive semireflexive space* [1].

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Relative torsion theories

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Abstract

We formulate necessary and sufficient conditions for a pair of subcategories to form a relative torsion theory.

Key words: reflective and coreflective subcategories, relative torsion theories, the right and left product of two subcategories, locally convex spaces.

1 Introduction

In the category $\mathcal{C}_2\mathcal{V}$ of the vectorial topological locally convex Hausdorff spaces we examined the subcategories: Γ_0 – the subcategory of the complete spaces, $\mathcal{S}$ – the subcategory of the spaces with weak topology, $\widetilde{\mathcal{M}}$ – the subcategory of the spaces with the Mackey topology (see [4]); the classes of morphisms: $\mathcal{M}_u$ – the class of universal monomorphisms (see [2]), $\mathcal{E}pi$ – the class of epimorphisms; $\mathcal{M}ono$ – the class of monomorphisms, $\mathcal{I}so$ – the class of isomorphisms, if $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ (respectively: $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$) is a reflector functor (respectively: coreflector), then: $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \mid r(e) \in \mathcal{I}so\}$, $\mu\mathcal{K} = \{m \in \mathcal{M}ono \mid k(m) \in \mathcal{I}so\}$. The factorization structures $(\mathcal{E}_u, \mathcal{M}_p)$, $(\mathcal{E}'(\mathcal{K}), \mathcal{M}'(\mathcal{K}))$, $(\mathcal{P}''(\mathcal{R}), \mathcal{P}''(\mathcal{R}))$ are described in [2], the right and left product of two subcategories are described in [3].

2 The right and left product of two subcategories and the relative torsion theories

Definition 1 [1]. *Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ be a reflective subcategory of category $\mathcal{C}$. The pair $(\mathcal{K}, \mathcal{R})$ is called a relative* *torsion theory (RTT), i.e. relative to the subcategory $\mathcal{K} \cap \mathcal{R}$, if the functors $k : \mathcal{C} \to \mathcal{K}$ and $r : \mathcal{C} \to \mathcal{R}$ verify the following two relations:*

1. *The functors k and r commute: $k \cdot r = r \cdot k$;*
2. *For any object X of category $\mathcal{C}$ the square $r^X \cdot k^X = k^{rX} \cdot r^{kX}$ is pull-back and puschout, where $k^X : kX \to X, k^{rX} : krX \to rX$ are the $\mathcal{K}$-coreplique, and $r^X : X \to rX$ and $r^{kX} : kX \to rkX = krX$ are $\mathcal{R}$-replique of the respective objects.*

Remark 1. *In the abelien categories a torsion theory $(\mathcal{T}, \mathcal{F})$ is a RTT relative to the intersection $\mathcal{T} \cap \mathcal{F} = 0$.*

Theorem 1 ([1]). *Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ – a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$ and $\Gamma_0 \subset \mathcal{R}$. The pair $(\mathcal{K}, \mathcal{R})$ forms a RTT iff the coreflector functor $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and the reflector $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ commute: $k \cdot r = r \cdot k$.*

Remark 2. *Examples of RTT and coreflective and reflective functors that commute, can be found in* [1].

Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ be a reflective subcategory of category $\mathcal{C}_2\mathcal{V}$. We examine the following conditions:

(S) The subcategory $\mathcal{K}$ is closed with respect to $(\varepsilon\mathcal{R})$-factorobjects.

(D) The subcategory $\mathcal{R}$ is closed with respect to $(\mu\mathcal{K})$-subobjects.

Lemma 1. *The subcategory $\mathcal{R}$ has the property (D), if for any object (E, u) and every locally convex topology v with properties $u \leq v \leq k(u)$, where $(E, k(u))$ is $\mathcal{K}$-coreplique of object (E, u), the object (E, v) also belongs to subcategory $\mathcal{R}$.*

Lemma 2. *For the subcategories $\mathcal{K}$ and $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$ the following affirmations are equivalent:*

1. *$\mathcal{K} *_s \mathcal{R} = \mathcal{K}$.*
2. *The subcategory $\mathcal{K}$ satisfies the condition (S).*

If the subcategory $\widetilde{\mathcal{M}} \subset \mathcal{K}$, then the previous conditions are equivalent to the condition:

3. *The subcategory $\mathcal{K}$ is closed with respect to $\mathcal{P}''(\mathcal{R})$-factorobjects.*

Dual statement.

Lemma 3. *For the subcategories $\mathcal{K}$ and $\mathcal{R}$ of category $\mathcal{C}_2\mathcal{V}$ the following conditions are equivalent:*

1. $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

2. *The subcategory $\mathcal{R}$ satisfies the condition (D).*

If $\mathcal{S} \subset \mathcal{R}$, then the previous conditions are equivalent to the condition:

3. *The subcategory $\mathcal{R}$ is closed with respect to $\mathcal{M}'(\mathcal{K})$-subobjects.*

Theorem 2. *Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ – a reflective subcategory. The following statements are equivalent:*

1. *The pair $(\mathcal{K}, \mathcal{R})$ forms a RTT.*

2. a) *The functors k and r commute: $k \cdot r = r \cdot k$;*

b) $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$;

c) $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

3. a) *The functors k and r commute: $k \cdot r = r \cdot k$;*

b) *The subcategory $\mathcal{K}$ possesses the property (S);*

c) *The subcategory $\mathcal{R}$ possesses the property (D).*

If $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\mathcal{S} \subset \mathcal{R}$ then the previous conditions are equivalent to the following:

4. a) *The functors k and r commute: $k \cdot r = r \cdot k$;*

b) *The subcategory $\mathcal{K}$ is closed with respect to $\mathcal{P}''(\mathcal{R})$-factorobjects;*

c) *The subcategory $\mathcal{R}$ is closed with respect to $\mathcal{M}'(\mathcal{K})$-subobjects.*

Theorem 3. *Let it be $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\Gamma_0 \subset \mathcal{R}$. Then:*

1. *The subcategory $\mathcal{K}$ is closed with respect to $(\mathcal{E}pi \cap \mathcal{M}_p)$-factorobjects. In other words, the subcategory $\mathcal{K}$ is closed with respect to extensions.*

2. *The subcategory $\mathcal{R}$ is closed with respect to $(\mu\widetilde{\mathcal{M}})$-subobjects. In other words, if the locally convex spaces (E, t) belong to the subcategory $\mathcal{R}$, then the space E belongs to the subcategory $\mathcal{R}$ with every locally convex topology u stronger than t, but compatible with the same duality: $t \le u \le m(t)$, where $(E, m(t))$ is the $\widetilde{\mathcal{M}}$-coreplique of the object (E, t).*

Remark 3. 1. *For some subcategories $\mathcal{K}$ with the property $\widetilde{\mathcal{M}} \subset \mathcal{K}$, in particular, for the subcategory $\widetilde{\mathcal{M}}$, it is well known that they are closed with respect to extensions* ([4], Affirmation IV.3.5.).

2. *Every locally convex complete space* (E, t) *remains complete in any topology* u *stronger than* t *but compatible with the same duality:* $t \leq u \leq m(t)$ ([4], VI Corollary of Proposition 3).

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On a poset of extensions with first-countable remainder

Laurenţiu Calmuţchi

Abstract

In the present paper we study the poset of extensions with first-countable remainders. We prove that for a Lindelöf (or a paracompact with the non-measurable Lindelöf number) space that poset is cofinal.

Keywords: extension, poset.

1 Introduction

Every space is considered to be a completely regular T_1-space. Denote by βX the Stone-Čech compactification of a space X. If X is a dense subspace of a space Y, then Y is called an extension of X and the subspace $Y \setminus X$ is called a remainder of X. If Y and Z are extensions of X and there exists a continuous mapping $f : Y \longrightarrow Z$ such that $f(x) = x$ for any $x \in X$, then we put $Z \leq Y$. Hence the set $E(X)$ of all extensions of X is a poset (a partially ordered set).

An extension Y of a space X is called: an fc-extension if $Y \setminus X \neq \emptyset$ and the space Y has a countable base at each point $p \in Y \setminus X$; a one-point fc-extension if $Y \setminus X$ is a singleton subset of Y.

Denote by $E_{fc}(X)$ the poset of all fc-extensions of the space X and $E^n_{fc}(X) = \{Y \in E_{fc}(X) : |Y \setminus X| \leq n\}$. Hence $E^1_{fc}(X)$ is the poset of all one-point fc-extensions of the space X.

The present research was motivated initially by the Bel'nov's study of the poset $M(X)$ of metric extensions of a locally compact metric space X [2] and by the M.Henriksen, L.Janos and R. G. Woods's study of the poset $S(X)$ of one-point metric extensions of a locally compact metric space X [4].

All one-point fc-extensions of the spaces are constructed in [1, 4], using the notion of the extension traces. The following statement is obvious.

Proposition 1.1. *Let X be a non-empty space. The following assertions are equivalent:*

1. X is not a pseudocompact space.

2. $E_{fc}(X) \neq \emptyset$.

3. $E^1_{fc}(X) \neq \emptyset$.

2 Construction of extensions with first-countable remainder

A subset L of a space X is bounded in X if any continuous function f on X is bounded on L.

Let Y be an extension of X. A point $y \in Y \setminus X$ is of countable type if there exists a compact subset $F \subseteq Y \setminus X$ of the countable character in Y such that $y \in F$.

We say that the family $\mathcal{F}$ of functions is discrete at infinity if the following conditions hold: the family $\mathcal{F}$ is non empty and each function $f \in \mathcal{F}$ is unbounded on $f^{-1}(0, +\infty)$; the family $\{f^{-1}(0, +\infty) : f \in \mathcal{F}\}$ is discrete in X.

Theorem 2.1. *Let $\mathcal{F}$ be a discrete at infinity family of continuous functions on a space X. Then there exists an fc-extension $e_{\mathcal{F}}X$ such that:*

1. $e_{\mathcal{F}}X \setminus X = \{y_f : f \in \mathcal{F}\}$ is a discrete closed subspace of $e_{\mathcal{F}}X$.

2. For each $f \in \mathcal{F}$ there exists an open subset V_f of $e_{\mathcal{F}}X$ such that $V_f \cap (e_{\mathcal{F}}X \setminus X) = \cap\{cl_{e_{\mathcal{F}}X} f^{-1}(n, +\infty) : n \in \mathbb{N}\} = \{y_f\}$ and $V_f \cap X \subseteq f^{-1}(0, +\infty)$.

Proof. We can assume that $f \geq 0$ for each $f \in \mathcal{F}$. Let $a\mathbb{R} = \mathbb{R} \cup \{\infty\}$ be the one-point Alexandroff compactification of the reals. Consider the continuous extension $\beta f : \beta X \to a\mathbb{R}$ of the function f. Then $W_f = \beta f^{-1}(0, +\infty]$ is an open subset of βX. Obviously,

$W_f \cap W_g = \emptyset$ for distinct $f, g \in \mathcal{F}$. The set $H(f) = f^{-1}[2, +\infty)$ is closed and unbounded in X. By construction, $\Phi(f) = \beta f^{-1}(\infty)$ is a non-empty compact subset of $e_{\mathcal{F}}X \setminus X$ and of countable character in $e_{\mathcal{F}}X$. We put $Z = X \cup \cup\{\Phi(f) : f \in \mathcal{F}\}$ as a subspace of βX. Then Z is an extension of the space X and $\{\Phi(f) : f \in \mathcal{F}\}$ is a cover of open and compact subsets of the space $Z \setminus X$. Moreover, that family is discrete in Z. Now, each set $\Phi(f)$ we identify in a singleton set and obtain the set $e_{\mathcal{F}}X$ and the mapping $p : Z \longrightarrow e_{\mathcal{F}}X$ with the properties: $p(x) = x$ for each $x \in X$; $p(\Phi(f))$ is a singleton set and $p^{-1}(p(\Phi(f))) = \Phi(f)$ for each $f \in \mathcal{F}$. On $e_{\mathcal{F}}X$ consider the quotient topology. Then p is a perfect mapping, $e_{\mathcal{F}}X$ is an extension of X and the space $e_{\mathcal{F}}X$ has a countable base at the point $y_f = p(\Phi(f))$. The set $\{y_f : f \in \mathcal{F}\}$ is discrete in $e_{\mathcal{F}}X$. The proof is complete.

3 On cofinal families of extensions

The family $\{Y_\alpha : \alpha \in A\}$ of extensions of a space X is cofinal if for any compactification bY_α of Y_α, $\alpha \in A$, we have $\beta X = sup\{bY_\alpha : \alpha \in A\}$.

We say that the space X is *pure non-pseudocompact* if X is not compact and has the properties: if $y \in \beta X \setminus X$, then either $y \in cl_{\beta X} L$ for some bounded subset L of X, or there exists a G_δ-subset H of βX such that $y \in H \subseteq \beta X \setminus X$; if F and L are non-compact closed bounded subsets of X, then $cl_{\beta X} L \cap cl_{\beta X} H \neq \emptyset$.

Let νX be the real-compactification of X. The space X is pure non-pseudocompact if and only if $|\nu X \setminus X| \leq 1$ and $\nu X \neq \beta X$. Any non-compact realy-compact space is pure non-pseudocompact. In particular, any Lindelöf non-compact space (or a Dieudonné complete space with the non-measurable Lindelöf number) is pure non-pseudocompact.

Theorem 3.1. *For a space X the following assertions are equivalent:*

1. *X is a pure non-pseudocompact space.*
2. *The family of extensions $E_{fc}(X) \neq \emptyset$ is cofinal.*
3. *The family of extensions $E^1_{fc}(X) \neq \emptyset$ is cofinal.*

Proof. If $\nu X \setminus X$ contains two distinct points a, b, then we identify

a, b in βX and obtain the compactification cX. For any $Y \in E_{fc}(X)$ there exists a compactification bY such that $bY \leq cX$. This fact proves the implications $3 \rightarrow 2 \rightarrow 1$.

Let X be a pure non-pseudocompact space. If $a \in \beta X \setminus \nu X$ and the set W is open in βX, then for Theorem 2.1 it follows that there exists $Y \in E^1_{fc}(X)$ such that for each compactification bY of Y we have $h^{-1}(Y \setminus X) \subseteq W$, where $h : \beta X \longrightarrow bY$ is the continuous mapping with $h(x) = x$ for each $x \in X$. From this fact we have implication $1 \rightarrow 3$. The proof is complete.

Remark 3.2. *If* $\nu X \neq \beta X$, *then* $\beta X = sup\{\beta Y : Y \in E^1_{fc}(X)\}$.

Example 3.3. Let X be a discrete space and X is not realcompact, i.e. $|X|$ is a measurable cardinal and on the algebra of all subsets of X there exists a σ-additive measure m with $m(X) = 1$. Then $\beta X \neq \nu X \neq \mu X = X$, $\beta X = sup\{\beta Y : Y \in E^1_{fc}(X)\} = sup\{\beta Y : Y \in E_{fc}(X)\}$ and the family of extensions $E_{fc}(X) \neq \emptyset$ is not cofinal.

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On the holomorph of π-quasigroups of type T_1

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Abstract

Quasigroups satisfying the identity $x\cdot(x\cdot(x\cdot y)) = y$ are called π-quasigroups of type T_1. Necessary and sufficient conditions for the holomorph of a π-quasigroup of type T_1 to be a π-quasigroup of type T_1 are established. Also, it is proved that the left (right) multiplication group of a π-quasigroup of type T_1 is isomorphic to some normal subgroup of the left (right) multiplication group of its holomorph, respectively.

Keywords: π-quasigroup of type T_1, holomorph, multiplication group, normal subgroup.

Quasigroups satisfying identities of length five, with two variables, are called π-quasigroups. V. Belousov [1] and, independently F. Bennett [2], have given a classification of minimal identities, consisting of seven identities. The general form of an identity of length five with two variables in a quasigroup (Q, A) is:

$$^{\alpha}A(x,^{\beta}A(x,^{\gamma}A(x,y))) = y, \tag{1}$$

where $\alpha, \beta, \gamma \in S_3$. In this case, the tuple $[\alpha, \beta, \gamma]$ is called the type of the identity (1), and the quasigroup (Q, A) which satisfies (1), is called a π-quasigroup of type $T = [\alpha, \beta, \gamma]$. Using the notations from [1], a quasigroup $(Q, \cdot)$ which satisfies the identity

$$x \cdot (x \cdot (x \cdot y)) = y \tag{2}$$

is called a π-quasigroup of type $T_1 = [\varepsilon, \varepsilon, \varepsilon]$, where ε is the identity of S_Q. If $(Q, \cdot)$ is a quasigroup, then the groupoid $(Hol(Q, \cdot), \circ)$, where $Hol(Q, \cdot) = Aut(Q, \cdot) \times Q$ and

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, \beta(x) \cdot y),$$

for $\forall(\alpha, x),(\beta, y) \in Hol(Q,\cdot)$, is called the holomorph of the quasigroup $(Q,\cdot)$ [3]. From the definition it follows that the holomorph of a quasigroup is a quasigroup. Moreover, the mapping $Q \mapsto Hol(Q,\cdot), x \mapsto (\varepsilon, x)$, is an embedding of the quasigroup $(Q,\cdot)$ into its holomorph $(Hol(Q,\cdot),\circ)$. Thus, denoting $Q_1 = \{(\varepsilon, x)|x \in Q\}$, we obtain $(Q,\cdot) \cong (Q_1,\circ)$. If the quasigroup $(Q,\cdot)$ has a right or a left unit, then the mapping $Aut(Q,\cdot) \mapsto Hol(Q,\cdot), \alpha \mapsto (\alpha, e)$ is an embedding, too. In particular, if $(Q,\cdot)$ is a group with an abelian group of automorphisms, then $(Hol(Q,\cdot),\circ)$ is a group, $(Q_1,\circ) \lhd Hol(Q,\cdot)$, where $Q_1 = \{(\varepsilon, x)|x \in Q\}$, and $Hol(Q,\cdot)/(Q_1,\circ) \cong Aut(Q,\cdot)$. This isomorphism is given by the surjection:

$$\xi : Hol(Q,\cdot) \to Aut(Q,\cdot), \xi(\varphi, x) = \varphi,$$

for which $Ker\xi = (Q_1,\circ)$.

Proposition 1. *The holomorph of a $\pi-$quasigroup of type T_1 is a $\pi-$quasigroup of type T_1, if and only if the following conditions hold:*
1)$\alpha^3 = \varepsilon, \forall\alpha \in Aut(Q,\cdot)$;
2)$x \cdot (\alpha^2(x) \cdot (\alpha(x) \cdot y)) = y, \forall\alpha \in Aut(Q,\cdot), \forall x, y \in Q$.

Proof. If $(Q,\cdot)$ is a $\pi-$quasigroup of type T_1 and $(\alpha, x),(\beta, y) \in Hol(Q,\cdot)$, then: $(\alpha, x) \circ ((\alpha, x) \circ ((\alpha, x) \circ (\beta, y))) = (\alpha, x) \circ ((\alpha, x) \circ (\alpha\beta, \beta(x) \cdot y)) = (\alpha, x) \circ (\alpha^2\beta, \alpha\beta(x) \cdot (\beta(x) \cdot y)) = (\alpha^3\beta, \alpha^2\beta(x) \cdot (\alpha\beta(x) \cdot (\beta(x) \cdot y))) = (\beta, y)$, hence:

$$\begin{cases} \alpha^3 = \varepsilon \\ \alpha^2\beta(x) \cdot (\alpha\beta(x) \cdot (\beta(x) \cdot y)) = y. \end{cases}$$

Making the replacement $x \mapsto \beta^{-1}\alpha(x)$, and using the equality $\alpha^3 = \varepsilon$, the second relation implies:

$$x \cdot (\alpha^2(x) \cdot (\alpha(x) \cdot y)) = y,$$

for $\forall x, y \in Q, \forall\alpha \in Aut(Q,\cdot)$.

$\square$

Remark 1. The holomorph of a $\pi-$quasigroup of type T_1 with a trivial group of automorphisms is a $\pi-$quasigroup of type T_1. This fact is used below to obtain $\pi-$quasigroups of type T_1, the holomorphs of which are from the same class.

Example. The quasigroup $(Q,\cdot)$, where $Q=\{1,2,3\}$, given by the left translations $L_1=(132)$, $L_2=(123)$, $L_3=\varepsilon$, is a $\pi-$quasigroup of type T_1 with $Aut(Q,\cdot)=\{\varepsilon\}$, so its holomorph $Hol(Q,\cdot)$ is a $\pi-$quasigroup of type T_1 as well.

Remark 2. 1. If $L^{(\circ)}_{(\beta,b)}$ and $R^{(\circ)}_{(\beta,b)}$ are the left and, respectively, the right translations with (β,b) in the holomorph $(Hol(Q,\cdot),\circ)$, then, for every $(\alpha,a)\in(Hol(Q,\cdot)$, we have:

$$L^{(\circ)-1}_{(\beta,b)}(\alpha,a)=(\beta^{-1}\alpha,\beta^{-1}\alpha(b)\backslash a)=L^{(\circ)}_{(\beta^{-1},b_1)}(\alpha,a),$$

where $\alpha(b_1)\cdot a=\beta^{-1}\alpha(b)\backslash a)$ and, respectively,

$$R^{(\circ)-1}_{(\beta,b)}(\alpha,a)=(\alpha\beta^{-1},\beta^{-1}(a/b))=R^{(\circ)}_{(\beta^{-1},b_2)}(\alpha,a),$$

where $\alpha(b_2)=\beta^{-1}(a/b)$.

2. Let $(Q,\cdot)$ be a finite $\pi-$quasigroup of type T_1. If $(Hol(Q,\cdot),\circ)$ is a $\pi-$quasigroup of type T_1, then there exists a positive integer k such that $|Aut(Q,\cdot)|=3^k$ and $|Hol(Q,\cdot)|\equiv 0(mod\ 3)$.

Proposition 2. *Let $(Q,\cdot)$ be a $\pi-$quasigroup of type T_1 and let $Q_1=\{(\varepsilon,x)|x\in Q\}$. Then $(Q,\cdot)\cong(Q_1,\circ)$ and the following statements hold:* 1)$LM(Q_1,\circ)\lhd LM(Hol(Q,\cdot),\circ)$; 2)$RM(Q_1,\circ)\lhd RM(Hol(Q,\cdot),\circ)$.

Proof. 1) $(Q,\cdot)\cong(Q_1,\circ)$ implies $LM(Q,\cdot)\cong LM(Q_1,\circ)$. Moreover, $LM(Q_1,\circ)$ is a subgroup of $LM(Hol(Q,\cdot),\circ)$. Now, let $L^{(\circ)}_{(\varepsilon,x)}\in LM(Q_1,\circ)$ and $L^{(\circ)}_{(\beta,b)}\in LM(Hol(Q,\cdot),\circ)$, then:

$$L^{(\circ)}_{(\beta,b)}L^{(\circ)}_{(\varepsilon,x)}L^{(\circ)-1}_{(\beta,b)}(\alpha,a)=L^{(\circ)}_{(\beta,b)}L^{(\circ)}_{(\varepsilon,x)}(\beta^{-1}\alpha,\beta^{-1}\alpha(b)\cdot a)=$$

$$L^{(\circ)}_{(\beta,b)}(\beta^{-1}\alpha,\beta^{-1}\alpha(x)\cdot(\beta^{-1}\alpha(b)\backslash a))=$$

$$(\alpha, \beta^{-1}\alpha(b) \cdot (\beta^{-1}\alpha(x) \cdot (\beta^{-1}\alpha(b)\backslash a))) = L^{(\circ)}_{(\varepsilon,c)}(\alpha, a),$$

for $\forall(\alpha, a) \in Hol(Q, \cdot)$, where $\alpha(c) \cdot a = \beta^{-1}\alpha(b) \cdot (\beta^{-1}\alpha(x) \cdot (\beta^{-1}\alpha(b)\backslash a))$. Thus,

$$L^{(\circ)}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)} L^{(\circ)-1}_{(\beta,b)} \in LM(Q_1, \circ). \quad (3)$$

Analogously,

$$L^{(\circ)-1}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)} L^{(\circ)}_{(\beta,b)}(\alpha, a) = L^{(\circ)-1}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)}(\beta\alpha, \alpha(b) \cdot a) =$$

$$L^{(\circ)-1}_{(\beta,b)}(\beta\alpha, \beta\alpha(x) \cdot (\alpha(b) \cdot a)) =$$

$$(\alpha, \alpha(b)\backslash(\beta\alpha(x) \cdot (\alpha(b) \cdot a))) = L^{(\circ)}_{(\varepsilon,g)}(\alpha, a),$$

for $\forall(\alpha, a) \in Hol(Q, \cdot)$, where $\alpha(g) \cdot a = \alpha(b)\backslash(\beta\alpha(x) \cdot (\alpha(b) \cdot a))$. So,

$$L^{(\circ)-1}_{(\beta,b)} L^{(\circ)}_{(\varepsilon,x)} L^{(\circ)}_{(\beta,b)} \in LM(Q_1, \circ). \quad (4)$$

(3) and (4) imply that $LM(Q_1, \circ) \triangleleft LM(Hol(Q, \cdot), \circ)$. The proof of the second relation is similar. $\square$

Remark 3. The function $\xi : LM(Hol(Q, \cdot)) \mapsto Aut(Q, \cdot)$, $\xi(L^{\delta_1}_{(\alpha_1,x_1)} L^{\delta_2}_{(\alpha_2,x_2)}, ..., L^{\delta_n}_{(\alpha_n,x_n)}) = \alpha_1^{\delta_1}\alpha_2^{\delta_2}...\alpha_n^{\delta_n}$, where $\delta_i = 1$ or -1, for every $i = 1, 2, ..., n$, is a surjective homomorphism with $Ker\xi = LM(Q_1, \circ)$, so $LM(Hol(Q, \cdot))/LM(Q_1, \circ) \cong Aut(Q, \cdot)$.

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Coloring hyperplanes of CAT(0) cube complexes

Victor Chepoi, Mark Hagen

Abstract

In these notes, we briefly describe the results which we will present in the talk *Coloring hyperplanes of CAT(0) cube complexes.* They are based on the papers [4] and [5].

Keywords: geometry, combinatorics, CAT(0) cube complexes, isometric embedding, hyperplanes.

1 CAT(0) cube complexes

In his seminal paper [7], among many other results, Gromov gave a nice combinatorial characterization of *CAT(0) cube complexes* as simply connected cube complexes in which the links of 0-cubes are *simplicial flag complexes.* Subsequently, Sageev [9] introduced and investigated the concept of *hyperplanes* of CAT(0) cube complexes, showing in particular that each hyperplane is itself a CAT(0) cube complex and divides the complex into two CAT(0) cube complexes. These two results identify CAT(0) cube complexes as the basic objects in geometric group theory. For instance, many well-known classes of groups are known to act nicely on CAT(0) cube complexes. On the other hand, [3] established that the 1-skeleta of CAT(0) cube complexes are exactly the median graphs, i.e. the graphs in which any triplet of vertices admit a unique median vertex. Median graphs and related median structures have been investigated in several contexts by quite a number of authors for more than half a century. They have many nice properties and admit numerous characterizations relating them to other discrete structures. Barthélemy and Constantin [1] showed that pointed median graphs are exactly the domains of event structures with binary conflict (investigated in computer science in concurrency theory [10, 8]).

All CAT(0) cube complexes $\mathbf{X}$ and median graphs – the 1-skeleta $G(\mathbf{X})$ of $\mathbf{X}$ – are intimately related to hypercubes: they are constituted of cubes and themselves embed isometrically into hypercubes. The minimum dimension of a hypercube into which $G(\mathbf{X})$ (or $\mathbf{X}$) isometrically embeds equals the number of hyperplanes of $\mathbf{X}$, or, equivalently, the number of equivalence classes of the transitive closure of the "opposite" relation of edges of $G(\mathbf{X})$ on 2-cubes of $\mathbf{X}$. While the dimension of the smallest hypercube into which the median graph $G(\mathbf{X})$ embeds is easy to determine, the problem of determining the least number $\tau(\mathbf{X}) = \tau(G(\mathbf{X}))$ of tree factors necessary for an isometric embedding of the 1-skeleton of $\mathbf{X}$ into a cartesian product of trees is hard. For arbitrary CAT(0) cube complexes $\mathbf{X}$, the value $\tau(\mathbf{X})$ is closely related to the chromatic number of the so-called *crossing graph* $\Gamma_{\#}(\mathbf{X})$ of $\mathbf{X}$. $\Gamma_{\#}(\mathbf{X})$ can be viewed as the intersection graph of the hyperplanes of $\mathbf{X}$: its vertices are the hyperplanes of $\mathbf{X}$ sensu [9] and two hyperplanes are adjacent in $\Gamma_{\#}(\mathbf{X})$ iff they cross (or, equivalently, they intersect).

Extending the fact that $\tau(\kappa(G)) = \chi(G)$, it can be shown that the equality $\tau(\mathbf{X}) = \chi(\Gamma_{\#}(\mathbf{X}))$ holds for all CAT(0) cube complexes $\mathbf{X}$. Since an arbitrary graph can be realized as the crossing graph of a CAT(0) cube complex $\mathbf{X}$, to better capture the structure of $\mathbf{X}$, the concept of the *contact graph* $\Gamma(\mathbf{X})$ of $\mathbf{X}$ introduced in [6] is useful: the vertices of $\Gamma(\mathbf{X})$ are the hyperplanes of $\mathbf{X}$ and two hyperplanes are adjacent in $\Gamma(\mathbf{X})$ iff they cross or osculate (i.e., their carriers touch each other). $\Gamma(\mathbf{X})$ can be also viewed as the intersection graph of the carriers of the hyperplanes of $\mathbf{X}$. The clique number $\omega(\Gamma(\mathbf{X}))$ of the contact graph of $\mathbf{X}$ is exactly the maximum degree in $G(\mathbf{X})$ of a 0-cube of $\mathbf{X}$, i.e., to the maximum number of 1-cubes incident to a 0-cube of $\mathbf{X}$. The contact graph $\Gamma(\mathbf{X})$ always contains the crossing graph $\Gamma_{\#}(\mathbf{X})$. $\Gamma(\mathbf{X})$ also hosts the *pointed contact graph* $\Gamma_{\alpha}(\mathbf{X})$ of the 1-skeleton $G_{\alpha}(\mathbf{X})$ of $\mathbf{X}$ pointed at arbitrary vertex α. The graph $\Gamma_{\alpha}(\mathbf{X})$ has hyperplanes of $\mathbf{X}$ as vertices and two hyperplanes H, H' are adjacent in $\Gamma_{\alpha}(\mathbf{X})$ if and only if they are adjacent in $\Gamma(\mathbf{X})$ and two incident 1-cubes, one crossed by H and another crossed by H', are directed away from the common origin.

Pairwise-independently, F. Haglund, G. Niblo, M. Sageev, and the first author of these notes asked the following question:

Question 1. *Is it true that all CAT(0) cube complexes* $\mathbf{X}$ *with uniformly bounded degrees can be isometrically embedded into a finite number of trees?*

2 Event structures

Question 1 is closely related with the conjecture of Rozoy and Thiagarajan [8] (also called the *nice labeling problem*) asserting that:

Question 2. *Any event structure with finite (out)degree admits a labeling with a finite number of labels.*

An *event structure* is a triple $\mathcal{E} = (E, \leq, \smile)$, where E is a set of *events*, $\leq$ is a partial order on E, called *causal dependency*, and $\smile$ is a symmetric, irreflexive binary relation on E called *conflict*. For all e, e', e'', if $e \smile e'$ and $e' \leq e''$, then $e \smile e''$. The events e and e' are *concurrent* if they are incomparable in the partial ordering $\leq$ and $e \not\smile e'$. The events e and e' are *independent* if they are either concurrent or in minimal conflict. An *independent set* is a set of pairwise independent events in E. The *degree* of E is the maximum cardinality of an independent set in E. In [8], Rozoy and Thiagarajan formulated the *nice labeling problem* for event structures (Question 2). A *labeling* is a map $\lambda : E \to \Lambda$, where Λ is some alphabet, and λ is a *nice labeling* if $\lambda(e) \neq \lambda(e')$ whenever e and e' are independent. Solving the nice labeling problem for $\mathcal{E}$ entails constructing a nice labeling λ such that Λ is finite. The *domain* $\mathcal{D}(\mathcal{E})$ of the event structure $\mathcal{E}$ is defined as follows. A *configuration* C is a subset $C \subseteq E$ of the set of events such that no two elements of C are in conflict, and, if $e \leq e' \in C$ are not in conflict, then $e \in C$. The domain $\mathcal{D}(\mathcal{E})$ is the set of all such configurations C, ordered by inclusion. This construction naturally gives rise to a median graph and an accompanying CAT(0) cube complex associated to $\mathcal{E}$. Indeed, let

$G = G(\mathcal{E})$ be the graph whose vertices are the elements of the domain $\mathcal{D}(\mathcal{C})$, with C and C' joined by an edge if and only if $C = C' \cup \{e\}$ for some $e \in E - C$. In this situation, the edge $C'C$ is directed from C' to C. In other words, an event $e \in E$ is viewed as a minimal change from one configuration to another [10].

As noted above, pointed median graphs are exactly the domains of event structures [1]. Then, in view of the bijection between median graphs and 1-skeleta of CAT(0) cube complexes, the nice labeling problem for such event structures can be equivalently viewed as the colouring problem of the pointed contact graph $\Gamma_\alpha(\mathbf{X})$ of the CAT(0) cube complex $\mathbf{X}$ associated to the domain of the event structure. Since $\chi(\Gamma_\alpha(\mathbf{X})) \leq \chi(\Gamma(\mathbf{X}))$ and $\chi(\Gamma_\#(\mathbf{X})) \leq \chi(\Gamma(\mathbf{X}))$, in relation with Questions 1 and 2, the following question is natural:

Question 3. *Is it true that the chromatic number $\chi(\Gamma(\mathbf{X}))$ of the contact graph of a CAT(0) cube complex $\mathbf{X}$ of degree Δ can be bounded by a function ϵ of Δ?*

3 Results

Since $\omega(\Gamma(\mathbf{X})) = \Delta$ and $\Gamma_\#(\mathbf{X}), \Gamma_\alpha(\mathbf{X})$ are subgraphs of $\Gamma(\mathbf{X})$, all three questions can be reformulated, namely: which of the classes of graphs $\Gamma_\#(\mathbf{X}), \Gamma_\alpha(\mathbf{X})$, and $\Gamma(\mathbf{X})$ are χ-bounded? A class $\mathcal{C}$ of graphs is called χ-*bounded* if there exists a function f such that $\chi(G) \leq f(\omega(G))$ for any graph G of $\mathcal{C}$. Via a series of nontrivial examples, Burling [2] showed that the class of intersection graphs of axis-parallel boxes of $\mathbb{R}^3$ is not χ-bounded. Based on Burling's examples, it was recently shown in [4] that for CAT(0) cube complexes the classes of graphs $\Gamma(\mathbf{X})$ and $\Gamma_\alpha(\mathbf{X})$ are not χ-bounded, thus disproving the nice labeling conjecture of [8]:

Theorem 1. *[4] There exists a pointed median graph $\widetilde{G}^*_\alpha$ of maximum out-degree 5 such that the chromatic number of its pointed contact graph $\Gamma(\widetilde{G}^*_\alpha)$ is infinite. In particular, any nice labeling of the event structure $\mathcal{E}_\alpha$ (of degree 5) whose domain is $\widetilde{G}^*_\alpha$, requires an infinite number of labels.*

We adapted this counterexample by using the recubulation technique from [6] to show that the class of crossing graphs $\Gamma_{\#}(\mathbf{X})$ of CAT(0) cube complexes is also not χ-bounded, thus answering in the negative the first open question:

Theorem 2. *[5] For any $n > 0$, there exists a CAT(0) cube complex $\mathbf{X}_n$ with constant maximum degree such that any colouring of the crossing graph of $\mathbf{X}_n$ requires more than n colours, i.e., any isometric embedding of $\mathbf{X}_n$ into a Cartesian product of trees requires $> n$ trees. There exists an infinite CAT(0) cube complex $\mathbf{X}$ with constant maximum degree which cannot be isometrically embedded into a Cartesian product of a finite number of trees, i.e., the chromatic number of its crossing graph is infinite.*

On the other hand, and this is the main contribution of [5], we show that in the case of 2-dimensional CAT(0) cube complexes $\mathbf{X}$ the contact graphs $\Gamma(\mathbf{X})$ (and therefore the crossing and the pointed contact graphs) are χ-bounded by a polynomial function in $\omega(\Gamma(\mathbf{X})) = \Delta$, thus showing that in the 2-dimensional case the three questions have positive answers; this is the content of our main result:

Theorem 3. *[5] Let $\mathbf{X}$ be a 2-dimensional CAT(0) cube complex such that the degrees of all its vertices are bounded by Δ. Then there exists $M < \infty$, independent of $\mathbf{X}$, such that $\chi(\Gamma(\mathbf{X})) \leq \epsilon(\Delta) = M\Delta^{26}$. In particular, $\tau(\mathbf{X}) \leq \epsilon(\Delta)$, i.e. the 1-skeleton of $\mathbf{X}$ isometrically embeds into the Cartesian product of at most $\epsilon(\Delta)$ trees. Finally, all event structures of (out)degree Δ_0, whose domains are 2-dimensional, admit a nice labeling with at most $\epsilon(\Delta_0 + 2)$ labels.*

We actually obtain the following bound: $\chi(\Gamma(\mathbf{X})) \leq \epsilon(\Delta) = 1165226\Delta^{26}$, or, simply $M = 1165226$.

Idea of the proof: To show that the chromatic number $\chi(\Gamma(\mathbf{X}))$ of the contact graph $\Gamma(\mathbf{X})$ is polynomially bounded in Δ, we show that the edges of $\Gamma(\mathbf{X})$ can be distributed over six spanning subgraphs of $\Gamma(\mathbf{X})$, such that the chromatic numbers of each of these subgraphs can be polynomially bounded. As a result, each vertex of $\Gamma(\mathbf{X})$ (hyperplane of $\mathbf{X}$) receives a sextuple of colours, each colour corresponding to the colour received by this vertex in the colouring of the corresponding subgraph. Since each edge of $\Gamma(\mathbf{X})$ is present in at least one spanning subgraph, the sextuple-colouring of the hyperplanes of $\mathbf{X}$ is a correct colouring of the contact graph $\Gamma(\mathbf{X})$. The number of colours is the product of the six numbers of colours used to colour the spanning subgraphs, whence it is polynomial in Δ. In Sections 4-6, one after another, we will define and colour the six spanning subgraphs. For this, we will study the geometrical and the combinatorial properties of contact graphs of 2-dimensional CAT(0) cube complexes.

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Study on properties of non-isomorphic finite quasigroups using the computer

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Abstract

In this paper we elaborate mathematical algorithms for generating non-isomorphic finite quasigroups using the computer. We study some algebraic properties of non-isomorphic quasigroups.

Keywords: Non-isomorphic TS-quasigroups, AG-quasigroups and AD-quasigroups.

1 Introduction

Our main goal is to elaborate algorithms for generating non-isomorphic finite TS-quasigroups, AG-quasigroups and AD-quasigroups using the computer. The results established here are related to the work in ([1,2,3,4]).

2 Basic notions

A non-empty set G is said to be a *groupoid* relatively to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

A groupoid $(G, \cdot)$ is called a *quasigroup* if for every $a, b \in G$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions.

A quasigroup $(G, \cdot)$ is called an *Abel-Grassmann quasigroup* or an *AG-quasigroup* if it satisfies the left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.

A quasigroup $(G, \cdot)$ is called *AD-quasigroup* if it satisfies the law $a \cdot (b \cdot c) = c \cdot (b \cdot a)$ for all $a, b, c \in G$.

A quasigroup $(G, \cdot)$ is called *TS-quasigroup* if it satisfies the laws $a \cdot (a \cdot b) = b$ and $a \cdot b = b \cdot a$ for all $a, b \in G$.

Let $(G, +)$ be a groupoid and let $n, m \geq 1$. The element e of the groupoid $(G, +)$ is called:

- an (n, m)-zero of G if $e + e = e$ and $n(e, x) = (x, e)m = x$ for every $x \in G$;

- an $(n, \propto)$-zero if $e + e = e$ and $n(e, x) = x$ for every $x \in G$;

- an $(\propto, m)$-zero if $e + e = e$ and $(x, e)m = x$ for every $x \in G$.

Clearly, if $e \in G$ is both an $(n, \propto)$-zero and an $(\propto, m)$-zero, then it is also an (n, m)-zero. If $(G, \cdot)$ is a multiplicative groupoid, then the element e is called an (n, m)-identity.

The notion of the $(n, m)-$identity was introduced by M. Choban and L. Chiriac in [2].

We consider the following problems:

Problem 1. How many non-isomorphic TS-quasigroups, AG-quasigroups and AD-quasigroups of order 3, 4, 5 do there exist?

Problem 2. How many types of $(n, m)-$identities of the non-isomorphic TS-quasigroups, AG-quasigroups and AD-quasigroups of order 3, 4, 5 do there exist?

3 Main Results

Applying the elaborated algorithms, we prove the following results:

Theorem 1. *There are exactly 2 non-isomorphic TS-quasigroups of order 3 such that:*

- 1 of them is a quasigroup where each element is $(2, 2)-$identity;

- 1 of them is a quasigroup which does not contain multiple identities.

Theorem 2. *There are exactly 2 non-isomorphic TS-quasigroups of order 4 such that:*

- 1 of them is an associative quasigroup;

- 1 of them is a quasigroup which contains one $(1, 2)-$identity.

Theorem 3. *There is exactly one TS-quasigroup of order 5, which contains one* $(2,2)-$ *identity.*

Theorem 4. *There are exactly 2 non-isomorphic AG-quasigroups of order 3 such that:*
- 1 of them is an associative quasigroup;
- 1 of them is a quasigroup which contains one $(1,2)-$*identity.*

Theorem 5. *There are exactly 6 non-isomorphic AG-quasigroups of order 4 such that:*
- 2 of them are associative quasigroups;
- 2 of them are quasigroups which contain one $(1,2)-$*identity;*
- 1 of them is a quasigroup where each element is $(3,3)-$*identity;*
- 1 of them is a quasigroup which does not contain $(n,m)-$*identities.*

Theorem 6. *There are exactly 5 non-isomorphic AG-quasigroups of order 5 such that:*
- 1 of them is an associative quasigroup;
- 1 of them is a quasigroup which contains one $(1,2)-$*identity;*
- 1 of them is a quasigroup, every element of which is $(2,4)-$*identity;*
- 1 of them is a quasigroup which contains one $(2,4)-$*identity;*
- 1 of them is a quasigroup which does not contain $(n,m)-$*identities.*

Theorem 7. *There are exactly 2 non-isomorphic AD-quasigroups of order 3 such that:*
- 1 of them is an associative quasigroup;
- 1 of them is a quasigroup which contains one $(2,1)-$ *identity.*

Theorem 8. *There are exactly 6 non-isomorphic AD-quasigroups of order 4 such that:*
- 2 are associative quasigroups;
- 2 are quasigroups which contain one $(2,1)-$*identity;*
- 1 is quasigroup where each element is $(3,3)-$*identity;*

- 1 is quasigroup which does not contain $(n, m)-$*identities.*

Theorem 9. *There are exactly 5 non-isomorphic AD-quasigroups of order 5 such that:*
- 1 of them is an associative quasigroup;
- 1 of them is a quasigroup which contains one $(2, 1)-$*identity;*
- 1 of them is a quasigroup, every element of which is $(4, 2)-$*identity;*
- 1 of them is a quasigroup which contains one $(4, 2)-$*identity;*
- 1 of them is a quasigroup which does not contain $(n, m)-$*identities.*

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Families of strictly p-subspaces

Mitrofan Choban, Ekaterina Mihaylova

Abstract

In the present paper we introduce and study special families of spaces and, in particular, the following notions: the notion of a family of p-subspaces; the notion of a family of strictly (complete) p-subspaces.

Keywords: metrizable families, p-space.

1 Introduction. Main results

Every space is considered to be a T_1-space.

In the present article we examine the collective conditions of location of a family of subspaces in topological spaces. We continue the investigations from [1, 2].

We say that a property $\mathcal{P}$ of spaces is of the compact type if: any space with property $\mathcal{P}$ is countably compact and the property $\mathcal{P}$ is hereditary with respect to closed subspaces. Consider the next properties of the compact type: s – the property to be a countably compact space; k – the property to be a compact space; Ω – the property to be a one point space.

Fix a property $\mathcal{P}$ of the compact type. Let d be a pseudometric on a space X. For all $x \in X$ and $\epsilon > 0$ we put: $V(x,d,\epsilon) = \{y \in X : d(x,y) < \epsilon\}$; $H(x,d) = \{y \in X : d(x,y) = 0\}$. There exist a metric space $(X/d, \bar{d})$ and a mapping $p_d : X \longrightarrow X/d$ such that $d(x,y) = \bar{d}(p_d(x), p_d(y))$ for all $x, y \in X$. On X/d we consider only the topology generated by the metric $\bar{d}$. The pseudometric d is continuous if the mapping p_d is continuous. If the metric space $(X/d, \bar{d})$ is complete, then we say that d is a complete pseudometric.

A family $\mathcal{A}$ of subsets of a space X is called *$\mathcal{P}$-metrizable by the pseudometric d* if d is a continuous pseudometric on the space X and for any $L \in \mathcal{A}$ and any point $x \in L$ the set $H(x,d) \cap L$ is a countably compact subset of X with the property $\mathcal{P}$, and for any open subset $U \supseteq H(x,d) \cap L$ of X there exist an open subset V of X and a number $\epsilon > 0$ such that $x \in V$ and $V(y,d,\epsilon) \cap M \subseteq U$ provided $M \in \mathcal{A}$ and $y \in M \cap V$. If $H(x,d) \cap L$ is a singleton set for all $L \in \mathcal{A}$ and $x \in L$, then we say that the family $\mathcal{A}$ is *metrizable by the pseudometric d*.

If $L \in \mathcal{A}$ and the metric $\bar{d}$ is complete on the subspace $p_d(L)$, then we say that the set L is complete relatively to the pseudometric d. If any set $L \in \mathcal{A}$ is complete relatively to the pseudometric d, then we say that the family $\mathcal{A}$ is complete metrizable by the pseudometric d.

Theorem 1. *Let $\mathcal{A}$ be a $\mathcal{P}$-metrizable by the pseudometric d family of subsets of a space X and $X = \cup\mathcal{A}$. Then there exist a T_1-space Y, a family $\mathcal{B}$ of subsets of the space Y metrizable by a pseudometric ρ and a continuous closed mapping $g : X \longrightarrow Y$ such that $g^{-1}(y)$ is a closed subset with the property $\mathcal{P}$ for each $y \in Y$, $\mathcal{A} = \{g^{-1}(L) : L \in \mathcal{B}\}$ and $d(x,z) = \rho(g(x), g(z))$ for all $x, z \in X$. Moreover, the pseudometric d is complete on the set $L \in \mathcal{A}$ if and only if the pseudometric ρ is complete on the set $g(L)$.*

Let X be a space, $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ be a sequence of open families of X, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\}$ be a sequence of mappings. A sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called a *c-sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every $n \in \mathbb{N}$. The sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called a *mc-sequence* if it is a c-sequence and $H(\alpha) = \cap\{U_{\alpha_n}; n \in \mathbb{N}\}$ is a non-empty subset of the space X.

Consider the following conditions: (SC1) $\cup\{U_\beta : \beta \in A_n\} = X$ for each $n \in \mathbb{N}$; (SC2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = \cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$; (SC3) for any mc-sequence $\alpha = \{\alpha_n \in A_n : n \in \mathbb{N}\}$, the set $H(\alpha) = \cap\{U_{\alpha_n}; n \in \mathbb{N}\}$ has the property $\mathcal{P}$ and any sequence $\{x_n \in U_{\alpha_n}; n \in \mathbb{N}\}$ has an accumulation point; (SC4) any c-sequence $\alpha = \{\alpha_n \in A_n : n \in \mathbb{N}\}$ is a mc-sequence.

The sequences γ and π are called an *A-sieve* if they have the Properties (SC1) and (SC2).

Let X be a space. A family $\mathcal{A}$ of subsets of the space X is called a family of *strong* $A(\mathcal{P})$*-subspaces* if there exists an A-sieve (γ, π) such that if $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is a c-sequence, $x \in L \in \mathcal{A}$ and $x \in H(\alpha) = \cap\{U_{\alpha_n} : \alpha_n \in \alpha\}$, then:

$(FS1)$ $L \cap H(\alpha)$ has the property $\mathcal{P}$;

$(FS2)$ any sequence $\{x_n \in L \cap U_{\alpha_n} : n \in \mathbb{N}\}$ has an accumulation point in L;

$(FS3)$ if U is an open subset of X and $H(\alpha) \subseteq U$, then there exist an open subset V of X and a natural number $m \in \mathbb{N}$ such that $x \in V$ and $U_{\alpha_m} \cap M \subseteq U$ provided $M \in \mathcal{A}$ and $M \cap V \neq \emptyset$.

A family $\mathcal{A}$ of subsets of the space X is called a family of *complete* $A(\mathcal{P})$*-subspaces* if there exists an A-sieve $\gamma = (\gamma, \pi)$ with properties $(FS1) - (FS3)$ and with the next property:

$(FS4)$ if $L \in \mathcal{A}$, $\beta = \{\beta_n : n \in \mathbb{N}\}$ is a c-sequence, $y \in H(\beta)$ and we have $U_{\beta_n} \cap L \neq \emptyset$ for each $n \in \mathbb{N}$, then $L \cap H(\beta) \neq \emptyset$.

In the above conditions we say that $\mathcal{A}$ is an (a complete) $A(\mathcal{P})$-family relatively to the A-sieve (γ, π).

A family $\mathcal{A}$ of subsets of a space X is called a *complete family of subsets* of X if there exists an A-sieve (γ, π) with the property:

$(FS5)$ $(\gamma, \pi)|L = (\{\gamma_n|L = \{L \cap U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}, \{\pi_n : A_{n+1} \to A_n : n \in \mathbb{N}\})$ is an A-sieve of the subspace L with the properties $(SC3)$ and $(SC4)$ for the property s and any $L \in \mathcal{A}$.

Theorem 2. *For any family $\mathcal{A}$ of subspaces of a regular space X the next assertions are equivalent:*

1. $\mathcal{A}$ is a family of complete $A(\mathcal{P})$-subspaces of the space X.

2. $\mathcal{A}$ is a complete family of subsets of X and a family of $A(\mathcal{P})$-subspace of the space X.

Let $\mathcal{A}$ be a family of subsets of a space X and $X \subseteq Z$, where Z be an arbitrary space. A countable family $\mathcal{F}$ of $\mathcal{A}$ in Z if:

$(FP1)$ $X \subseteq \cup\xi$ for every $\xi \in \mathcal{F}$;

$(FP2)$ for all $L \in \mathcal{A}$ and $x \in L$ there exists an element $\xi \in \mathcal{F}$ such that the set $L \cap cl_Z St(x, \xi)$ is closed in X;

$(FP3)$ for all $L \in \mathcal{A}$, $x \in L$ and $z \in Z \setminus X$ there exist an open subset V of X and $\xi \in \mathcal{F}$ such that $x \in V$ and $z \notin St(x, \xi) \cap cl_Z M$

provided $M \in \mathcal{A}$ and $V \cap M \neq \emptyset$.

Let X be a subspace of a space Z, $\mathcal{A}$ be a family of subsets of a space X and a countable family $\mathcal{F}$ of families of open subsets of Z be a plumage of $\mathcal{A}$ in Z. For any point $x \in X$ we put $H(x, \mathcal{F}) = \cap\{St(x, \mathcal{F}) : \xi \in \mathcal{F}\}$ and $\bar{H}(x, \mathcal{F}) = \cap\{cl_Z St(x, \mathcal{F}) : \xi \in \mathcal{F}\}$.

A countable family $\mathcal{F}$ of families of open subsets of Z is said to be a *strictly plumage* (Arhangel'skii for one subspace) of $\mathcal{A}$ in Z if it is a plumage of $\mathcal{A}$ in Z and has the following two properties:

$(FP4)$ For all $L \in \mathcal{A}$, $x \in L$ and an open subset $U \supseteq \bar{H}(x, \mathcal{F})$ of the space Z there exist an open subset V of X and $\xi \in \mathcal{F}$ such that $x \in V$ and $St(y, \xi) \cap M \subseteq U$ provided $M \in \mathcal{A}$ and $y \in V \cap M$.

$(FP5)$ $H(x, \mathcal{F}) \cap L = H(x, \mathcal{F}) \cap M$ provided $L, M \in \mathcal{A}$ and $x \in L \cap M$.

A family $\mathcal{A}$ of subsets of a space X is called a *family of (strictly) p-subspaces* of X if the space X is completely regular and there exists a (strictly) plumage of $\mathcal{A}$ in some compact Hausdorff space $Z \supseteq X$.

Theorem 3. *Any family $\mathcal{A}$ of strictly p-subspaces of a space X is a family of $A(k)$-subspaces of the space X.*

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On skew polynomial rings and some related rings

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Abstract

For a ring A with identity and a monoid G we consider "monoid rings" with respect to G over A where the multiplication $(a \cdot x)(b \cdot y)$ $(a, b \in A,\ x, y \in G)$ is determined by a monoid homomorphism $G \to End(A)$. Examples include various skew polynomial rings. There is also a link to $\mathbb{Z}_2-$ graded rings.

Keywords: Derivation, higher derivation, ring, monoid algebra.

A system called a D-structure in [3] and introduced in [2] consists of a ring A with identity 1, a monoid G with identity e and mappings $\sigma_{x,y} : A \to A$ for $x,\ y \in G$ satisfying the following condition:

Condition (A)

(0) For each $x \in G$ and $a \in A$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.

(i) Each $\sigma_{x,y}$ is an additive endomorphism.

(ii) $\sigma_{x,y}(ab) = \sum_{z \in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.

(iv$_1$) $\sigma_{x,y}(1) = 0$ if $x \neq y$; (iv$_2$) $\sigma_{x,x}(1) = 1$;

(iv$_3$) $\sigma_{e,x}(a) = 0$ if $x \neq e$; (iv$_4$) $\sigma_{e,e}(a) = a$.

In [2] a sort of "skew" or "twisted" monoid ring associated with A and G was constructed by means of the mappings $\sigma_{x,y}$. Examples include group rings, skew polynomial rings, the Weyl algebras and other related ones. There are also connections with gradings of rings [3].

One way of getting a D-structure is from a monoid homomorphism $G \to End(A)$: we define

$$\sigma_{x,y} = \begin{cases} \sigma(x) & \text{if} \quad x = y, \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

There is also a converse.

Theorem 1. *For a monoid G and a unital ring A, a D-structure has all $\sigma_{x,y}$ for $x \neq y$ equal to the zero map if and only if there is a homomorphism $\sigma : G \to End(A)$ with $\sigma_{x,x} = \sigma(x)$ for all $x \in G$.*

The modified monoid ring $A < G; \sigma >$ in this case has the multiplication

$$(a \cdot x)(b \cdot y) = (a\sigma(x)(b)) \cdot xy$$

for $a, b \in A$, $x, y \in G$, rather than $(a \cdot x)(b \cdot y) = ab \cdot xy$ as in the usual monoid ring $A[G]$.

Proposition 1. *If G' is another monoid, $\sigma' : G' \to End(A)$ is a monoid homomorphism and $\varphi : G \to G'$ is a monoid homomorphism, then there is a unique ring homomorphism*

$$\psi : A < G; \sigma > \longrightarrow A < G'; \sigma' >$$

such that $\psi(ax) = a\varphi(x)$ for all $a \in A$, $x \in G$.

Thus in a suitable sense the correspondence $(G; \sigma) \to A < G'; \sigma' >$ is functorial.

For any endomorphism f of A there is a homomorphism from the free monoid $< x >$ on a single generator to $End(A)$ given by $x^n \mapsto f^n$. The associated monoid ring in this case is a skew polynomial ring of some kind.

Example 1. Let G be the infinite cyclic monoid

$$\left\{x^0\left(=e\right), x^1, x^2, ..., x^n, ...\right\},$$

R a ring with identity, $R\left[t\right]$ the usual polynomial ring.

We define $\sigma : G \rightarrow EndR[t]$ by $\sigma(x^n)(p(t)) = p\left(t^{2^n}\right)$. Then $\sigma(x^n)$ as defined is indeed a ring endomorphism, and σ is a monoid homomorphism. Let

$$\sigma_{mn} = \sigma_{x^m,x^n} = \begin{cases} \sigma(x^n) & if \quad m = n, \\ 0 & if \quad m \neq n. \end{cases}$$

In $R[t]\langle G;\sigma\rangle$ we have $xt = x1 \cdot tx^0 = 1\sigma_{11}(t)\,xx^0 = t^2x$.

Thus we get Example 2.5, [3] by a simpler construction.

Example 2. Similarly if K is a field of prime characteristic p, and for our endomorphism we take the one for which $a \mapsto a^p$ for all $a \in K$, then $K < G; \sigma >$ is the Frobenius polynomial ring in x over K in which $xa = a^p x$ for all $a \in K$.

In these examples we have D-structures essentially defined by individual endomorphisms. There is another way to get D-structures from endomorphisms. In [2] it was shown that if f is a homomorphism, δ an $(f, id)-$ derivation of A, i.e. $\delta(ab) = \delta(a)b + f(a)\delta(b)$, and $\delta \circ f = f \circ \delta$, then we get a D-structure using the free monoid on x and defining $\sigma_{x^m x^n} = \binom{n}{m}\delta^{n-m} \circ f^m$ for $n \geq m$ and all others to be zero. (If $\delta \circ f \neq f \circ \delta$ there is a more complicated D-structure.)

Proposition 2. *Let $f : A \rightarrow A$ be an endomorphism, and let $\delta(a) = a - f(a)$ for all $a \in A$. Then δ is an (f, id) and an (id, f) derivation and $\delta \circ f = f \circ \delta$.*

As above we get a D-structure from f and δ and hence, in effect, from f. As a simple illustration we have

Example 3. In $\mathbb{C}$, if $f(x + yi) = x - yi$, then $\delta(x + yi) = 2yi$. Let us note three things about this elementary example.

(1) $f^2 = id$;

(2) $\frac{1}{2}\delta$ exists and is also an (f, id) and an (id, f) derivation which commutes with f and

(3) $\mathbb{C}$ is graded by $\mathbb{Z}_2$.

More generally we have:

Theorem 2. *The following conditions are equivalent for a ring A.*

(i) A has an automorphism f of order ≤ 2 such that $a - f(a) \in 2A$

for all $a \in A$.

(ii) A has an automorphism f of order ≤ 2 and an idempotent (f, id) and (id, f) derivation δ such that $a = f(a) + 2\delta(a)$ for all $a \in A$.

(iii) A is $\mathbb{Z}_2-$ graded.

The following special cases have been proved by Yu. A. Bahturin and M. M. Parmenter:

(1) If $2A = 0$, then $f = id$ and $\mathbb{Z}_2-$ gradings correspond to idempotent derivations. [4]

(2) If A is $2-$ torsion free, then $\mathbb{Z}_2-$ gradings correspond to automorphisms f of order ≤ 2 such that $a - f(a) \in 2A$ for all $a \in A$ [1].

Full details of our results will appear elsewhere.

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Star non-compact and compact hyperbolic lens polytopes

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Abstract

In the present work the authors discuss the construction of star complexes over regular maps on hyperbolic manifolds analogously with the construction of star polytopes over regular maps on two- and three-dimensional spheres. In particular, the construction of star polytopes which correspond to equidistant regular hyperbolic polytopes and their factors (lens polytopes) are considered.

Keywords: regular maps, hyperbolic manifolds, star polytopes, lens polytopes, star complexes.

1. Introduction

Coxeter and Moser [1] have noted that the incidence structure of faces of the regular star polytope {5, 5/2} (or its dual {5/2, 5}) gives one of the most symmetrical regular maps of genus 4. The manifold $\mathcal{M}^2$ is geodesically immersed into the well-known hyperbolic 3-manifold of the regular dodecahedron (Seifert–Weber manifold [2]).

2. Seifert–Weber and Davis manifolds's similarities

A natural 4-dimensional analogue of Seifert–Weber manifold is the hyperbolic space of the regular 120-cells D^4 (Davis manifold $\mathcal{D}^4$ [3]). $\mathcal{D}^4$ can be obtained by gluing the opposite hyperfaces (3-faces) of the regular hyperbolic 120-cells D^4, with dihedral angle of $2\pi/5$, by translations. The hyperplanes of these 3-faces form a hyperbolic bundle. The base 3-plane of the bundle perpendicularly intersects D^4 in a regular dodecahedron D^3_N with dihedral angle of $2\pi/5$. The identification of opposite 3-faces of the polytope D^4 induces an identification of 2-faces of narrow dodecahedra. The latter yields a hyperbolic 3-manifold $\mathcal{M}^3_1$, geodesically immersed into $\mathcal{D}^4$. On the manifold $\mathcal{M}^3_1$ this construction gives a regular self-dual map from 120 dodecahedra.

3. Star polytope {5, 3, 5/2} and corresponding 3-manifold

We see that the manifold $\mathcal{M}^3_1$ is immersed into the Davis manifold $\mathcal{D}^4$ in the same way as the manifold $\mathcal{M}^2$ is immersed into the Seifert–Weber manifold. The regular map on the manifold $\mathcal{M}^2$ is well represented by the star polytope {5, 5/2}. In [4] it is proved that the manifold $\mathcal{M}^3_1$ has an analogous representation by the 4-dimensional regular star polytope {5, 3, 5/2}. We remark that both the regular star polytope {5, 3, 5/2} and the regular star polytope {3, 5, 5/2} lead us to hyperbolic 3-manifold with rich symmetry group (of icosahedral type).

4. Manifolds and their representations on lens polytopes

In topology a three-dimensional manifold is often given by the indication of the way how to identify pairwise faces of polytopes of some homogeneous complex. Already A. Poincare noticed that one polytope is sufficient. In [5] and present paper, we discussed an "intermediate" way which will be illustrated by examples.

We take the identification of opposite faces of the regular dodecahedron that gives the Seifert–Weber manifold, and transfer it on the map of the star dodecahedron {5, 5/2}, i.e. will pair the opposite faces of the star dodecahedron with the same motions. As the map of the incidences of the star dodecahedron coincides with the map of the manifold under consideration, we obtain the pairwise correspondence of cells of the map of the manifold. This pairwise correspondence gives cycles consisting of three edges. Choosing the height of the lens so that the dihedral angle be equal to $2\pi/3$ we obtain from the lens polytope over the surface of genus 4 a three-dimensional manifold. And this is just the same manifold we obtained from the truncated icosahedron.

If we "reconstruct" the map {5, 5} taking the Dirichlet tiling for the centres of the edges of the initial map, then on the same manifold we obtain the map from 15 squares meeting by 5 at vertices. If we construct the lens polytope over this map, we can easy find a pairwise correspondence of faces which gives cycles of edges by three. It turned out that the corresponding manifold can be given by an identification of faces of the truncated rombic triacontahedron.

If we use the Poincare identification given on the spherical dodecahedron we obtain a 3-manifold with orthogonal boundary. Its boundary consists of two surfaces of genus 4 congruent to the initial surface. There are different methods to eliminate boundary for obtaining a new manifold. We have not managed yet to present it in any nice way via an identification of faces of a convex polytope.

5. Equidistant regular lens star polytopes

We have shown above some examples when finite regular lens polytopes can be obtained from infinite equidistant regular polytopes in Lobachevski spaces, and that these regular lens polytopes can be used to obtain new three-dimensional hyperbolic manifolds. On the other hand, we have shown in [6] that a regular lens polytope of type {2m+1, 3} can always be "stellate", using the Coxeter terminology, yielding infinite regular star equidistant polytopes with convex 2-faces and stellar vertices (we obtain also the dual star equidistant regular polytopes with stellar faces and convex honohedra). Obviously, using factorization of a base, the star regular lenses can be transformed into regular finite star lens polytopes, analogously as for usual convex regular infinite star lenses. One of the most simple and interesting cases is the Klein surface of genus 3 with the regular map {7, 3}.

In an analogous way we can transfer this method to regular maps on equidistant surfaces or on their factors (lens polytopes). Having the corresponding combinatorial "stellating" schemes we can consider the possibility of their metrical realization.

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On Some Classes of Quasigroups

Ivan I. Deriyenko, Wieslaw A. Dudek

Abstract

We present a short survey of our recent results on two classes of quasigroups. Firstly we characterize loops with the antiautomorphic inverse property, next – quasigroups having only one autotopism.

Keywords: *D*-loop, rigid quasigroup, super rigid quasigroup.

1 Introduction

One of methods used for examination of finite quasigroups is based on investigations of some special permutations determined on a given quasigroup. The main role plays *right middle translations* (*tracks*) of a quasigroup $Q(\cdot)$ defined as permutations φ_a of Q satisfying the identity $x \cdot \varphi_a(x) = a$, where $a \in Q$. The composition $\varphi_i \varphi_j^{-1}$ of two tracks of a quasigroup $Q(\cdot)$ is called a *spin* of $Q(\cdot)$ and is denoted by φ_{ij}. Obviously $\varphi_{ii} = \varepsilon$ for $i \in Q$ and $\varphi_{ij} \neq \varphi_{ik}$ for $j \neq k$, but the situation where $\varphi_{ij} = \varphi_{kl}$ for some $i, j, k, l \in Q$ also is possible (cf. [2]). Hence the collection Φ of all spins of a given quasigroup $Q(\cdot)$ can be divided into disjoint subsets $\Phi_i = \{\varphi_{ij} : j \in Q\}$ (called *spin-basis*) in which all elements are different. Generally, Φ_i are not closed under the composition of permutations but in some cases Φ_i are groups.

Theorem 1 [2]. *A quasigroup $Q(\cdot)$ is isotopic to some group if and only if its spin-basis Φ_1 is a group. In this case $\Phi_1 = \Phi_i$ for all $i \in Q$.*

2 D-loops

A loop $Q(\circ)$ is an *IP-loop*, if for each $a \in Q$ there exists a uniquely determined *inverse element* $a' \in Q$ such that $a' \circ (a \circ b) = (b \circ a) \circ a' = b$.

On the other hand, in any loop $Q(\circ)$ for each $a \in Q$ there are uniquely determined *left* and *right loop-inverse* elements $a_L^{-1}, a_R^{-1} \in Q$ for which we have $a_L^{-1} \circ a = a \circ a_R^{-1} = 1$. A two-sided loop-inverse element to $a \in Q$ is denoted by a^{-1}. Clearly, $(a^{-1})^{-1} = a$. In a loop each inverse element is loop-inverse but a loop-inverse element may not be inverse.

Recall that a loop $Q(\circ)$ satisfies the *antiautomorphic inverse property* (or the *dual automorphic property*) if for each $x \in Q$ there exists a two-sided loop-inverse element x^{-1} such that $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$ holds for all $x, y \in Q$. Loops with this property are also called *D-loops*. The class of D-loops is much larger than the class of IP-loops and contains as a proper subclass the class of middle Bol loops. The smallest D-loop which is not an IP-loop has six elements.

Theorem 2 [4]. *A loop $Q(\cdot)$ is a D-loop if and only if $\varphi_1 \varphi_a \varphi_1 = \varphi_{a^{-1}}^{-1}$ holds for all $a \in Q$, where a^{-1} is the inverse of a.*

Theorem 3 [4]. *Let $Q(\cdot)$ be an IP-loop and let $a \in Q$ be fixed. If an element $a' \in Q$ is inverse to a in $Q(\cdot)$, then $Q(\circ)$ with the operation $x \circ y = R_{a'}(x) \cdot L_a(y)$ is a D-loop with the same identity as in $Q(\cdot)$. In this case, an element $a \in Q$ has the same inverse in $Q(\cdot)$ and $Q(\circ)$ if and only if $L_a L_a = L_{a^2}$ and $R_a R_a = R_{a^2}$, where L_a and R_a are left and right translations of $Q(\cdot)$.*

Theorem 4. *A D-loop is isotopic to a group if and only if its spin-basis Φ_1 is closed under the composition of permutations, or equivalently, if and only if for all $i, j \in Q$ there exists $k \in Q$ such that $\varphi_i \varphi_1 \varphi_j = \varphi_k$.*

Theorem 5. *A principal isotope $Q(\cdot)$ of a D-loop $Q(\circ)$ is a D-loop if and only if each $a \in Q$ has the same inverse element in $Q(\cdot)$ and $Q(\circ)$, or equivalently, if and only if they have the same tracks induced by the identity of $Q(\circ)$.*

Theorem 6. *If a quasigroup $Q(\cdot)$ is isotopic to a D-loop $Q(\circ)$, then there exists a permutation σ of Q and an element $p \in Q$ such that for all tracks φ_i of $Q(\cdot)$ we have $\varphi_p \varphi_i^{-1} \varphi_p = \varphi_{\sigma(i)}$.*

Let $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ be tracks of a D-loop $Q(\cdot)$ with the identity 1, $Q = \{1, 2, 3, ..., n\}$. We say that tracks φ_i, φ_j, where $i \neq j \neq 1$, are *decomposable* if there exist two nonempty subsets X, Y of Q such that $Q = X \cup Y$, $X \cap Y = \emptyset$, $1 \in X$ and $\varphi_i = \bar{\varphi}_i \hat{\varphi}_i$, $\varphi_j = \bar{\varphi}_j \hat{\varphi}_j$, where $\bar{\varphi}_i$, $\bar{\varphi}_j$ are permutations of X, $\hat{\varphi}_i, \hat{\varphi}_j$ are permutations of Y.

Putting $\psi_i = \bar{\varphi}_i \hat{\varphi}_j$, $\psi_j = \bar{\varphi}_j \hat{\varphi}_i$and $\psi_k = \varphi_k$ for $k \notin \{i, j\}$ we obtain the new system of tracks which defines on Q the new loop $Q(\circ)$ with the same identity as in $Q(\cdot)$.

Using different pairs of decomposable tracks we obtain different loops which may not be isotopic. Obtained loops may not be isotopic to the initial loop $Q(\cdot)$, too.

Theorem 7. *Let $Q(\cdot)$ be a D-loop with the identity 1. If φ_i, φ_j, where $i \cdot j = 1$ and $i \neq j$, are decomposable tracks of $Q(\cdot)$, then a loop $Q(\circ)$ obtained from $Q(\cdot)$ by exchanging of tracks is a D-loop.*

The assumption $i \cdot j = 1$ is essential.

3 Super rigid quasigroups

Autotopies of a quasigroup form a group. Isotopic quasigroups have isomorphic groups of autotopies but groups of automorphisms of such quasigroups may not be isomorphic.

A quasigroup having only one automorphism is called *rigid.*

A quasigroup isotopic to a rigid quasigroup may not be rigid. Quasigroups of order two are rigid. No rigid quasigroups of order three [1], but for every $k > 3$ there exists at least one rigid quasigroup of order k [5].

The next interesting class of quasigroups is a class of quasigroups having only one (trivial) autotopism. Quasigroups with this property are called *super rigid.* Clearly, a super rigid quasigroup has only one automorphism. Hence a super rigid quasigroup is rigid. The smallest super rigid quasigroup has 7 elements. We have known only two examples (given below) of super rigid quasigroups.

$\cdot$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	1	7	6	4	5	3
3	3	6	1	2	7	4	5
4	4	5	2	1	3	7	6
5	5	7	4	3	6	2	1
6	6	3	5	7	2	1	4
7	7	4	6	5	1	3	2

$\cdot$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3	1	8	6	7	5	9	4
3	3	1	2	9	7	5	6	4	8
4	4	5	6	7	9	8	1	3	2
5	5	6	4	2	1	9	8	7	3
6	6	4	5	3	8	1	9	2	7
7	7	8	9	5	3	2	4	6	1
8	8	9	7	1	4	3	2	5	6
9	9	7	8	6	2	4	3	1	5

Open problem 1. *Give more examples of super rigid quasigroups.*

Open problem 2. *Describe the class of super rigid quasigroups.*

4 Indicators of quasigroups

Any permutation $\varphi \in S_n$ can be decomposed into disjoint cycles. Denote by $C(\varphi)$ the sequence $l_1, l_2, \ldots, l_n$, where l_i denotes the number of cycles of the length i. By the *indicator* of a permutation φ with $C(\varphi) = \{l_1, l_2, ..., l_n\}$ we mean the polynomial $w(\varphi) = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$.

For example, for

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 4 & 6 & 7 \end{pmatrix}$$

and

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 8 & 6 & 4 & 7 & 5 & 3 \end{pmatrix}$$

we have $C(\varphi) = \{2,1,1,0,0,0,0\}$ and $C(\psi) = \{0,2,0,1,0,0,0,0\}$. Hence, $w(\psi) = x_1^2 x_2 x_3$ and $w(\psi) = x_2^2 x_4$.

Consider the following three matrices:

$$\Phi = [\varphi_{ij}], \qquad L = [L_{ij}], \qquad R = [R_{ij}],$$

where $\varphi_{ij} = \varphi_i \varphi_j^{-1}$, $L_{ij} = L_i L_j^{-1}$, $R_{ij} = R_i R_j^{-1}$ for all $i, j \in Q$. Obviously, $\varphi_{ii}(x) = L_{ii}(x) = R_{ii}(x) = x$ and $\varphi_{ij}(x) \neq x$, $L_{ij}(x) \neq x$, $R_{ij}(x) \neq x$ for all $i, j, x \in Q$ and $i \neq j$.

By the *indicator of the matrix* Φ we mean the polynomial

$$w(\Phi) = \sum_{i=1}^{n} w(\Phi_i),$$

where $\Phi_i = \{\varphi_{i1}, \varphi_{i2}, \ldots, \varphi_{in}\}$ and

$$w(\Phi_i) = \sum_{j=1,\, j\neq i}^{n} w(\varphi_{ij}).$$

Indicators of the matrices L and M are defined analogously.

Theorem 8 [3]. *Isotopic quasigroups have the same indicators of the matrices* Φ*,* L *and* R*.*

For a quasigroup isotopic to a group we have $w(\Phi) = w(\Phi_1)$.

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About extensions of mappings into topologically complete spaces

Radu N. Dumbrăveanu

Abstract

In the present paper, for a subspace Y of a space X and a space E the conditions, for which any continuous mapping $f : Y \longrightarrow E$ is continuous extendable on X, are determined.

Keywords: collectionwise normal space, continuous extension.

1 Introduction

Every space is considered to be a completely regular T_1-space. Consider a topological space X. We denote by $cl_X A$ the closure of any set A from X. A subset B of X is clopen if it is simultaneously closed and open. A regular space X is said to be *zero-dimensional* if it is of small inductive dimension zero ($indX = 0$), i.e. X has a base of clopen sets. A normal space X has large inductive dimension zero ($IndX = 0$) if and only if for any two disjoint closed subsets A and B of X there is a clopen set C such that $A \subseteq C$ and $B \subseteq (X \setminus C)$. A normal space X has Lebesgue covering dimension zero ($dimX = 0$) if any finite open cover of X can be refined to a partition of X into clopen sets. It is well known that $IndX = dimX$ for any metric space X. Also if X is Lindelöf, then $indX = 0$ if and only if $IndX = 0$ [2, Theorem 1.6.5] and if X is normal, then $IndX = 0$ if and only if $dimX = 0$ [2, Theorem 1.6.11]. A topological space X is Dieudonné complete if there exists a complete uniformity on the space X [3]. A space X is topologically complete if X is homeomorphic to a closed subspace of a product of metrizable spaces. The Dieudonné completion μX of a space X is a topological complete space, for which X is a dense subspace of μX and each continuous mapping g from X into a topologically complete space Y admits a continuous extension μg over μX.

2 On extension of discrete-valued mappings

All spaces considered in this section are assumed to be zero-dimensional. We are going to use the symbol $\mathcal{D}_\tau$ (τ is any cardinal number) for the discrete space consisting of τ elements. As usual, we write $\mathcal{D}$ instead of $\mathcal{D}_2$ and it is very convenient to view $\mathcal{D}$ as being $\{0, 1\}$ endowed with the discrete topology. Also $C(X, Z)$ is the set of all continuous functions defined on the topological space X and with values in a space Z. If $Y \subseteq X$ and $f \in C(Y, Z)$, then we say that f extends to a function $g \in C(X, Z)$ if $g(x) = f(x)$, for every $x \in Y$.

Theorem 1.1. *Let $Y \subseteq X$, X normal and $dimX = 0$, then the following assertions are equivalent:*

(i) For every clopen subset U of Y the set $cl_X U$ is clopen in $cl_X Y$.

(ii) For every clopen partition $\gamma = \{U, V\}$ of Y there exists a clopen partition $\gamma' = \{U', V'\}$ of X such that $U = U' \cap Y$ and $V = V' \cap Y$.

(iii) Every function $f \in C(Y, \mathcal{D})$ extends to a function in $C(X, \mathcal{D})$.

Proof. (i)→(ii). Let X be normal and $dimX = 0$. Then $IndX = 0$, which means that any two disjoint closed subsets of X can be separated by disjoint clopen subsets of X, i.e., X is ultranormal [1]. Let $\gamma = \{U, V\}$ be a clopen partition of Y. Then, by assumption, $cl_X U$ is clopen in $cl_X Y$. As $cl_X Y$ is a closed subset of X and $dimX = 0$, we can find a clopen subset U' of X, such that $cl_X U = U' \cap cl_X Y$. On the other hand, $U \subseteq cl_X U$ and U is clopen in Y, therefore $U' \cap Y = U$. The collection $\{U', X \setminus U'\}$ is the desired partition.

(ii)→(i). Let U be a clopen subset of Y. Then the collection $\{U, V = Y \setminus U\}$ is a clopen partition of Y. Therefore, by assumption, we can find a clopen partition $\{U', V'\}$ of X such that $U = U' \cap Y$ and $V = V' \cap Y$. Now, as $\{U, V\}$ is partition of Y, we have that

$cl_X Y = cl_X U \cup cl_X V$. On the other hand, $U \subseteq cl_X U \subseteq U'$, therefore $cl_X U = U' \cap cl_X Y$, i.e. $cl_X U$ is clopen in $cl_X Y$.

(ii)$\leftrightarrow$(iii). This is obvious.

Example. Let $Y = \mathbb{N}$ with the discrete topology and $X = \beta\mathbb{N}$. Then X is normal and $dimX = 0$. Let $\tau < \omega$. Then every continuous function from Y into $\mathcal{D}_2$ extends to a continuous function on X. But if $\tau \geq \omega$ then, since a continuous function on a compact space must be bounded, not every continuous function from Y into $\mathcal{D}_\tau$ extends to a continuous function on X. Thus, in case of continuous functions on an infinite discrete space, the conditions for X to be normal and $dimX = 0$ are not enough.

Theorem 1.2. *Let $Y \subseteq X$, X be a collectionwise normal space and $dimX = 0$. Then the following assertions are equivalent:*

(i) For every cardinal τ and a discrete collection $\{U_\alpha : \alpha \in \mathcal{D}_\tau\}$ of clopen subsets of Y, the collection $\{cl_X U_\alpha : \alpha \in \mathcal{D}_\tau\}$ is discrete in X.

(ii) For every clopen subset U of Y, the set $cl_X U$ is clopen in $cl_X Y$ and every discrete collection $\{U_\alpha : \alpha \in A\}$ of clopen subsets of Y is locally finite in X.

(iii) For each discrete space Z every function $f \in C(Y, Z)$ extends to a function in $C(X, Z)$.

(iv) If Z is a topologically complete space and $f \in C(Y, Z)$, then there exists $g \in C(cl_X Y, Z)$ such that $f = g|Y$.

(v) $cl_{\mu X} Y \subseteq \mu Y$.

3 Extension of mappings into metric spaces

Theorem 3.1. *Let Y be a subspace of the space X, E be a topologically complete space and for each closed subspace Z of X and any continuous mapping $g : Z \longrightarrow E$ there exists a continuous extension $\bar{g} : X \longrightarrow E$. If $\mu Y = cl_{\mu X} Y$, then for each continuous mapping $g : Y \longrightarrow E$ there exists a continuous extension $\bar{g} : X \longrightarrow E$.*

A family $\{F_\alpha : \alpha \in A\}$ of the space X is functionally discrete if there exists a family $\{f_\alpha : \alpha \in A\}$ of continuous functions on X such that the family $\{f_\alpha^{-1}(0,2) : \alpha \in A\}$ is discrete in X and $F_\alpha \subseteq f_\alpha^{-1}(1)$ for each $\alpha \in A\}$.

Theorem 3.2. *Let Y be a subspace of the space X, and for any continuous mapping $g : Z \longrightarrow E$ of a closed subspace Z of X into a Banach space E there exists a continuous extension $\bar{g} : X \longrightarrow E$. Then the following assertions are equivalent:*

(i) $\mu Y = cl_{\mu X} Y$,

(ii) For each continuous mapping $g : Y \longrightarrow E$ into a Banach space E there exists a continuous extension $\bar{g} : X \longrightarrow E$.

(iii) For each continuous mapping $g : Y \longrightarrow E$ into a metrizable space E there exists a continuous extension $\mu g : cl_X Y \longrightarrow E$.

(iv) For each functionally discrete family $\{F_\alpha : \alpha \in A\}$ of the space Y the family $\{cl_X F_\alpha : \alpha \in A\}$ is discrete in X.

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On a construction of orthogonal operations

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Abstract

We propose a construction method of orthogonal operations. It is a generalization of the recursive method. We call it *step algorithm*. For this purpose we introduce a notion of retract orthogonality and give its construction method. This notion does not coincide with usual notion of orthogonality.

Keywords: orthogonality of operations, retract orthogonal operations, recursive method, step algorithm.

1 Introduction

A problem of construction of MDS-codes, hash-functions and secret-sharing schemes is connected with the construction of orthogonal operations, partial orthogonal operations, orthogonal quasigroups. Some applications of orthogonal operations in cryptography are described in survey [2].

P.N.Syrbu [3] proposed a system of identities which warrants orthogonality of n-ary operations. Couselo E., Gonzalez S., Markov V., Nechaev A. [1] and Belyavskaya G., Mullen G.L. [4] considered a method for construction of n-ary orthogonal operations. We call it *recursive method*.

Here, we propose a new method for the construction of n-ary orthogonal operations which is a generalization of recursive method and is called a *step algorithm*.

2 Preliminaries

By x_i^j we denote the sequence $x_i, x_{i+1}, \dots, x_j$, when $i \leqslant j$ and empty sequence otherwise. An operation f is called: *i-invertible* if for arbitrary elements a_1^{i-1}, b, a_{i-1}^n there exists a unique element x such that $f(a_1^{i-1}, x, a_{i+1}^n) = b$; and *invertible* or *quasigroup operation*, if it is i-invertible for all i. σ-parastrophe of f is an operation ${}^{\sigma}\!f$ which is defined by

$$ {}^{\sigma}\!f(x_{1\sigma}, \dots, x_{n\sigma}) = x_{(n+1)\sigma} :\Longleftrightarrow f(x_1, \dots, x_n) = x_{n+1}. $$

Following [4], a tuple of n-ary operations $(f_1, \dots, f_k)$ $(n \geqslant 2, k \leqslant n)$ defined on Q $(m := |Q|)$ is called *orthogonal*, if for an arbitrary $a_1, \dots, a_k \in Q$ the system $\left\{ f_i(x_1^n) = a_i |_1^k \right.$ has exactly m^{n-k} solutions.

3 Retract orthogonality

Let $\delta := \{i_1, \dots, i_s\} \subseteq \overline{1,n}$ and f be n-ary operation on Q. An s-ary operation g being obtained from $f(x_1, \dots, x_n)$ by replacing all variables from $\{x_1, \dots, x_n\} \setminus \{x_{i_1}, \dots, x_{i_s}\}$ with some elements from Q, is called *δ-retract* of f. δ-retracts of operations $f_1, \dots, f_s$ are called *similar* if the same variables are replaced with the same elements.

Consider a sequence of operations $f_1, \dots, f_s$. If all sequences of operations being similar δ-retracts of $f_1, \dots, f_s$ are orthogonal, then the operations $f_1, \dots, f_s$ are called *δ-recract orthogonal.*

Theorem 1. *Let $p_1, \dots, p_s$ be arbitrary 1-invertible $(n-s+1)$-ary operations, $h_1, \dots, h_s$ arbitrary operations, and let operations $f_1, \dots, f_s$ be defined by*

$$ \begin{cases} f_1(x_1, \dots, x_n) := p_1(h_1(x_1, \dots, x_s), x_{s+1}, \dots, x_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_s(x_1, \dots, x_n) := p_s(h_s(x_1, \dots, x_s), x_{s+1}, \dots, x_n). \end{cases} $$

Then $f_1, \dots, f_s$ are $\overline{1,s}$-retract orthogonal if and only if $h_1, \dots, h_s$ are orthogonal.

The notion of orthogonality and retract orthogonality do not coincide. The following example confirms this.

Example. Let operations $f_1(x,y,z) = x + 2y + z$, $f_2(x,y,z) = x + 2y + 3z$ be defined on $\mathbb{Z}_5$.[1] They are orthogonal, but their $\{1,2\}$-retracts are not orthogonal.

4 Step algorithm of construction of orthogonal operations

Let $f_1, \ldots, f_n$ be n-ary operations on Q and let $\pi := \{\pi_1, \pi_2, \ldots, \pi_s\}$ be a partition of $\overline{1,n}$, where $\pi_i := \{n_{i-1}+1, \ldots, n_i\}$, $n_0 := 0$, $n_s := n$, $i = 1, \ldots, s$. Define operations $g_1, \ldots, g_n$ by

$$\begin{cases} g_1(x_1^n) := f_1(x_1^n), \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ g_{n_1}(x_1^n) := f_{n_1}(x_1^n), \\ g_{n_1+1}(x_1^n) := f_{n_1+1}(g_1(x_1^n), \ldots, g_{n_1}(x_1^n), x_{n_1+1}^n), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_{n_2}(x_1^n) := f_{n_2}(g_1(x_1^n), \ldots, g_{n_1}(x_1^n), x_{n_1+1}^n), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_{n_{s-1}+1}(x_1^n) := f_{n_{s-1}+1}(g_1(x_1^n), \ldots, g_{n_{s-1}}(x_1^n), x_{n_{s-1}+1}^n), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ g_n(x_1^n) := f_n(g_1(x_1^n), \ldots, g_{n_{s-1}}(x_1^n), x_{n_{s-1}+1}^n), \end{cases}$$

This construction method of a tuple $(g_1, \ldots, g_n)$ of operations is called π-*step algorithm* or *step algorithm.*

Theorem 2. *If for every $i \in \overline{1,s}$ n-ary operations $f_{n_{i-1}+1}, \ldots, f_{n_i}$ are π_i-retract orthogonal, then the operations $g_1, \ldots, g_n$, defined by π-step algorithm, are orthogonal.*

[1] $\mathbb{Z}_5$ denotes ring of integers modulo 5.

5 Conclusion

If the partition π is trivial, i.e. for every class of the partition $\pi_i = \{i\}$, then retract orthogonality of f_i means i-invertibility of f_i. So, Theorem 2 implies Theorem 3 from [4]. In other words, step algorithm is a generalization of recursive algorithm.

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Limit Theorems For Random Walks In The General Linear Group

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Abstract

Let $G_n = g_n \dots g_1$ be a random walk in the general linear group. We study the asymptotic of the exit time from the complement $\mathbb{B}^c$ of a closed unit ball $\mathbb{B}$ of the associated walk $G_n v$ in the vector space $\mathbb{V} = \mathbb{R}^d$, where v is any starting vector in $\mathbb{B}^c$. We establish a limit theorem for this walk conditioned to stay in $\mathbb{B}^c$.

Keywords: general linear group, random walks, conditional random walk, limit theorems.

1 Introduction and previous results

Let $\mathbb{G} = GL(d, \mathbb{R})$ be the general linear group of $d \times d$ invertible matrices w.r.t. ordinary matrix multiplication. If g is an element of $\mathbb{G}$ by $\|g\|$ we mean the operator norm and if v is an element of the vector space $\mathbb{V} = \mathbb{R}^d$, the norm $\|v\|$ is Euclidean. Endow the group $\mathbb{G}$ by the usual Borel σ-algebra w.r.t. $\|\cdot\|$. Let $\boldsymbol{\mu}$ be a probability measure on $\mathbb{G}$ and suppose that on the probability space $(\Omega, \mathcal{F}, \mathbf{Pr})$ we are given an i.i.d. sequence $(g_n)_{n\geq 1}$ of $\mathbb{G}$-valued random elements of the same law $\mathbf{Pr}(g_1 \in dg) = \boldsymbol{\mu}(dg)$. A random walk in $\mathbb{G}$ is the product $G_n = g_n \dots g_1$. Let $v \in \mathbb{V} \setminus \{0\}$ be any starting point. The object of interest is the size of the vector $G_n v$ which is controlled by the quantity $\log \|G_n v\|$. It follows from the results of Le Page [3] that, under appropriate assumptions, the sequence $(\log \|G_n v\|)_{n\geq 1}$ behaves like a sum of i.i.d. r.v.'s and satisfies standard classical properties such as the law of large numbers, law of iterated logarithm and the central limit theorem. There is a vaste literature on this subject. We refer to Bougerol and Lacroix [1] and to the references therein.

Introduce the following conditions. Let $N(g) = \max\left\{\|g\|, \|g\|^{-1}\right\}$, $\mathrm{supp}\boldsymbol{\mu}$ be the support of the measure $\boldsymbol{\mu}$ and $\mathbb{P}(\mathbb{V})$ be the projective space of $\mathbb{V}$.

P1. *There exists* $\delta_0 > 0$ *such that*

$$\int_{\mathbb{G}} N(g)^{\delta_0} \boldsymbol{\mu}(dg) < \infty.$$

The next condition requires, roughly speaking, that the dimension of the support of $\mathrm{supp}\boldsymbol{\mu}$ cannot be reduced.

P2 (Strong irreducibility). *The support* $\mathrm{supp}\boldsymbol{\mu}$ *of* $\boldsymbol{\mu}$ *acts strongly irreducibly on* $\mathbb{V}$, *i.e. no proper union of finite vector subspaces of* $\mathbb{V}$ *is invariant with respect to all elements* g *of the group generated by* $\mathrm{supp}\boldsymbol{\mu}$.

We say that the sequence $(h_n)_{n\geq 1}$ of elements of $\mathbb{G}$ is contracting for the projective space $\mathbb{P}(\mathbb{V})$ if $\lim_{n\to\infty} \log \frac{a_1(n)}{a_2(n)} = \infty$, where $a_1(n) \geq \ldots \geq a_d(n)$ are the eigenvalues of the symmetric matrix $h_n' h_n$ and h_n' is the transpose of h_n.

P3 (Proximality). *The closed semigroup generated by* $\mathrm{supp}\boldsymbol{\mu}$ *contains a contracting sequence for the projective space* $\mathbb{P}(\mathbb{V})$.

For example **P3** is satisfied if the closed semigroup generated by $\mathrm{supp}\boldsymbol{\mu}$ contains a matrix with a unique simple eigenvalue of maximal modulus.

In the sequel for any $v \in \mathbb{V} \setminus \{0\}$ we denote by $\overline{v} = \mathbb{R}v \in \mathbb{P}(\mathbb{V})$ its direction and for any direction $\overline{v} \in \mathbb{P}(\mathbb{V})$ we denote by v a vector in $\mathbb{V} \setminus \{0\}$ of direction $\overline{v}$. Define the function $\rho : \mathbb{G} \times \mathbb{P}(\mathbb{V}) \to \mathbb{R}$ called norm cocycle by setting

$$\rho(g, \overline{v}) := \log \frac{\|gv\|}{\|v\|}, \text{ for } (g, \overline{v}) \in \mathbb{G} \times \mathbb{P}(\mathbb{V}).$$

It is well known (see Le Page [3] and Bougerol and Lacroix [1]) that under conditions **P1-P3** there exists a unique $\boldsymbol{\mu}$-invariant measure $\boldsymbol{\nu}$ on $\mathbb{P}(\mathbb{V})$ such that, for any continuous function φ on $\mathbb{P}(\mathbb{V})$,

$$(\boldsymbol{\mu} * \boldsymbol{\nu})(\varphi) = \boldsymbol{\nu}(\varphi).$$

Moreover the upper Lyapunov exponent

$$\gamma = \gamma_{\boldsymbol{\mu}} = \int_{\mathbb{G}\times\mathbb{P}(\mathbb{V})} \rho(g, \overline{v})\, \boldsymbol{\mu}(dg)\, \boldsymbol{\nu}(d\overline{v})$$

is finite and there exists a constant $\sigma > 0$ such that for any $v \in \mathbb{V} \setminus \{0\}$ and any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{Pr}\left(\frac{\log \|G_n v\| - n\gamma}{\sigma\sqrt{n}} \leq t\right) = \Phi(t),$$

where $\Phi(\cdot)$ is the standard normal distribution.

2 Main results

Denote by $\mathbb{B}$ the closed unit ball in $\mathbb{V}$ and by $\mathbb{B}^c$ its complement. For any $v \in \mathbb{B}^c$ define the exit time of the random process $G_n v$ from $\mathbb{B}^c$ by

$$\tau_v = \min\{n \geq 1 : G_n v \in \mathbb{B}\}.$$

In the sequel we consider that the upper Lyapunov exponent γ is equal to 0. The fact that $\gamma = 0$ does not imply that the events

$$\{\tau_v > n\} = \{G_k v \in \mathbb{B}^c : k = 1, ..., n\},\ n \geq 1$$

occur with positive probability for any $v \in \mathbb{B}^c$. To ensure this we need the following additional condition:

P4. *There exists $\delta > 0$ such that*

$$\inf_{s\in\mathbb{S}^{d-1}} \boldsymbol{\mu}(g : \log\|gs\| > \delta) > 0.$$

Under conditions **P1-P4** we prove that, for any $v \in \mathbb{B}^c$,

$$\mathbf{Pr}\left(\tau_v > n\right) = \frac{2V\left(v\right)}{\sigma\sqrt{2\pi n}}\left(1 + o\left(1\right)\right) \text{ as } n \to \infty,$$

where V is a positive function on $\mathbb{B}^c$. Moreover, we prove that the limit law of the quantity $\frac{1}{\sigma\sqrt{n}}\log\|G_n v\|$, given the event $\{\tau_v > n\}$, coincides with the Rayleigh distribution $\Phi^+\left(t\right) = 1 - \exp\left(-\frac{t^2}{2}\right)$: for any $v \in \mathbb{B}^c$ and for any $t \geq 0$,

$$\lim_{n\to\infty} \mathbf{Pr}\left(\frac{\log\|G_n v\|}{\sigma\sqrt{n}} \leq t \,\middle|\, \tau_v > n\right) = \Phi^+\left(t\right).$$

Our proofs rely upon the strong approximation result for Markov chains established in [2].

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On multiplication groups of isostrophic quasigroups

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Abstract

Relations between the multiplication groups of loops which are isostrophes of quasigroups are studied in the present work. We prove that, if $(Q,\cdot)$ is a quasigroup and its isostrophe $(Q,\circ)$, where $x\circ y=\psi(y)\setminus\varphi(x)$, $\forall x,y\in Q$, is a loop, then the right multiplication group of $(Q,\circ)$ is a subgroup of the left multiplication group of $(Q,\cdot)$. Moreover, if $\varphi\in Aut(Q,\circ)$, then $RM(Q,\circ)$ is a normal subgroup of $LM(Q,\cdot)$. As a corollary from this result we get that the right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop.

Keywords: Right (left, middle) Bol loop, isostrophy, (right, left) multiplication group.

Recall that the isotopes of parastrophes of a quasigroup $(Q,\cdot)$ are called isostrophes of $(Q,\cdot)$. A loop satisfying the identity $(xy\cdot z)y=x(yz\cdot y)$ is called a right Bol loop. A loop $(Q,\cdot)$ is called a middle Bol if it satisfies the identity $x(yz\backslash x)=(x/z)(y\backslash x)$, where "\" ("/") is the right (respectively, left) division in the loop $(Q,\cdot)$. Moreover, the middle Bol identity is a necessary and sufficient condition for the universality of the anti-automorphic inverse property $(x\cdot y)^{-1}=y^{-1}\cdot x^{-1}$ [1]. According to [2], a loop $(Q,\circ)$ is middle Bol if and only if there exists a right Bol loop $(Q,\cdot)$ such that

$$x\circ y=y^{-1}\setminus x, \tag{1}$$

for all $x,y\in Q$. Middle Bol loops are studied also in [4,5].

Let $(Q, \cdot)$ be a quasigroup and let S_Q be the symmetric group on Q. Denote by $LM(Q, \cdot) =< L_a^{(\cdot)} | a \in Q >$ (resp. $RM(Q, \cdot) = < R_a^{(\cdot)} | a \in Q >$, $M(Q, \cdot) =< L_a^{(\cdot)}, R_a^{(\cdot)} | a \in Q >$) the left multiplication group (resp. the right multiplication group, multiplication group), where $L_a^{(\cdot)}(x) = a \cdot x, R_a^{(\cdot)}(x) = x \cdot a, \forall a, x \in Q$. K.Shchiukin proved in [3] that, if a loop $(Q, \circ)$ is isotopic to a quasigroup $(Q, \cdot)$, then the left (right) multiplication group of $(Q, \circ)$ is a subgroup in the left (right) multiplication group of $(Q, \cdot)$. Moreover, if the second (first) component of the isotopy is an automorphism of $(Q, \circ)$, then the left (right) multiplication group of $(Q, \circ)$ is a normal subgroup in the left (right) multiplication group of $(Q, \cdot)$. We prove in the present work some analogous results for the multiplication groups of loops which are isostrophes of a quasigroup. Namely, we prove that, if $(Q, \cdot)$ is a quasigroup and its isostrophe $(Q, \circ)$, where $x \circ y = \psi(y) \setminus \varphi(x)$, $\forall x, y \in Q$, is a loop, then the right multiplication group of $(Q, \circ)$ is a subgroup of the left multiplication group of $(Q, \cdot)$. Moreover, if $\varphi \in Aut(Q, \circ)$, then $RM(Q, \circ)$ is a normal subgroup of $LM(Q, \cdot)$. As a corollary from this result we get that the right multiplication group of a middle Bol loop coinsides with the left multiplication group of the corresponding right Bol loop.

Theorem. *Let $(Q, \cdot)$ be a quasigroup and let $\varphi, \psi \in S_Q$, such that the isostrophe $(Q, \circ)$, where $x \circ y = \psi(y) \backslash \varphi(x)$, $\forall x, y \in Q$, is a loop. Denoting $I_x^{(\cdot)}(y) = y \setminus x$, $\forall x, y \in Q$, the following statements are true:*

1. $LM(Q, \circ) =< I_x^{(\cdot)} \psi | x \in Q >$;
2. $RM(Q, \circ) =< L_x^{(\cdot)} \varphi | x \in Q >=< L_x^{(\cdot)-1} L_y^{(\cdot)} | x, y \in Q >$;
3. $M(Q, \circ) =< I_x^{(\cdot)} \psi, L_y^{(\cdot)} \varphi | x, y \in Q >=$
$=< I_x^{(\cdot)} \psi, L_y^{(\cdot)-1} L_z^{(\cdot)} | x, y, z \in Q >$;
4. $RM(Q, \circ) \trianglelefteq LM(Q, \cdot)$ if φ *is an automorphism of* $(Q, \circ)$;
5. $LM(Q, \cdot) =< RM(Q, \circ), \varphi >$.

Proof. 1. According to the definition of " $\circ$ ", for every $x, y \in Q$, we have: $x \circ y = \psi(y) \backslash \varphi(x)$, which implies $L_x^{(\circ)}(y) = I_{\varphi(x)}^{(\cdot)} \psi(y)$, so $LM(Q, \circ) =< I_x^{(\cdot)} \psi | x \in Q >$.

2. Let $e \in Q$ be the unit of the loop $(Q, \circ)$. Then $x = x \circ e = \psi(e)\backslash\varphi(x) \Rightarrow \varphi(x) = \varphi(e)\cdot x$, so $L^{(\cdot)}_{\psi(e)}(x) = \varphi(x), \forall x \in Q$ and $\varphi = L^{(\cdot)}_{\psi(e)}$. Hence, from $x \circ y = \psi(y)\backslash\varphi(x)$ follows $R^{(\circ)}_y(x) = L^{(\cdot)-1}_{\psi(y)}\varphi(x)$, $\forall x \in Q$, which implies

$$R^{(\circ)}_y = L^{(\cdot)-1}_{\psi(y)}\varphi = L^{(\cdot)-1}_{\psi(y)}L^{(\cdot)}_{\psi(e)} \tag{2}$$

and, in particular, we get that, $\forall y \in Q$, the following equality holds:

$$L^{(\cdot)-1}_{\psi(y)} = R^{(\circ)}_y\varphi^{-1}. \tag{3}$$

Using (2),we have: $RM(Q, \circ) =< L^{(\cdot)}_x\varphi | x \in Q >\subseteq< L^{(\cdot)-1}_x L^{(\cdot)}_y | x, y \in Q >$. On the other hand, $L^{(\cdot)-1}_x L^{(\cdot)}_y = L^{(\cdot)-1}_{\psi(\psi^{-1}(x))}L^{(\cdot)}_{\psi(e)}L^{(\cdot)-1}_{\psi(e)}L^{(\cdot)}_{\psi(\psi^{-1}(y))} = R^{(\circ)}_{\psi^{-1}(x)}R^{(\circ)-1}_{\psi^{-1}(y)} \in RM(Q, \circ)$, so $RM(Q, \circ) =< L^{(\cdot)-1}_x L^{(\cdot)}_y | x, y \in Q >$.

3. Follows from 1 and 2.

4. Let φ be an automorphism of $(Q, \circ)$, then $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$, for every $x, y \in Q$. The last equality implies: $\varphi R^{(\circ)}_y(x) = R^{(\circ)}_{\varphi(y)}\varphi(x), \forall x, y \in Q$ and $\varphi R^{(\circ)}_y = R^{(\circ)}_{\varphi(y)}\varphi, \forall y \in Q$, so

$$\varphi R^{(\circ)}_y\varphi^{-1} = R^{(\circ)}_{\varphi(y)}, \tag{4}$$

for every $y \in Q$. Hence, for $L^{(\cdot)}_x \in LM(Q, \cdot)$ and $R^{(\circ)}_y \in RM(Q, \circ)$, we'll use (3) and (4) to show that $L^{(\cdot)}_x R^{(\circ)}_y L^{(\cdot)-1}_x \in RM(Q, \circ)$:

$$L^{(\cdot)}_x R^{(\circ)}_y L^{(\cdot)-1}_x = \varphi R^{(\circ)-1}_{\psi^{-1}(x)}R^{(\circ)}_y R^{(\circ)-1}_{\psi^{-1}(x)}\varphi^{-1} =$$
$$\varphi R^{(\circ)-1}_{\psi^{-1}(x)}\varphi^{-1}\varphi R^{(\circ)}_y\varphi^{-1}\varphi R^{(\circ)-1}_{\psi^{-1}(x)}\varphi^{-1} =$$
$$R^{(\circ)-1}_{\varphi(\psi^{-1}(x))}R^{(\circ)}_{\varphi(y)}R^{(\circ)-1}_{\varphi(\psi^{-1}(x))} \in RM(Q, \circ).$$

Analogously, using (3) and (4) we shall prove that $L^{(\cdot)-1}_x R^{(\circ)}_y L^{(\cdot)}_x \in RM(Q, \circ)$:

$$L^{(\cdot)-1}_x R^{(\circ)}_y L^{(\cdot)}_x = R^{(\circ)}_{\psi^{-1}(x)}\varphi^{-1}R^{(\circ)}_y\varphi R^{(\circ)-1}_{\psi^{-1}(x)} =$$
$$R^{(\circ)-1}_{\psi^{-1}(x)}R^{(\circ)}_{\varphi^{-1}(y)}R^{(\circ)-1}_{\psi^{-1}(x)} \in RM(Q, \circ).$$

So as

$$L^{(\cdot)}_x R^{(\circ)-1}_y L^{(\cdot)-1}_x = (L^{(\cdot)}_x R^{(\circ)}_y L^{(\cdot)-1}_x)^{-1} =$$

$$(R^{(\circ)-1}_{\varphi(\psi^{-1}(x))}R^{(\circ)}_{\varphi(y)}R^{(\circ)-1}_{\varphi(\psi^{-1}(x))})^{-1} \in RM(Q,\circ)$$

and
$$L^{(\cdot)-1}_x R^{(\circ)-1}_y L^{(\cdot)}_x = (L^{(\cdot)-1}_x R^{(\circ)}_y L^{(\cdot)}_x)^{-1} =$$
$$(R^{(\circ)-1}_{\psi^{-1}(x)}R^{(\circ)}_{\varphi^{-1}(y)}R^{(\circ)-1}_{\psi^{-1}(x)})^{-1} \in RM(Q,\circ).$$

So, we proved that $RM(Q,\circ) \trianglelefteq LM(Q,\cdot)$.

5. Follows from 2. □

Corollary *Let* $(Q,\circ)$ *be a middle Bol loop and let* $(Q,\cdot)$ *be the corresponding right Bol loop. Then the following equalities are true*
$$RM(Q,\circ) =< L^{(\cdot)}_x | x \in Q >= LM(Q,\cdot).$$

Proof. From (1) follows $x \circ y = y^{-1}\backslash x = I(y)\backslash x$, for every $x, y \in Q$. From the previous Theorem, for $\varphi = \varepsilon$ and $\psi = I$, we obtain $RM(Q,\circ) =< L^{(\cdot)}_x | x \in Q >= LM(Q,\cdot)$. □

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On the completion of incomplete non-orientable hyperbolic 3-manifolds

I.S. Gutsul

Abstract

The present work is devoted to the study of the geometry of the completion of non-orientable hyperbolic 3-manifolds.

Keywords: hyperbolic geometry, 3-manifold, discrete group.

1 Introduction

In the work [1] W.Thurston developed the theory of the completion of orientable hyperbolic 3-manifold. But he did not consider the completion of incomplete non-orientable hyperbolic manifolds. It is possible to develop the theory of the completion of such non-orientable manifolds, too, but the method of Thurston cannot be used in this case. For this case the hyperbolic space should be considered from the point of view of synthetic, i.e. Poincare models, or some other models of the hyperbolic space cannot be used.

In the present communication we consider the completion of incomplete non-orientable hyperbolic 3-manifolds with manifolds being non-compact but having finite volume.

2 Completion of 3-manifolds

Non-compact non-orientable hyperbolic 3-manifolds with finite volume can have cusps of two kinds: orientable and non-orientable. An orientable cusp is the set $T^2 \times [0, \infty)$, i.e. the product of a two-dimensional torus and a half-line. An non-orientable cusp is the set $K^2 \times [0, \infty)$, i.e. the product of a Kleins bottle and a half-line.

Consider a complete non-orientable non-compact hyperbolic 3-manifold M with finite volume. As it is shown in [2] it is rigid which means that two such manifolds with isomorphic fundamental groups are homeomorphic. If we begin to deform the manifold M it becomes incomplete but its completion is possible.

Let M be a complete non-orientable non-compact hyperbolic 3-manifold with finite volume. It can be obtained by the identification of faces of a polyhedron R in the hyperbolic space, some vertices of R being infinitely removed, i.e. they lie on the absolute. Consider an horosphere S centered at an infinitely removed vertex A of the polyhedron R. Then motions which identify faces of this polyhedron and retain the center of the horosphere S induce on the horosphere S a discrete two-dimensional group Γ of transformations. As the metric of horosphere is Euclidean and the group, generated by all the motions identifying the faces of the polyhedron R, does not contain elements of finite order, the group Γ is isomorphic to the fundamental group either of torus or of Kleins bottle. If we begin to deform the polyhedron, we obtain a similarity group Γ^1 on the horosphere S. If the group Γ^1 is non-orientable, the completion of such an end of the manifold M cannot yield a manifold. It is so because of all the non-orientable two-dimensional similarity symmetry groups for which tile-transitive tilings of the punctured plane with compact convex polygons exist, contain either rotations or reflections [3]. But if the group Γ^1 is orientable, the completion of such an end of the manifold M yields a countable series of non-orientable manifolds M_i. Moreover, the volumes of the manifolds M_i converge to the volume of the manifold M.

As an illustration consider a manifold obtained by the identification of faces of the octahedron O with all the vertices being on the absolute. Let us label infinitely removed vertices of the octahedron O by numbers $1, 2, 3, 4, 5, 6$. It can be easy proved that the set of such octahedra is a family with 6 parameters. Identify faces of the octahedron by motions using the following scheme:

$$(1,2,5)\,\varphi_1(4,3,5);(2,3,5)\,\varphi_2(1,4,5);$$

$$(1,2,6)\,\varphi_3(4,1,6);(3,4,6)\,\varphi_4(2,3,6).$$

As a result we obtain a non-orientable manifold with one orientable cusp and two non-orientable cusps. To do metrical calculations, we partition the octahedron O into 4 simplexes:

$T1(1,2,3,5), T2(1,3,4,5); T3(1,3,4,6); T4(1,3,2,6)$

Let dihedral angles of these tetrahedra be:

$T1(\alpha_1,\beta_1,\gamma_1)$; $T2(\alpha_2,\beta_2,\gamma_2$; $T3(\alpha_3,\beta_3,\gamma_3)$; $T4(\alpha_4,\beta_4,\gamma_4)$.

Then the identifications $\varphi_1,\varphi_2,\varphi_3,\varphi_4$ lead to a non-complete non-orientable manifold M if the dihedral angles of the above simplexes satisfy the following system of equations:

$$\alpha_1+\beta_1+\gamma_1=\pi\,,\alpha_2+\beta_2+\gamma_2=\pi\,;\ \alpha_3+\beta_3+\gamma_3=\pi;$$

$$\alpha_4+\beta_4+\gamma_4=\pi\,;\ \alpha_1+\alpha_2+\alpha_3+\alpha_4=2\pi\,;$$

$$sin\beta_1/sin\gamma_1\times sin\beta_2/sin\gamma_2=1\ sin\beta_3/sin\gamma3\times sin\beta_4/sin\gamma_4=1;$$

$$sin\beta_1/sin\alpha_1\times sin\beta_2/sin\alpha_2\times sin\alpha_3/sin\gamma_3\times sin\alpha_4/sin\gamma_4=1.$$

The obtained manifold has an orientable cusp given by the vertex 5, a non-orientable cusp of the vertex 6 and a non-orientable cusp given by the vertices $1,2,3,4$. If we require that the non-orientable cusps be complete (i.e. the discrete groups on the respective horospheres be isometry groups), then the parameters of the octahedron give the equalities: $\alpha_3=\alpha_4,\ \beta_3=\beta_4,\ \gamma_3=\gamma_4$.

Then obtain the following system of equations:

$$\begin{gathered}\alpha_1+\beta_1+\gamma_1=\pi\,,\ \alpha_2+\beta_2+\gamma_2=\pi\,,\ \alpha_3+2\times\beta_3=\pi,\\ \alpha_1+\alpha_2+2\times\alpha_3=2\pi\,,\ sin\beta_1/sin\gamma_1\times sin\beta_2/sin\gamma_2=1\,,\\ sin\beta_1/sin\alpha_1\times sin\beta_2/sin\alpha_2\times sin^2\alpha_3/sin^2\beta_3=1.\end{gathered}\tag{1}$$

Thus we obtained that 8 parameters are connected by 6 equations. Therefore we have two free parameters which can be used for the completion of the orientable cusp connected with the vertex 5.

Consider the horosphere S centered at the vertex 5. On S we obtain a two-dimensional similarity symmetry group induced by the motions φ_1 and φ_2. In order that this group to be discrete, it is necessary that it should be generated by two spiral rotations f_1 and f_2. Then to the completion of the cusp to the system of equations (1) the following equations should be added:

$$m \times \psi_1 + n \times \psi_2 = 2\pi; \; k_1^m = k_2^n,$$

where m and n are natural coprime numbers and $m + n \geqslant 5$. In these equations $\psi_!$, ψ_2 are rotation angles, k_1, k_2 are coefficients of spiral rotations f_1 and f_2. Then to the completion of the cusp to the system of equations (1) the following equations should be added:

$$m(\beta_1 - \beta_2) + n(\gamma_2 - \gamma_1) = 2\pi,$$
$$((sin\gamma_1/sin\alpha_1)\times(sin\alpha_2/sin\gamma_2))^m = ((sin\alpha_1/sin\beta_1)\times(sin\beta_2/sin\alpha_2)^n,$$

where m and n are natural coprime numbers and $m + n \geqslant 5$. Varying the numbers m and n, we obtain a countable series of non-orientable non-compact hyperbolic 3-manifolds M_{mn}, and the volumes of manifolds M_{mn} are bounded by the volume of the regular hyperbolic octahedron, all the vertices being on the absolute.

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An introduction to Hilbert depth

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Abstract

In the first part of this paper we present the Hilbert decompositions of a module M which depends only on the Hilbert function of M and an analogous notion of depth, called Hilbert depth.

In the second part we implemented in CoCoA an algorithm for computing the Hilbert depth of a module.

Keywords: commutative algebra, computer algebra, Stanley depth, Hilbert depth.

1 Introduction

Stanley decompositions of multigraded modules over polynomial rings $R = K[X_1,\dots,X_n]$ have been introduced by Stanley in [8]. They break the module M into a direct sum of graded vector subspaces, each of which is of type Sx, where x is a homogeneous element and $S = K[X_{i_1},\dots,X_{i_d}]$ is a polynomial subalgebra. Stanley conjectured that one can always find such a decomposition in which $d \geq \operatorname{depth} M$ for each summand.

One says that M has *Stanley depth* m, $\operatorname{sdepth} M = m$, if one can find a Stanley decomposition in which $d \geq m$ for each polynomial subalgebra involved, but none with m replaced by $m+1$.

In [1] the authors introduce a weaker type of decomposition in which we no longer require the summands to be submodules of M, but only vector spaces isomorphic to polynomial subrings. Evidently, such decompositions depend only on the Hilbert series of M, and therefore they are called *Hilbert decompositions*. The *Hilbert depth* $\operatorname{hdepth} M$ is defined accordingly.

In [4] the authors introduce a method for computing the Hilbert depth of a multigraded module and this work was extended in [5] where the authors improved the method and introduced an effective algorithm for performing computations. Using it they completely resolved the following questions asked by Herzog in [3]:

Problem 1. *[3, Problem 1.66] Find an algorithm to compute the Stanley depth for finitely generated multigraded R-modules M with* $\dim_K M_a \leq 1$ *for all* $a \in \mathbb{Z}^n$.

The answer can be found in [[5], Algorithm 2].

Problem 2. *[3, Problem 1.67] Let M and N be finitely generated multigraded R-modules. Then*

$$\operatorname{sdepth}(M \oplus N) \geq \operatorname{Min}\{\operatorname{sdepth}(M), \operatorname{sdepth}(N)\}.$$

Do we have equality?

Problem 3. *[3, Text following Problem 1.67] In the particular case, where* $I \subset R$ *is a monomial ideal, does* $\operatorname{sdepth}(R \oplus I) = \operatorname{sdepth} I$ *hold?*

The answear of both questions is *No*, you cand find the counterexamples in [5] Example 14 and Example 16.

2 An algorithm for computing Hdepth

A first algorithm to compute the Hilbert depth of a standard graded module was presented in [6]. We next present a CoCoA [2] function which computes the Hilbert Depth of a multigraded module using recursive backtracking, you can find the program here:

https://dl.dropboxusercontent.com/s/urhrasy5ntgbwzf/Hdepth.htm.

We performed some tests to compute the stanley depth or hilbert depth, they are equal in this case, for the maximal ideal and got better results than the algorithm implemented by Rinaldo [7] which uses iterative backtracking, see [5] Table 1. Below we present the CheckHilbertDepth function implemented in CoCoa [2] from [[5], Algorithm 1] which is called recursevely.

```
Define CheckHilbertDepth(P,E,F)
TopLevel CurrentRing;
N:=NumIndets(CurrentRing);
If TestNatural(P) = false Then Return false; EndIf; – P ∉ ℕ[X_1,...,X_N]
st:=NewList(N,1);
If len(E) = 0 Then Return true; – check if E = 0
Else
   For i:=1 To len(E) Do
      C:=FPC(E[i],F); – Find the elementes that cover E[i] with ρ = s
      If len(C) = 0 Then Return false; EndIf; – check if C = 0
      For j:=1 to Len(C) Do
         P2:=0;
         D:=E[i]-st;
         BuildInterval(C[j],D,D,0,Ref P2); – build P2 = poly[E[i],C[j]]
         P1:= P - P2;
        t:= CheckHilbertDepth(P1,Deduction(P1,P2,E),Deduction2(P1,C[j],F));
         If t= true Then PrintLn P2;
            Return true;
         EndIf;
      EndFor;
   EndFor;
Return false;
EndIf;
EndDefine;
```

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On homogeneity of topological spaces

Dumitru Ipate

Abstract

In the present paper we study distinct conditions of homogeneity of spaces. We prove that the topological product of strongly homogeneous compact spaces is strongly homogeneous.

Keywords: homogeneous space, strongly homogeneous space.

1 Introduction

Every space is considered to be a completely regular T_1-space. Give a topological space X. We denote by $cl_X A$ the closure of the subset $A \subseteq X$ in X. A subset B of X is called clopen if it is simultaneously closed and open. Denote by $w(X)$ the weight of X. A space X is said to be *zero-dimensional* if it is of small inductive dimension zero ($indX = 0$), i.e. X has a base of clopen sets. A normal space X has large inductive dimension zero ($IndX = 0$) if and only if for any two disjoint closed subsets A and B of X there is a clopen set C such that $A \subseteq C$ and $B \subseteq (X \setminus C)$. A normal space X has Lebesgue covering dimension zero ($dimX = 0$) if any finite open cover of X can be refined to a partition of X into clopen sets. It is well known that $indX \leq IndX$ for each normal space X. Moreover, $IndX = dimX$ for any metric space X and $dimX \leq indX$ for any Lindelöf space X.

A space X is called:

- a homogeneous space if for any two points $a, b \in X$ there exists a homeomorphism $g : X \longrightarrow X$ such that $g(a) = b$;

- a weight homogeneous (briefly, w-homogeneous) space if X has a base $\mathcal{B}$ such that $w(U) = w(X)$ for each $U \in \mathcal{B}$;

- a strictly homogeneous (briefly, sw-homogeneous) space if $indX = 0$ and for any non-empty clopen subset U of X there exists a homeomorphism $g : X \longrightarrow U$;

- a weakly strictly homogeneous (briefly, wsw-homogeneous) space if X has a base $\mathcal{B}$ of clopen sets such that for any two non-empty clopen subsets $U, V \in \mathcal{B}$ there exists a homeomorphism $g : V \longrightarrow U$.

The notion of strictly homogeneous space was introduced by J. van Mill in [6]. Distinct notions of homogeneity were introduced and studied in [1, 2, 3, 4].

Denote by $\mathbb{D}_\tau$ (τ is a cardinal number) the discrete space consisting of τ elements.

Any infinite compact homogeneous space is w-homogeneous. Any sw-homogeneous space is w-homogeneous.

The following assertions are obvious:

Proposition 1.1. *Let X be a sw-homogeneous space and $|X| \geq 2$. Then:*

1. The spaces X and $X \times D_\tau$ are homeomorphic for any finite cardinal $\tau \geq 1$.

2. If X is a paracompact space with a closed discrete subspace of cardinality $\tau \geq 2$, then the spaces X and $X \times D_\tau$ are homeomorphic.

2 Examples

The following examples show that the class of sw-homogeneous spaces is large. If $\tau \geq 2$, then the space $\mathbb{D}_\tau$ is homogeneous, wsw-homogeneous and not sw-homogeneous.

Example 2.1. Let $\mathbb{D} = \mathbb{D}_2$ and $\mathbb{C} = \mathbb{D}^{\aleph_0}$ be the Cantor set. The Cantor set is a compact homogeneous sw-homogeneous space.

Example 2.2. Let $\mathbb{Q}$ be the space of rational numbers and $\mathbb{J}$ be the space of irrational numbers as a subspace of real numbers $\mathbb{R}$. Then $\mathbb{Q}$ and $\mathbb{J}$ are homogeneous sw-homogeneous non-compact spaces.

Example 2.3. Let $\mathbb{A} = C_0 \cup C_1$, $C_0 = \{(t, 0) : 0 < t \leq 1\}$, $C_1 = \{(t, 1) : 0 \leq t < 1\}$ and let topology on X be generated by the base consisting of sets of the form $O_n(x, 0) = \{(x, 0)\} \cup \{(t, 0), (t, 1) \in \mathbb{A} :$

$x - 2^{-n} < t < x\}$ and $O_n(y,1) = \{(y,1)\} \cup \{(t,0),(t,1) \in \mathbb{A} : y < t < y + 2^{-n}\}$, where $(x,0),(y,1) \in \mathbb{A}$ and $n \in \mathbb{N} = \{1,2,...\}$. The space $\mathbb{A}$ is called the Alexandroff-Urysohn two arrows space (see [5], p. 270). The space $\mathbb{A}$ is compact, homogeneous and sw-homogeneous.

Example 2.4. The subspaces C_0 and C_1 of the space $\mathbb{A}$ from Example 2.3 are homeomorphic to the Sorgenfrey line $\mathbb{K}$ (see [5], Example 1.2.2, p. 39). The space $\mathbb{K}$ is Lindelöf, homogeneous, sw-homogeneous and non-compact.

3 Product of homogeneous spaces

The following theorem is the main result.

Theorem 3.1. *Let $\{X_\alpha : \alpha \in A\}$ be a family of non-empty sw-homogeneous compact spaces. Then $X = \sqcap\{X_\alpha : \alpha \in A\}$ is a sw-homogeneous space.*

Proof. It is sufficient to prove the assertion of Theorem for the case when $|A| = 2$. Assume that $A = \{1,2\}$. The natural projection $p_\alpha : X \longrightarrow X_\alpha$, $\alpha \in A$, is an open and closed continuous mapping.

Fix a non-empty clopen subset U of X. The set $U_\alpha = p_\alpha(U)$ is clopen in X_α for each $\alpha \in A$. Fix a point $(x,y) \in X_1 \times X_2$. Let $\{(V_\beta(x,y), W_\beta(x,y)) : \beta \in B(x,y)$ be the family of all pairs (V,W) of sets with the properties:

- $x \in V$ and the set V is open in X_1;
- $y \in W$ and the set V is open in X_1;
- $V \times W \subseteq W$ and $U \cap (V \times \{y\}) \subseteq (V \times W)$.

Put $V_(x,y) = \cup : V_\beta(x,y) : \beta \in B(x,y)\}$, $W_(x,y) = \cup : W_\beta(x,y) : \beta \in B(x,y)\}$ and $U(x,y) = V(x,y) \times W(x,y)$. The sets $U(x,y)$, $V_(x,y)$ and $W_(x,y)$ are clopen in X, X_1 and X_2 respectively. For any two points $(x,y),(u,v) \in U$ either $U(x,y) = U(u,v)$, or $U(x,y) \cap U(u,v) = \emptyset$. Hence, there exists a finite subset $L \subseteq X$ such that $U = \cup\{U(x,y) : (y,x) \in L\}$ and $\{U(x,y) : (y,x) \in L\}$ is a disjoint family of subsets of U. On L consider the discrete topology. The spaces X and $X \times L$ are homeomorphic. The spaces $U(x,y)$ and $X \times \{(x,y)\}$ are homeomorphic too. Therefore the spaces U, $X \times L$ and X are homeomorphic. The proof is complete.

The following assertion is obvious.

Theorem 3.2. *Let $\{X_\alpha : \alpha \in A\}$ be a family of non-empty wsw-homogeneous spaces. Then $X = \sqcap\{X_\alpha : \alpha \in A\}$ is a wsw-homogeneous space.*

Question 3.3. *Is $\mathbb{K} \times \mathbb{K}$ a sw-homogeneous space?*

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Mappings compatible with equivalence relations

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Abstract

In the present paper we consider mappings compatible with given equivalence relations and calculate their number.

Keywords: equivalence, mapping, binary relation, partition.

1 Introduction

In the theory of algebraic systems, congruences, i.e. equivalence relations which are compatible with operations from the signature of the system, play an important role. For many algebraic systems, compatibility of equivalence relations with respect to operations of algebraic system is determined by its compatibility with some mappings on the basic set of the system. This idea is used in [1] and [4]. Following the ideas formulated in [4], we show how to build all applications that are compatible with the given equivalence and evaluate the number of such applications.

2 Definitions and notations

Relations and mappings. Any ordered triplet $\alpha = (A, B, G_\alpha)$, where G_α is a subset of the Cartesian product $A \times B$ is called a binary relation between elements of the sets A and B, in that order. Often the relation is identified by G_α, called the graph of this relation. Notations $(a, b) \in \alpha$, $(a, b) \in G_\alpha$, $a\alpha b$ are equivalent and express that a and b are in the relation α. The inverse relation α^{-1} of the relation α is defined by $a\alpha b \Leftrightarrow b\alpha^{-1}a$. Denote by $a\alpha$ or $(a)\alpha$ the image of a, defined as

$a\alpha = \{b \in B | a\alpha b\}$. A binary relation $\alpha = (A, B, G_\alpha)$ is called mapping from A to B if and only if: a) $a\alpha \neq \emptyset, \forall a \in A$ and b) $a\alpha b$ and $a\alpha c$ implies $b = c$, which is equivalent to the fact that $a\alpha$ consists only of a single element, which will be denoted allso by $a\alpha$ and called the value of α in a. For mappings the common notation is $\alpha : A \to B, a \to a\alpha$. A binary relation $\alpha = (A, A, G_\alpha)$ is called a binary relation defined on A.

Relational algebra. The known Boolean operations: intersection $-$ $\cap$, union $-$ $\cup$, difference $-$ $\setminus$, complementarity denoted by "$^-$", or $\complement$ are defined on binary relations between elements of sets A and B. Also we say that $\alpha \subseteq \beta$ if and only if $G_\alpha \subseteq G_\beta$.

Composition of relations $\alpha = (A, B, G_\alpha)$ and $\beta = (C, D, G_\beta)$ is denoted by $\alpha \circ \beta$ and defined by $\alpha \circ \beta = (A, D, G_{\alpha\circ\beta})$:

$$\forall a \in A, \forall d \in D, \quad a(\alpha \circ \beta)d \Leftrightarrow (\exists b \in B \cap C, \quad a\alpha b, \quad b\beta d)$$

Any mapping $\varphi : A \to A, \ a \to a\varphi$ is a binary relation $\varphi = (A, A, G_\varphi)$ with the graph $G_\varphi = \{(a, a\varphi) | a \in A\}$. Denote by i_A the mapping $x \to x$ from A to A, or the binary relation $i_A = (A, A, G_{i_A})$ by $G_{i_A} = \{(x, x) | x \in A\}$.

Equivalences and partitions. An equivalence relation defined on A is a binary relation θ which is: reflexive ($i_A \subseteq \theta$); symmetric ($\theta \subseteq \theta^{-1}$); and transitive ($\theta \circ \theta \subseteq \theta$). In this case the image of $a\theta$ is called the equivalence class generated by the element a, or $\theta-$ class and is denoted usually by $[a]_\theta$. Two θ classes often coincide or do not intersect. The set of all equivalence classes of θ is denoted by $A_{/\theta}$ and is called the factor set of θ. The factor set $A_{/\theta}$ forms a partition P_θ of the set A, i.e.: a) $[a]_\theta \neq \emptyset, \forall a \in A$; b) $[a]_\theta = [b]_\theta \Leftrightarrow a\theta b, \ \forall a, b \in A$; c) $\bigcup_{a\in A}[a]_\theta = A$. If $\alpha : A \to A$ is a mapping, then the relation $\epsilon_\alpha = (A, A, G_{\epsilon_\alpha}) \quad G_{\epsilon_\alpha} = \{(x, y) | x\alpha = y\alpha \ x, y \in A\}$ is an equivalence relation. The factor set of this equivalence relation is denoted by $A_{/\epsilon_\alpha}$ and forms the partition P_{ϵ_α}.

Definition 1. *Let α be a mapping and θ be an equivalence relation defined on A. The mapping α is called compatible with θ (preserve relation θ), if an only if $a\theta b \Rightarrow (a\alpha)\theta(b\alpha)$, for $\forall a \forall b \in A$.*

3 Preliminary results

Proposition 1.[4] *Let σ be a mapping defined on A and θ be an equivalence relation defined on A. Then the following statements are equivalent:*
a) $a\theta b \Rightarrow (a\alpha)\theta(b\alpha)$, *for* $\forall a \forall b \in A$.
b) $\sigma^{-1} \circ \theta \circ \sigma \subseteq \theta$;
c) if $a\theta = b\theta$, *then* $a(\sigma \circ \theta) = b(\sigma \circ \theta)$, *for* $\forall a, \forall b \in A$;
d) the correspondence $\overline{\sigma} : A_{/\theta} \to A_{/\theta}$, $(a\theta)\overline{\sigma} = a(\sigma \circ \theta)$ *is a mapping on the set* $A_{/\theta}$.

We note that the condition b) transfers calculations in the algebra of binary relations and the condition c) suggests how to build mappings compatible with the equivalence relation θ.

4 Main result

We show how to find all mappings that preserve an equivalence relation and will calculate their number.

The following statements are true.

Proposition 2. *For any mapping α and any equivalence relation θ defined on the set A the equality $a(\alpha \circ \theta) = (a\alpha)\theta$ is true for $\forall a \in A$.*

Proposition 3. *For any equivalence relation θ defined on the set A and any mapping $\gamma : A_{/\theta} \to A_{/\theta}$ defined on the set $A_{/\theta}$ there is a mapping σ defined on A and compatible with θ, such that $(x\theta)\gamma = (x\theta)\overline{\sigma}$.*

Proof. For each element $Z \in A_{/\theta}$ choose one unic element $m_z \in Z$. Define the mapping $\sigma : A \to A$, $x\sigma = m_{(x\theta)\gamma}$, for all $x \in A$. The mapping σ is compatible with the equivalence θ. In fact, for $x, y \in A$ and $x\theta y$ we have $x\theta = y\theta$ and $(x\theta)\gamma = (y\theta)\gamma$. This means $m_{(x\theta)\gamma} = m_{(y\theta)\gamma}$, so $x\sigma\theta y\sigma$ since θ is reflexive. It is easy to see that $(x\theta)\gamma = (m_{(x\theta)\gamma})\theta = (x\sigma)\theta = (x\theta)\overline{\sigma}$. This means that $\gamma = \overline{\sigma}$. $\square$

So any mapping on the set $A_{/\theta}$ can be obtained in the previously indicated manner, from a mapping on the set A that preserves equivalence θ. If the set A is finite, then we can evaluate the set of all mappings defined on A which preserves the equivalence relation θ. Namely, the following theorem is true.

Theorem 1. *Let A be a finite nonempty set, θ an equivalence relation defined on A and $n = |A_{/\theta}|$, $A_{/\theta} = \{A_1, A_2, ..., A_n\}$. Then the following equality*

$$|F_\theta(A)| = \sum_{f \in F_n} \left(\prod_{k=1}^{n} \left(|A_k|^{\sum\limits_{i \in \overline{1,n}, f(i)=k} |A_i|} \right) \right),$$

holds, where F_n is the set of all mappings defined on $\{1, 2, 3, ..., n\}$, and $F_\theta(A)$ is the set of all mappings defined on A which preserve equivalence θ.

Theorem 2. *Mapping $\phi : F_\theta(A) \to F(A_{/\theta})$, $\sigma\phi = \overline{\sigma}$, $\forall \sigma \in F_\theta(A)$ is a homomorphism.*

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On absence of finite approximation relative to model completeness in the propositional provability logic

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Abstract

In the present paper we consider expressibility of formulas in the provability logic GL and related to it questions of model completeness of system of formulas. We prove the absence of finite approximation relative to model completeness in GL.

Keywords: expressibility of formulas, model completeness, provabilty logic, diagonalizable algebra.

1 Introduction

Artificial Intelligence (AI) systems simulating human behavior are often called intelligent agents. These intelligent agents exhibit somehow human-like intelligence. Intelligent agents typically represent human cognitive states using underlying beliefs and knowledge modeled in a knowledge representation language, specifically in the context of decision making [1]. In the present paper we investigate some functional properties of the underlying knowledge representation language of intelligent agents which are based on the provability logic GL [2].

The notion of expressibility of formulas was proposed in [6, 7]. In the present paper we prove that the propositional provability logic of Gödel-Löb (GL) is not finitely approximable relative to model completeness.

2 Definitions and notations

Provability logic. We consider the propositional provability logic GL, the formulas of which are based on propositinal variables $p, q, r, \ldots$ and logical connectives $\&, \vee, \supset, \neg, \Delta$, its axiomes are the classical ones together with the following Δ-formulas:

$$\Delta(p \supset q) \supset (\Delta p \supset \Delta q), \quad \Delta(\Delta p \supset p) \supset \Delta p, \quad \Delta p \supset \Delta\Delta p,$$

and the rules of inference are the rules of: 1) substitution; 2) the modus ponens, and 3) the rule of necessity, which allows to get formula ΔA if we already get formula A. The normal extensions of the propositional provability logic GL are defined as usual [2].

Diagonalizable algebras. A diagonalizable algebra [4] is a universal algebra of the form $\mathfrak{A} =< M; \&, \vee, \supset, \neg, \Delta >$, where $< M; \&, \vee, \supset, \neg >$ is a boolean algebra, and the unary operation Δ satisfies the relations

$$\Delta(\Delta x \supset x) = \Delta x,\ \Delta(x \& y) = (\Delta x \& \Delta y),\ \Delta 1_{\mathfrak{A}} = 1_{\mathfrak{A}},$$

where $1_{\mathfrak{A}}$ is the unit of $\mathfrak{A}$, which is denoted also by 1 in case the confusion is avoided.

Diagonalizable algebras are known to be algebraic models for provability logic and its extensions [5]. Obviously we can interpret any formula of the calculus of GL on any diagonalizable algebra $\mathfrak{A}$. As usual a formula F is said to be valid on $\mathfrak{A}$ if for any evaluation of variables of F with elements of $\mathfrak{A}$ the value of the formula on $\mathfrak{A}$ is $1_{\mathfrak{A}}$. The set of all valid formulas on $\mathfrak{A}$, denoted by $L\mathfrak{A}$, and referred to as the logic of the algebra $\mathfrak{A}$, forms an extension $L\mathfrak{A}$ of the provability logic GL [5].

An extension L of GL is called tabular if there is a finite diagonalizable algebra $\mathfrak{A}$ such that $L = L\mathfrak{A}$.

Expressibility and model completeness. The formula $F(p_1, \ldots, p_n)$ is a model for the Boolean function $f(x_1, \ldots, x_n)$ if for any ordered set $(\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \{0, 1\}, i = 1, \ldots, n$, we have

$F(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n)$, where logical connectors from F are interpreted in a natural way on the two-valued Boolean algebra [6, 7].

They say the formula F is expressible in the logic L via a system of formulas Σ if F can be obtained from variables and Σ, applying finitely many times 2 kinds of rules: a) the rule of weak substitution, b) the rule of passing to equivalent formula in L [3].

The system of formulas Σ is called model complete in the logic L if at least a model for every Boolean function is expressible via Σ in the logic L. System Σ is model pre-complete in L if Σ is not model complete in L but for any formula F which is not expressible in L via Σ, the system $\Sigma \cup \{F\}$ is already model complete in L [8].

The logic L is finitely approximable with respect to model completeness if for any system of formulas Σ which is not model complete in L there is a tabular extension of L in which Σ is model incomplete too.

3 Main result

Now we are able to formulate the main result of the present work.

Theorem 1. *The propositional provability logic GL is not finitely approximable with respect to model completeness.*

4 Conclusion

Taking into account our previous result [9] together with these new findings we can conclude that traditional algorithm for determining model completeness of systems of formulas in GL is impossible to find out.

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Closure operators in R-Mod: the main operations and their properties

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Abstract

The main operations in the class of all closure operators of R-Mod are studied.

Keywords: category, module, closure operator.

Let $\mathbb{CO}$ be the class of all closure operators of a module category R-Mod ([1, 2]). The main operations in $\mathbb{CO}$ are:

1) *the meet* $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$, where $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \bigcap\limits_{\alpha\in\mathfrak{A}} [(C_\alpha)_M(N)]$;

2) *the join* $\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha$, where $\big(\bigvee\limits_{\alpha\in\mathfrak{A}} C_\alpha\big)_M(N) = \sum\limits_{\alpha\in\mathfrak{A}} [(C_\alpha)_M(N)]$;

3) *the product* $C \cdot D$, where $(C \cdot D)_M(N) = C_M\big(D_M(N)\big)$;

4) *the coproduct* $C \# D$, where $(C \# D)_M(N) = C_{D_M(N)}(N)$,

for any operators $C_\alpha\,(\alpha \in \mathfrak{A}), C, D \in \mathbb{CO}$ and any submodule N of $M \in R$-Mod.

The properties of these operations are studied. In particular, the following *relations of distributivity* are proved.

Theorem 1. *For every family* $\{C_\alpha \in \mathbb{CO} \mid \alpha \in \mathfrak{A}\}$ *and* $D \in \mathbb{CO}$ *the following relations are true:*

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big) \cdot D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D), \qquad \big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\big) \cdot D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \cdot D);$$

$$(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha) \# D = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha \# D), \qquad (\bigvee_{\alpha\in\mathfrak{A}} C_\alpha) \# D = \bigvee_{\alpha\in\mathfrak{A}} (C_\alpha \# D).$$

Theorem 2. a) *If $C \in \mathbb{CO}$ is* ***hereditary****, then* $C \# \big(\bigwedge_{\alpha\in\mathfrak{A}} D_\alpha\big) = \bigwedge_{\alpha\in\mathfrak{A}} (C \# D_\alpha)$;

b) *If $C \in \mathbb{CO}$ is* ***minimal****, then* $C \cdot \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big) = \bigvee_{\alpha\in\mathfrak{A}} (C \cdot D_\alpha)$, *for every family* $\{D_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}$.

The other discussed question is the preservation of types of closure operators when the studied operations in $\mathbb{CO}$ are applied.

Theorem 3. a) *If $C_\alpha (\alpha \in \mathfrak{A}) \subseteq \mathbb{CO}$ are weakly hereditary (maximal, minimal, cohereditary), then the operator $\bigvee_{\alpha\in\mathfrak{A}} C_\alpha$ possesses the respective property.*

b) *If $C, D \in \mathbb{CO}$ are maximal (minimal, cohereditary), then the operator $C \cdot D$ possesses the respective property.*

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Generic Variational Principles in General Topological Spaces

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Abstract

Consider a bounded from below lower semicontinuous real-valued function f, defined in a completely regular topological space X. Let Y be a set of bounded continuous real-valued functions in X endowed with the topology of uniform convergence. A *variational principle* for the pair (f, Y) is any statement of the type "*the set* $S(f) := \{g \in Y : f + g$ *attains its infimum in* $X\}$ *is dense in* Y". We give here sufficient conditions for the validity of different types of variational principles for the case when Y coincides with $C^*(X)$, the space of all bounded continuous functions on X. The presentation is based on the joint work with M. M. Choban and J. P. Revalski published in [1].

Keywords: Topology, Variational principles, Optimization, Topological games.

1 Introduction

The famous Bishop-Phelps theorem [3] seems to be the first variational principle of this kind: *The set of continuous linear functionals in a real Banach space* Z *attaining their infimum on a closed bounded convex set* $X \subset Z$ *is dense in the dual Banach space* Z^*. In this case $f \equiv 0$ and $Y = Z^*$. Other examples are the Ekeland variational principle [4], Stegall variational principle [5], the smooth variational principles established in [6] (by Borwein and Preiss) and in [7] (by Deville, Godefroy and Zizler). Further we restrict our considerations only to the case when Y coincides with $C^*(X)$, the space of all bounded continuous functions on X.

2 Variational principle for completely regular topological spaces.

The topological spaces considered here are assumed to be completely regular. $\overline{A}$ denotes the closure of the set A. It is assumed also that the bounded from below lower semicontinuous function f may take $+\infty$ as value but the set $\operatorname{dom}(f) := \{x \in X : f(x) < +\infty\}$ must be nonempty. In the sequel such functions f will be called *proper*. The infimum of a function f over the set $A \subset X$ will be denoted by $\inf_A f$.

The following statement, proved in [2], is in the base of our considerations.

Proposition 1. *Let the proper function f, the point $x_0 \in \operatorname{dom}(f)$ and the number $\varepsilon > 0$ satisfy the inequality $f(x_0) < \inf_X f + \varepsilon$. Then, there exists a continuous function $g \in C^*(X)$ such that $-\varepsilon \le g(x) \le 0$ for every $x \in X$ and the function $f + g$ attains its infimum at x_0.*

Corollary 1. *Let f be a proper function. Then the set $S(f) = \{g \in C^*(X) : f + g$ attains its infimum on $X\}$ is dense in $C^*(X)$ and, for every $x_0 \in \operatorname{dom}(f)$ there exists $g \in C^*(X)$ such that $(f + g)(x_0) = \inf_X (f + g)$ (i.e. x_0 is a minimizer for the "perturbed function" $f+g$).*

Consider the set-valued mapping M_f which puts into correspondence to each function $g \in C^*(X)$ the (possibly empty) set of minimizers in X of the perturbed function $f + g$:

$$M_f(g) := \{x \in X : (f + g)(x) \le (f + g)(y) \quad \forall y \in X\}.$$

Proposition 2. *Let f be a proper function in X. Then:*

(a) *the graph $Gr(M_f) := \{(g, x) \in C^*(X) \times X : x \in M_f(g)\}$ is closed in the product topology in $C^*(X) \times X$;*

(b) *for any two open sets U of $C^*(X)$ and V of X such that $M_f(U) \cap V \neq \emptyset$, there is a nonempty open set $U' \subset U$ for which $M_f(U') \subset V$;*

(c) *if $(U_n)_{n\ge1}$ is a base of neighborhoods of $g_0 \in C^*(X)$, then $M_f(g_0) = \cap_n \overline{M(U_n)}$;*

3 Generic and almost generic variational principles

We give here sufficient conditions under which the set $S(f)$ contains a dense G_δ-subset of $C^*(X)$. In such a case the corresponding variational principle is called *Generic variational principle*. The variational principle is called *Almost generic*, if the set $S(f)$ is of the second Baire category in every open subset of $C^*(X)$. The sufficient conditions presented here for the validity of a generic (almost generic) variational principle are in terms of a two-player topological game $G(X)$ played in the space X. This game was introduced by E. Michael [8] in the study of completeness properties of metric spaces. Two players, which are denoted by Σ and Ω, play a game in the topological space X in the following way: Σ starts by choosing a nonempty subset A_1 of X and Ω makes his/her first move by choosing a nonempty relatively open subset B_1 of A_1, i.e. $B_1 = A_1 \cap W$, where W is an open subset of X. On the n-th stage, $n \geq 2$, the player Σ chooses some nonempty set A_n contained in the previous choice B_{n-1} of Ω and Ω chooses a nonempty relatively open subset B_n of A_n. Playing in this way the players generate an infinite sequence of sets $\{A_n, B_n\}_{n\geq 1}$ which is called *a play*. The player Ω is said to have won this play, if $\cap_n \overline{A}_n = \cap_n \overline{B}_n \neq \emptyset$. Otherwise Σ wins. The notion of *winning strategy* for either of the players is defined in the traditional (for the topological games) way.

Theorem 1. *Let f be a proper function in X and player Ω has a winning strategy in the game $G(X)$. Then the set $S(f)$ contains a dense G_δ-subset of $C^*(X)$.*

Theorem 2. *Let f be a proper function in X and player Σ does not have a winning strategy in the game $G(X)$. Then the set $S(f)$ is of the second Baire category in every open subset of $C^*(X)$.*

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Invariants of parastrophic equivalency

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Abstract

Some invariants of parastrophic equivalency of quasigroup functional equations are found. The number of general quadratic equations in n individual variables up to parastrophic equivalency (namely, two if $n = 2$, three if $n = 3$ and eighteen if $n = 4$) is established.

Keywords: quasigroup, functional equation, invariants, parastrophic equivalence.

1 Introduction

Let Q be a set. An operation f is called a *quasigroup* if each of the equations $f(x; a) = b$ and $f(a; y) = b$ has a unique solution for all a, $b \in Q$. *σ-parastrophe* ${}^{\sigma}f$ of f is defined by

$$ {}^{\sigma}f(x_{1\sigma}; x_{2\sigma}) = x_{3\sigma} \;:\Longleftrightarrow\; f(x_1; x_2) = x_3. $$

Two functional equations are called *parastrophically equivalent* [5] if one of them can be obtained from the other by applying a finite number of the renamings of individual or functional variables or parastrophic transformations (replacement of a functional variable with its parastrophe and the corresponding permutation of subterms).

2 Invariants of parastrophic equivalency

An element e is called a *left* (*right*, *middle*) *neutral element* of an operation f if the identity $f(e; x) = x$ (correspondingly $f(x; e) = x$, $f(x; x) = e$) is true.

Proposition 1. *Property "to have unilateral neutral element" is an invariant under parastrophic equivalency.*

We say that i-th component of solutions of functional equation $\omega = \upsilon$ has a *property of neutrality*, if i-th component of each solution of $\omega = \upsilon$ has a left, right or middle neutral element. The number of components in the solution of $\omega = \upsilon$, which have the property of neutrality, will be called a *characteristic of neutrality* of the equation.

Proposition 2. *Characteristic of neutrality is an invariant.*

We say that i-th component of equation $\omega = \upsilon$ has the *property of (commutative) group isotopy*, if i-th component of each its solution is isotopic to a (commutative) group. The number of such components is a *characteristic of (commutative) group isotopy.*

Proposition 3. *Characteristic of group isotopy is an invariant.*

Proposition 4. *Characteristic of commutative group isotopy is an invariant.*

Sequence $(n_1, n_2, \ldots, n_p)$ is called a *type* of functional equation if for all $s = 1, 2, \ldots, p$ the number n_s is equal to the number of minimal parastrophically self-sufficient sub-terms in s individual variables.

Theorem 1. *Type of a functional equation is an invariant.*

Two collections $(\mathcal{A}; \subseteq)$ and $(\mathcal{B}; \subseteq)$ of subsets of sets A and B are called *strongly isomorphic*, if there is a one-to-one correspondence φ between the sets A and B, such that a correspondence φ' defined by $\varphi'(x) := \{\varphi(x) \mid x \in X\}$, is an isomorphism between ordered collections $(\mathcal{A}; \subseteq)$ and $(\mathcal{B}; \subseteq)$.

Theorem 2. *If two functional equations are parastrophically equivalent, then their collections of parastrophically self-sufficient subsets of individual variables are strongly isomorphic.*

Corollary. *The number of (minimal) parastrophically self sufficient subsets of individual variables of a functional equation is invariant under parastrophic equivalency.*

3 Classification of functional equations

Full classification of all general quadratic equations, which have two and three individual variables was done in [3], namely, the following theorems were proved.

Theorem 3. *Each general quadratic functional equation in two individual variables is parastrophically equivalent to exactly one of the equations* $F_1(x;x) = F_2(y;y)$, $F_1(x;y) = F_2(x;y)$.

Theorem 4. *Each general quadratic functional equation in three individual variables is parastrophically equivalent to exactly one of the equations*

$$F_1(F_2(x;x);F_3(y;y)) = F_4(z;z),$$

$$F_1(F_2(x;y);F_3(x;y)) = F_4(z;z); \quad F_1(F_2(x;y);z) = F_3(F_4(x;y);z).$$

Theorem 5. *There exist eighteen general quadratic parastrophically non-equivalent functional equations: two equations are irreducible (mediality and pseudo-mediality); four equations are parastrophically reducible, but they have no self-sufficient sub-term and twelve functional equations which have self-sufficient sub-terms.*

A quasigroup functional equation

$$F_1(F_2(F_3(x;y);F_4(z;t));u) = F_5(F_6(F_7(x;z);u);F_8(y;t))$$

is parastrophically irreducible according to Krapež and Živković's definition of parastrophical irreducibility [4] and according to Belousov's definition, but it is reducible according to Sokhatsky's definition:

an equation is called *reducible*, if it has a non-trivial sequence of self-sufficient sub-terms. Functional equation is called *parastrophically reducible*, if it is parastrophically equivalent to a reducible equation.

Theorem 6. *Each parastrophically irreducible quadratic functional equation in five individual variables is parastrophically equivalent to exactly one of the following functional equations*

$$F_1(F_2(F_3(x;y);F_4(z;t));u) = F_5(F_6(F_7(x;z);F_8(y;u));t),$$

$$F_1(F_2(F_3(x;y);F_4(z;t));u) = F_5(F_6(F_7(x;z);t);F_8(y;u)),$$
$$F_1(F_2(F_3(x;y);F_4(z;t));u) = F_5(F_6(F_7(x;u);z);F_8(y;t)).$$

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About classification of quasigroups according to symmetry groups

Halyna Krainichuk

Abstract

Necessary and sufficient conditions for a group isotopes to be totally symmetric, commutative, left symmetric, right symmetric, skew symmetric and asymmetric are defined. An identity which describes a class of skew symmetric quasigroups is found. The respective variety is not a subvariety of the variety of totally symmetric quasigroups.

Keywords: qroup, quasigroup, isotope, totally symmetric, (left, right, skew) symmetric, asymmetric, commutative, identity.

1 Introduction

A groupoid $(Q;\cdot)$ is called a quasigroup if for all $a, b \in Q$ every of $x \cdot a = b$ and $a \cdot y = b$ has a unique solution. For every $\sigma \in S_3$ σ-parastrophe $(\overset{\sigma}{\cdot})$ is defined by

$$x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} \iff x_1 \cdot x_2 = x_3.$$

A groupoid $(Q;\cdot)$ is called an *isotope of a groupoid* $(Q;+)$, if and only if there exists a triple (α, β, γ) of bijections, called an *isotope* such that the relation $x \cdot y := \gamma^{-1}(\alpha x + \beta y)$ holds. An isotope of a group is called a *group isotope*.

Many authors consider classification of quasigroups according to the number of their different parastrophes (see, for example, [2], [1]). But as it was shown in [3] the number depends on the symmetry group $\mathrm{Sym}(\cdot)$ of a quasigroup $(Q;\cdot)$. Since $\mathrm{Sym}(\cdot)$ is a subgroup of the symmetric group $S_3 := \{\iota, \ell, r, s, s\ell, sr\}$, where $s := (12)$, $\ell := (13)$, $r := (23)$, then there are six classes of quasigroups according to the classification:

- *totally symmetric*, if $\mathrm{Sym}(\cdot) = S_3$ it is described by laws $xy = yx$ and $x \cdot xy = y$, all parastrophes coincide;
- *skew symmetric* or *semi-symmetric*, if $\mathrm{Sym}(\cdot) \supseteq A_3$ it is described by $x \cdot yx = y$, all parastrophes coincide with $(Q;\cdot)$ or $(Q;\overset{s}{\cdot})$;
- *commutative*, if $\mathrm{Sym}(\cdot) \supseteq \{\iota, s\}$ it is described by $xy = yx$, all parastrophes coincide with $(Q;\cdot)$ or $(Q;\overset{\ell}{\cdot})$ or $(Q;\overset{r}{\cdot})$;
- *left symmetric*, if $\mathrm{Sym}(\cdot) \supseteq \{\iota, r\}$ it is described by $x \cdot xy = x$, all parastrophes coincide with $(Q;\cdot)$ or $(Q;\overset{\ell}{\cdot})$ or $(Q;\overset{s}{\cdot})$;
- *right symmetric*, if $\mathrm{Sym}(\cdot) \supseteq \{\iota, \ell\}$ it is described by $xy \cdot y = y$, all parastrophes coincide with $(Q;\cdot)$ or $(Q;\overset{s}{\cdot})$ or $(Q;\overset{r}{\cdot})$;
- *asymmetric*, if $\mathrm{Sym}(\cdot) = \{\iota\}$, all parastrophes are different.

The following problem is natural:

In every of these classes to describe a subclass of group isotopes.

Partial answer was given in [1].

2 Isotopes of groups

Theorem 1. *A group isotope $(Q;\cdot)$ is totally symmetric if and only if there exists a commutative group $(Q;+)$ and an element a such that $x \cdot y = -x - y + a$.*

Theorem 2. *A group isotope $(Q;\cdot)$ is skew symmetric if and only if there exists a group $(Q;+)$, its anti-automorphism α and an element*

$a \in Q$ such that $x \cdot y = \alpha x + a + \alpha^{-1} y$ and $\alpha^3 = I_a^{-1} J$, $\alpha a = -a$, where $J(x) := -x$ and $I_a(x) := -a + x + a$.

Theorem 3. *A quasigroup $(Q; \cdot)$ satisfies the identity $xy = (y \cdot xz) \cdot z$ if and only if there exists a group $(Q; +)$ of exponent 2, its automorphism α and an element $a \in Q$ such that $x \cdot y = \alpha x + a + \alpha^{-1} y$ and $\alpha^3 = \iota$, $\alpha a = -a$.*

Corollary. *Every quasigroup $(Q; \cdot)$ which satisfies the identity $xy = (y \cdot xz) \cdot z$ is skew symmetric.*

A variety of quasigroups $(Q; \cdot)$ satisfying $xy = (y \cdot xz) \cdot z$ is not a subvariety of the variety of totally symmetric quasigroups because $(Q; \cdot)$, which is defined by $x \cdot y := 3x + 5y$ over Z_7, satisfies the identity $xy = (y \cdot xz) \cdot z$ and is not commutative.

Theorem 4. *A group isotope $(Q; \cdot)$ is commutative if and only if there exists a commutative group $(Q; +)$, a unitary permutation α and an element $a \in Q$ such that $x \cdot y = \alpha x + \alpha y + a$.*

Theorem 5. *A group isotope $(Q; \cdot)$ is left symmetric if and only if there exists an abelian group $(Q; +)$ and a permutation α such that $x \cdot y = \alpha x - y$.*

Theorem 6. *A group isotope $(Q; \cdot)$ is right symmetric if and only if there exists an abelian group $(Q; +)$ and a permutation β such that $x \cdot y = -x + \beta y$.*

Theorem 7. *A group isotope $(Q; \cdot)$ is asymmetric if and only if there exists a group $(Q; +)$, unitary permutations α, β and an element $a \in Q$ such that $x \cdot y = \alpha x + a + \beta y$ holds and $\alpha \neq \beta$, $\alpha \neq \beta^{-1}$, $\alpha \neq J$, $\beta \neq J$, where $\alpha 0 = \beta 0 = 0$.*

3 Conclusion

In this paper varieties of symmetric, commutative, left symmetric, right symmetric, skew symmetric and asymmetric group isotopes are described. An identity which describes a class of skew symmetric quasigroups is found. This variety is not a subvariety of the variety of totally symmetric quasigroups. This is a solution of Fedir Sokhatsky's problem (see [4]).

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Autotopies and automorphisms of distinguished loop transversal in a sharply 2-transitive permutation group

Eugene Kuznetsov

Abstract

The properties of one distinguished loop transversal in a sharply 2-transitive permutation group are investigated. The groups of automorphisms and all autotopies of loop transversal mentioned above are studied.

Keywords: transversal, loop transversal, automorphism, autotopy, permutation group.

1 Introduction

Sharply 2-transitive permutation groups on a finite set of symbols were described by Zassenhaus in [1, 2]. He proved (see [4]), that sharply 2-transitive permutation group G on a finite set of symbols E is a group G^* of linear transformations of some near-field $< E, +, \bullet >$:

$$G^* = \{\alpha_{a,b} \mid \alpha_{a,b}(t) = a \cdot t + b, \;\; a \neq 0, \;\; a, b, t \in E\}.$$

In the case when the set E is infinite, the problem of description of sharply 2-transitive permutation groups on E is still opened. Some investigations in this direction were pursued in [5, 6]. The same problem was formulated by Mazurov in [7], problem 11.52.

In this work the author studies this problem by the help of transversals in groups and their invariant transformations.

2 A partition on cases

Let G be a sharply 2-transitive permutation group on an arbitrary set E.

Lemma 1. *All elements of order* 2 *from* G *are in one and the same class of conjugate elements.*

Proof. It was given in [4]. $\square$

Since G is a sharply 2-transitive permutation group, then only the identity permutation **id** fixes more than one symbol from E. So we obtain the following two cases:

Case 1. *Every element of order 2 from* G *is a fixed-point-free permutation on* E.

Case 2. *Every element of order 2 from* G *has exactly one fixed point from* E.

Lemma 2. *Let* α *and* β *be distinct elements of order 2 from* G. *Then the permutation* $\gamma = \alpha\beta$ *is a fixed-point-free permutation on* E.

Proof. It was given in [4]. $\square$

Let 0 be some distinguished element from E. Denote $H_0 = St_0(G)$.

3 A loop transversal in group G and its properties

Lemma 3. *The following properties are true for the group* G*:*

1. *In both cases 1 and 2 there exists a left transversal* T *in* G *to* H_0*, which consists from* ***id*** *and fixed-point-free permutations;*
2. *Transversal* T *is a normal (invariant) subset in the group* G*;*
3. *Set* T *is a loop transversal in* G *to* H_0*, and the corresponding transversal operation* $< E, \cdot, 0 >$ *is a loop.*

Proof. It was given in [3]. □

In this report the following results are shown.

Theorem 1. *The following properties are fulfilled on* $< E, \cdot, 0 >$*:*

1. *The system* $< E, \cdot, 0 >$ *is a left G-loop.*

2. *The system* $< E, \cdot, 0 >$ *is a left special loop.*

Proof. In the case 1 it was given in [3]. □

Theorem 2. *Automorphism group* $Aut\,(< E, \cdot, 0 >)$ *contains a sharply transitive subgroup* Aut_0 *such that* $Aut_0 \simeq H_0$*.*

Theorem 3. *Group of autotopies* $Avt\,(< E, \cdot, 0 >)$ *contains a sharply 2-transitive subgroup* Avt_0 *such that* $Avt_0 \simeq G$ *or system* $< E, \cdot, 0 >$ *is a group.*

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A loop which can be presented as a semidirect product of two groups

Eugene Kuznetsov

Abstract

In this work an example of right Bol loop, which can be considered as a semidirect product of two groups, is given.

Keywords: transversal, loop transversal, semidirect product.

1 Introduction

A construction of a semidirect product of two algebraic systems in general case is a conversion of a construction of factorisation of one algebraic object by its suitable subobject. A construction of a semidirect product of two algebraic systems is well known as for groups (standard semidirect product) as for loops and suitable permutation groups (Sabinin's product [2]). The result of this product is always a group. So it would be very interesting to generalize a construction of semidirect product on a class of loops such that the result of the product would be a non-associative loop.

Some investigations in this brunch can be found in [3], [4], [5].

2 A transversal in loop to its suitable subloop

A semidirect product of a left loop and suitable permutation group is a conversion of a construction of left transversal in a group to its subgroup (see [1], [2]). So in order to construct a semidirect product of a left loop and suitable loop it is necessary to study a construction of left transversal in a loop to its suitable subloop. It was done by the author in [6].

In order to define correctly the notion of a left transversal in a loop to its proper subloop, it is necessary that the following condition be fulfilled.

Definition 1. *(**Left Condition A**) A product at the left of an arbitrary element a of a loop L on an arbitrary left coset R_i of a loop L by its proper subloop R is a left coset of the loop L by its proper subloop R too, i.e. for every $a, b \in L$ there exists an element $c \in L$ such that*

$$a(bR) = cR. \tag{1}$$

Definition 2. *([1]) Let $\langle L, \cdot, e\rangle$ be a loop, $\langle R, \cdot, e\rangle$ be its subloop and a left **Condition A** is fulfilled. Let $\{R_x\}_{x\in E}$ is a set of all left cosets in L to R that form a left coset decomposition of the loop L. A set $T = \{t_x\}_{x\in E} \subset L$ is called a **left transversal** in L to R if T is a complete set of representatives of the left cosets R_x in L to R, i.e. there exists a unique element $t_x \in T$ such that $t_x \in R_x$ for every $x \in E$ (we assume that $t_1 = e$).*

Theorem 1. *There exists an example of the right Bol loop L of order 8 such that the following conditions hold:*

1. *There exists the subgroup R of order 2 in the loop L;*
2. *A left Condition A is fulfilled for the loop L and its subloop R;*
3. *There exists the transversal T in L to R such that the transversal operation $\langle E, \overset{(T)}{\cdot}, 1\rangle$ is a group of order 4.*

As a construction of left transversal in a loop to its suitable subloop can be potentially converted by the help of a construction of semidirect product of a left loop and suitable permutation loop, so the Bol loop mentioned above may be represented as a semidirect product of two groups. So the last theorem may be reformulated by the following way.

Theorem 2. *There exist two finite groups: a cyclic group R of order 2 and non-cyclic abelian group T of order 4 such that the following conditions hold:*

1. *There exists a loop L of order 8 which contains a subgroup R' isomorphic to the group R;*
2. *A left Condition A is fulfilled for the loop L and its subloop R';*
3. *There exists the transversal T' in L to R' such that the transversal operation $\langle E, \overset{(T')}{\cdot}, 1\rangle$ is a group isomorphic to the group T;*
4. *The loop L may be represented as a "semidirect" product of two groups R and T.*

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Weakly prime rings

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Abstract

The communication contains some results on weakly prime rings.

Keywords: ideal, weakly prime ring, quiver.

1 Preliminary results

Main definition. *A ring A is called weakly prime if the product of any two nonzero ideals not contained in R is nonzero, where R is the Jacobson radical of a ring A* [1].

Proposition 1. *If e is a nonzero idempotent of a weakly prime ring, then eAe is weakly prime.*

2 Main results

Main theorem for weakly prime semiperfect rings. *Let $1 = e_1 + \ldots + e_n$ be a decomposition of the unity of a semiperfect ring A into a sum of mutually orthogonal local idempotents and $A_{ij} = e_i A e_j$ $(i, j = 1, \ldots, n)$. The ring A is weakly prime if and only if $A_{ij} \neq 0$ for $(i, j = 1, \ldots, n)$.*

Theorem 1. *The quiver $Q(A)$ of a weakly prime semiperfect Noetherian ring A is strongly connected.*

Theorem 2. *Let A be a tiled order, then the quotient ring $B = A/\pi A$ is weakly prime Noetherian and semiperfect.*

Theorem 3. *Let Q be an arbitrary simply laced quiver without loops. There exists a weakly prime semidistributive Artinian ring B such that $Q(B) = Q$.*

Theorem 3. *For any simply laced quiver Q with n vertices and without loops and for any field k there exists a weakly prime semidistributive n^2-dimensional algebra B over k such that $Q(B) = Q$.*

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On recursive differentiability of binary quasigroups

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Abstract

A quasigroup is called recursively n-differentiable if its first n recursive derivatives are quasigroups. The class of recursively differential quasigroups is arisen in the theory of MDS codes, in early 2000. Connections between recursive derivatives of different order are found in the present work. It is shown that isomorphic quasigroups have isomorphic recursive derivatives of any order. Also, it is proved that, if the recursive derivative of order one of a finite quasigroup $(Q,\cdot)$ is commutative, then its group of inner mappings is a subgrup of the group of inner mappings of $(Q,\cdot)$, of the same index as their corresponding multiplication groups.

Keywords: recursively differential quasigroup, recursive derivative, the group of inner mappings.

If $(Q,\cdot)$ is a binary quasigroup, then the operations $(\overset{i}{\cdot})$, defined as follows:

$$x \overset{0}{\cdot} y = x\cdot y,\; x \overset{1}{\cdot} y = y\cdot(x \overset{0}{\cdot} y),\; x \overset{2}{\cdot} y = (x \overset{0}{\cdot} y)\cdot(x \overset{1}{\cdot} y),\; \ldots,$$
$$x \overset{n}{\cdot} y = (x \overset{n-2}{\cdot} y)\cdot(x \overset{n-1}{\cdot} y),$$

$\forall n \geq 2$, and $\forall x,y \in Q$, are called the recursive derivatives of the operation $(\cdot)$, or of the quasigroup $(Q,\cdot)$. A quasigroup $(Q,\cdot)$ is called recursively n-differentiable if its first n recursive derivatives are quasigroup operations. The notions of recursive derivative and recursively differentiable quasigroup arose in the theory of recursive MDS (maximum distance separable) codes [2]. The recursive derivatives of a quasigroup are not always quasigroups. A necessary and sufficient condition when a finite abelian group is recursively s-differentiable is given in [3], for an arbitrary positive integer s. It is known that there exist recursively 1-differentiable finite quasigroups of order q, for every integer q, excepting $q = 6$ and possibly $q \neq 14, 18, 26$ (see [2,3]). Connections between recursive derivatives of different order are found in the present work. We show that isomorphic quasigroups have isomorphic recursive derivatives of any order and that the group of automorphisms of a quasigroup is a subgroup of the group of automorphisms of all its recursive derivatives. Also, it is proved that, if the recursive derivative $(Q, \circ)$ of order one of a finite quasigroup $(Q, \cdot)$ is commutative, then its group of inner mappings is a subgroup of the group of inner mappings of $(Q, \cdot)$ (with respect to the same element $h \in Q$) and $|M(Q, \cdot) : M(Q, \circ)| = |I_h^{(\cdot)} : I_h^{(\circ)}|$.

Proposition 1. *Let $(Q, \cdot)$ be a binary groupoid and $n \geq 2$ be a fixed positive integer. Then, for $\forall j = 1, \ldots, n-1$ and for $\forall x, y \in Q$, the following equality holds:*

$$x \overset{n}{\cdot} y = (x \overset{j-1}{\cdot} y) \overset{n-j-1}{\cdot} (x \overset{j}{\cdot} y). \tag{1}$$

Proof. We will use the mathematical induction. If $n = 2$, then $j = 1$ and: $x \overset{2}{\cdot} y = (x \overset{0}{\cdot} y) \overset{0}{\cdot} (x \overset{1}{\cdot} y)$, $\forall x, y \in Q$. Suppose that (1) is true for all natural numbers $2 \leq n \leq k$. Then, for $n = k+1$, we get:

$$x \overset{k+1}{\cdot} y = (x \overset{k-1}{\cdot} y) \overset{0}{\cdot} (x \overset{k}{\cdot} y) =$$
$$[(x \overset{j-1}{\cdot} y) \overset{(k-1)-(j+1)}{\cdot} (x \overset{j}{\cdot} y)] \overset{0}{\cdot} [(x \overset{j-1}{\cdot} y) \overset{k-(j+1)}{\cdot} (x \overset{j}{\cdot} y)] =$$
$$(x \overset{j-1}{\cdot} y) \overset{(k+1)-(j+1)}{\cdot} (x \overset{j}{\cdot} y). \ \square$$

Proposition 2. *Let $(Q, \cdot)$ be a binary groupoid. Then, for every positive integer n and $\forall x, y \in Q$, the following equality holds:*

$$x \overset{n}{\cdot} y = y \overset{n-1}{\cdot} (x \overset{0}{\cdot} y) \tag{2}$$

Proof. We will use the mathematical induction. If $n = 1$, then $x \overset{1}{\cdot} y = y \overset{0}{\cdot} (x \overset{0}{\cdot} y)$. Suppose that the equality (2) is true for all positive integers $n \leq k$. Then, for $n = k+1$, we get:

$$x \overset{k+1}{\cdot} y = (x \overset{k-1}{\cdot} y) \overset{0}{\cdot} (x \overset{k}{\cdot} y) =$$
$$= [y \overset{k-2}{\cdot} (x \overset{0}{\cdot} y)] \overset{0}{\cdot} [y \overset{k-1}{\cdot} (x \overset{0}{\cdot} y)] = y \overset{k}{\cdot} (x \overset{0}{\cdot} y),$$

for every $x, y \in Q$. $\square$

Proposition 3. *If two binary quasigroups $(Q, \cdot)$ and $(Q_1, \circ)$ are isomorphic, then their recursive derivatives $(Q, \overset{n}{\cdot})$ and $(Q_1, \overset{n}{\circ})$ are isomorphic, for every natural n.*

Proof. If φ is an isomorphism from $(Q, \cdot)$ to $(Q_1, \circ)$, then $\varphi(x \overset{1}{\cdot} y) = \varphi[y \cdot (x \cdot y)] = \varphi(y) \circ (\varphi(x) \circ \varphi(y)) = \varphi(x) \overset{1}{\circ} \varphi(y)$, $\varphi(x \overset{2}{\cdot} y) = \varphi[(x \cdot y) \cdot (x \overset{1}{\cdot} y)] = [(\varphi(x) \cdot \varphi(y)) \cdot (\varphi(x) \overset{1}{\cdot} \varphi(y))] = \varphi(x) \overset{2}{\circ} \varphi(y)$, ..., $\varphi(x \overset{n}{\cdot} y) = \varphi[(x \overset{n-2}{\cdot} y) \cdot (x \overset{n-1}{\cdot} y)] = (\varphi(x) \overset{n-2}{\circ} \varphi(y)) \circ (\varphi(x) \overset{n-1}{\circ} \varphi(y)) = \varphi(x) \overset{n}{\circ} \varphi(y)$, $\forall x, y \in Q$, i.e. φ is an isomorphism from $(Q, \overset{n}{\cdot})$ to $(Q_1, \overset{n}{\circ})$. $\square$

Proposition 4. *If $(Q, \cdot)$ is a quasigroup, then $Aut(Q, \cdot)$ is a subgroup of $Aut(Q, \overset{n}{\cdot})$, for every natural n.*

Proof. Let $\varphi \in Aut(Q, \cdot)$. Then $\varphi(x \overset{1}{\cdot} y) = \varphi(y \cdot (x \cdot y)) = \varphi(y) \cdot (\varphi(x) \cdot \varphi(y)) = \varphi(x) \overset{1}{\cdot} \varphi(y)$. So $Aut(Q, \cdot) \leq Aut(Q, \overset{1}{\cdot})$. Now, suppose that $Aut(Q, \cdot) \leq Aut(Q, \overset{i}{\cdot})$, $\forall i = 0, 1, ..., n-1$. Then,

$$\varphi(x \overset{n}{\cdot} y) = \varphi((x \overset{n-2}{\cdot} y) \cdot (x \overset{n-1}{\cdot} y)) = \varphi(x \overset{n-2}{\cdot} y) \cdot \varphi(x \overset{n-1}{\cdot} y) =$$
$$(\varphi(x) \overset{n-2}{\cdot} \varphi(y)) \cdot (\varphi(x) \overset{n-1}{\cdot} \varphi(y)) = \varphi(x) \overset{n}{\cdot} \varphi(y),$$

hence $Aut(Q, \cdot) \leq Aut(Q, \overset{n}{\cdot})$, for every natural number n. $\square$

Let $(Q, \cdot)$ be a quasigroup and let $M(Q, \cdot)$ be its multiplication group. Following [1], we will denote the group of inner mappings of $(Q, \cdot)$, with respect to an element $h \in Q$, by $I_h^{(\cdot)}$.

Proposition 5. *If $(Q, \cdot)$ is a quasigroup with a commutative recursive derivative $(Q, \circ)$ of order one, then $I_h^{(\cdot)}$* is a subgroup of $I_h^{(\circ)}$.

Proof. So as $R_x^{(\circ)}(y) = y \circ x = x \cdot (y \cdot x) = L_x^{(\cdot)} R_x^{(\cdot)}(y)$, $\forall x, y \in Q$, we get that $RM(Q, \circ) \subseteq M(Q, \cdot)$. If the recursive derivative $(Q, \circ)$ is commutative, then $M(Q, \circ) = RM(Q, \circ) \subseteq M(Q, \cdot)$. So $I_h^{(\circ)} = M(Q, \circ)_h \leq M(Q, \cdot)_h = I_h^{(\cdot)}$, where $M(Q, \circ)_h$ (resp. $M(Q, \cdot)_h$) is the centralizer of h in $M(Q, \circ)$ (resp., in $M(Q, \cdot)$). $\square$

Corollary. *If $(Q,\cdot)$ is a finite quasigroup with a commutative recursive derivative $(Q,\circ)$ of order one, then*

$$|M(Q,\cdot):M(Q,\circ)| = |I_h^{(\cdot)} : I_h^{(\circ)}|.$$

Proof. According to the previous proposition, if the recursive derivative $(Q,\circ)$ of order one is commutative, then $I_h^{(\circ)}$ is a subgroup of $I_h^{(\cdot)}$ and $M(Q,\circ)$ is a subgroup of $M(Q,\cdot)$. Now, using the equality $|M(Q,\cdot)| = |Q|\cdot|I_h^{(\cdot)}|$, we have:

$$|Q| = \frac{|M(Q,\cdot)|}{|I_h^{(\cdot)}|} = \frac{|M(Q,\circ)|}{|I_h^{(\circ)}|} \Rightarrow \frac{|M(Q,\cdot)|}{|M(Q,\cdot)|} = \frac{|I_h^{(\circ)}|}{|I_h^{(\circ)}|},$$

which implies $|M(Q,\cdot):M(Q,\circ)| = |I_h^{(\cdot)} : I_h^{(\circ)}|$. $\square$

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The minor groups of $\bar{6}$-symmetry, generated by tablet groups

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Abstract

In this paper all possible minor groups of $\bar{6}$-symmetry which are generated by the discrete tablet groups are drawn and are described.

Keywords: generalized symmetries, groups, right quasi-homomorphisms.

1 Introduction

The theory of symmetry of the real crystals gives rise to new generalizations of classical symmetry: the Shubnikov's antisymmetry, the multiple antisymmetry, the Belov's colour symmetry, the magnetic symmetry, the Zamorzaev's P-symetry [1], the cryptosymmetry, e.t.c. We shall discuss briefly the essence of the $\overline{P}$-symmetry [2-4].

2 General theory

Let P be a finite, transitive group of permutations on the set $N = \{1, 2, ..., m\}$ and let G be the discrete symmetry group of a geometrical figure F. The "indexes" r of the set N have a non-geometrical nature. Ascribe to each point M of the FG-domain F_i (to each fixed i) at least one "index" r from the set N. We obtain an "indexed" geometrical figure $F^{(N)}$. Let the "indexes" r from the set N be homogeny oriented magnitudes (vectors, tensors) and they are rigidly connected with the points.

The transformation of $\overline{P}$-symmetry is an isometric mapping $g^{(p)} = pg$ of $F^{(N)}$ onto itself in which the geometrical component g operates both on points and on "indexes" by the given rule, which does not depend on the points, but the permutation p is only a supplementary permutation of "indexes" which maps $F^{(N)}$ onto itself and $p \in P$. The components p and g of the transformation $g^{(p)}$, in general, do not commute: $pg \neq gp$, that is why $p \neq gpg^{-1}$.

The set $G^{(P)}$ of transformations of $\overline{P}$-symmetry of any "indexed" figure $F^{(N)}$ forms a group with the operation $g_i^{(p_i)} * g_j^{(p_j)} = g_k^{(p_k)}$, where $g_k = g_i g_j$, $p_k = p_i g_i p_j g_i^{-1}$. The groups $G^{(P)}$ of $\overline{P}$-symmetry are subgroups, which verify certain conditions, of the right semi-direct products of the defining group P with the generating group G, accompanied by a homomorphism $\varphi : G \rightarrow AutP$, where $Ker\varphi = H$ and $G/H \cong \Phi \leq AutP$. The set $P' = \{p|g^{(p)} \in G^{(P)}\}$ is a subset of the group P. Moreover, $e \subseteq P' \subseteq P$.

Let us have two groups G and P and a homomorphism $\varphi : G \rightarrow AutP$. The mapping ψ of the group G onto the subset P' of the group P, defined by the rule $\psi(g) = p$, is called a *right quasi-homomorphism* if, for any g_i and g_j from G,

$$\psi(g_i g_j) = \psi(g_i) \overrightharpoon{\varphi_{g_i}} [\psi(g_j)] = p_i \overrightharpoon{\varphi_{g_i}} (p_j) = p_k, \tag{1}$$

where $p_i, p_j, p_k \in P'$, $\overrightharpoon{\varphi_{g_i}} = \varphi(g_i)$ and $\overrightharpoon{\varphi_{g_i}} (p_j) = g_i p_j g_i^{-1}$. The mapping φ is called the accompanying homomorphism of right conjugation.

If ψ is a right quasi-homomorphism, in general, the image of G $\psi(G) = P' \subset P$ is not a group, but P' always contains the unit of the group P. The kernel H of the right quasi-homomorphism ψ of the group G into the group P is a subgroup in G; the index of this subgroup coincides with the order of $\psi(G) = P'$.

Let $G^{(P)}$ be a group of $\overline{P}$-symmetry with the defining group P, generating group G, the subset $P' = \{p|g^{(p)} \in G^{(P)}\}$, the kernel H of accompanying homomorphism $\varphi : G \rightarrow AutP$, the symmetry subgroup H' ($H' = G^{(P)} \bigcap G$) and the subgroup Q of P-identical transformations ($Q = G^{(P)} \bigcap P' = G^{(P)} \bigcap P$). Then: 1) the mapping φ of the group $G^{(P)}$ onto the generating group G, defined by the rule $\varphi[g^{(p)}] = g$, is a homomorphism with the kernel Q; 2) the group $G^{(P)}$ contains, as its subgroup, a group $H^{(P)}$ of P-symmetry (which is determined by initial defining group P of permutations) with the symmetry subgroup $H^{''} = H \bigcap H^{'}$ and with the same subgroup Q of P-identical transformations.

Moreover, if P is a finite group of permutations, then $\overrightarrow{\varphi_g}(Q) = p^{-1}Qp$, where g and p are components of the transformation $g^{(p)}$ from $G^{(P)}$.

3 A method of deducing minor groups

A group $G^{(P)}$ of $\overline{P}-$symmetry is called a minor group if it satisfies the following conditions: $e = Q < P' = P$.

The minor groups of $\overline{P}-$symmetry are derived from the groups P and G, when the kernel H of accompanying homomorphism φ is known, by the following steps: 1) we find in G all proper subgroups H' with the index equal to the order of P and for which there is an isomorphism χ of the quotient-group H/H'' to P'' ($\chi : H/H'' \to P''$ defined by the rule $\chi(hH'') = p$), where $e \leq P'' \leq P$ and $H'' = H' \cap H$; 2) we construct a right quasi-homomorphism ψ of the group G onto P by the rule $\psi(gH') = p$, and which preserves the correspondence between the elements of H and P'', defined by the isomorphism χ; 3) we combine pairwise each g' of gH' with $p = \psi(g')$; 4) we introduce on the set of all these pairs the operation $p_i g_i * p_j g_j = p_k g_k$, where $g_k = g_i g_j$, $p_k = p_i \overrightarrow{\varphi_{g_i}}(p_j)$, $\overrightarrow{\varphi_{g_i}} = \varphi(g_i)$ and $\overrightarrow{\varphi_{g_i}}(p_j) = g_i p_j g_i^{-1}$.

For the minor groups of $\overline{P}$-symmetry the polynomial symbols are elaborated. These symbols describe their structure. Namely: $G|H'[(P,P_i)|P''; H/H'''/H'']$, where G is the generating group of classical symmetry, H' is the subgroup of symmetry, (P,P_i) is the symbol of the defining group P with the stationary subgroup P_i (we use the symbols of Schonflies for the crystallographic point groups isomorphic to them), the set $P'' = \{p|g^{(p)} \in H^{(P)}\}$ is a subgroup of the group P, $H/H'''/H''$ is the trinomial symbol of the subgroup $H^{(P)}$ of P-symmetry. Remark that $H/H'' \cong P''$ and $H'''/H'' \cong P_i^{'} (= P_i \cap P'')$.

4 Conclusion

From the 31 crystallographic tablet generating groups G we obtained 118 different minor groups of $\bar{6}$-symmetry ($P \cong C_6$), only 21 of which are nonequivalent groups.

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Provably Sender-Deniable Encryption Scheme

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Abstract

The use of the well known provably secure public-key cryptoscheme by Rabin is extended for the case of the deniable encryption. In the proposed new sender-deniable encryption scheme the cryptogram is computed as coefficients of the quadratic congruence, roots of which are two simultaneously encrypted texts. One of the texts is a fake message and the other one is the ciphertext produced by the public-key encryption of the secret message. The proposed deniable encryption method produces the ciphertext that is computationally indistinguishable from the ciphertext produced by some probabilistic public-key encryption algorithm applied to the fake message.

Keywords: cryptography, ciphering, deniable encryption, public key, public encryption, probabilistic encryption

1 Introduction

The article [1] introduces the notion of public-key deniable encryption as cryptographic primitive that resist the attacks performed by the coercive adversary that intercepts the ciphertext and has power to force sender, receiver, or the both parties to open the ciphertext and disclose the encryption key and the random values used while encrypting the plaintext. There are considered sender-deniable [2], receiver-deniable [3], and sender- and- receive-deniable (bi-deniable) [4] schemes in which coercive adversary attacks the party sending message, the party receiving message, and the both parties, respectively. The encryption is deniable if the sender or/and receiver have possibility not to open the secret message, i.e. to lie, and the coercer is not able to disclose their lie. In particular the public-key encryption protocol (algorithm) is called sender-deniable, if the sender is able to disclose the fake plaintext and the fake random value that defines transformation of the fake plaintext into the cipher text intercepted by the adversary with using the receiver's public key. One of important problems relating to the deniable encryption schemes is justifying their deniability [5].

The present paper proposes a new sender-deniable encryption scheme that has provable deniability. The scheme is based on using the computational indistinguishability between the deniable encryption procedure and the probabilistic ciphering of the fake message.

2 A new method for public-key encryption

In the proposed public-key encryption method it is used computational difficulty of finding square roots modulo composite number $n = pq$, where p and q are two strong primes composing the private key, like in the Rabin public-key cryptoscheme [6]. The value n represents the public key. If the values x_1 and x_2 are roots of the quadratic congruence $x^2 - Ax + B \equiv 0$ modulo n, then this polynomial can be written as $(x - x_1)(x - x_2)$, i.e. $x^2 - Ax + B \equiv (x - x_1)(x - x_2) \equiv x^2 - (x_1 + x_2)x + x_1x_2 \mod n$, $A \equiv (x_1 + x_2) \mod n$ and $B \equiv x_1x_2 \mod n$.

Thus, if it is required that two messages R and Z to be roots of some quadratic congruence, the coefficients should be given as follows $A = (R + Z) \mod n$ and $B = RZ \mod n$.

The last two formulas define an encryption procedure of simultaneous encryption of the messages R and Z, which output the cryptogram in the form of the pair of numbers (A, B). Decryption of the cryptogram $C = (A, B)$ can be performed as solving the following quadratic congruence

$$x^2 - Ax + B \equiv 0 \mod n. \tag{1}$$

Congruence (1) can be solved using the secret key (p, q) in the following way. The following two congruences are solved: $x^2 - Ax + B \equiv 0 \mod p$

and $x^2 - Ax + B \equiv 0 \mod q$, each of which has two roots. The roots of the first congruence are the following two values:

$$x_{p1} = \frac{A}{2} + \sqrt{\frac{A^2}{4} - B} \mod p \quad \text{and} \quad x_{p2} = \frac{A}{2} - \sqrt{\frac{A^2}{4} - B} \mod p.$$

The roots of the second congruence have the following two values:

$$x_{q1} = \frac{A}{2} + \sqrt{\frac{A^2}{4} - B} \mod q \quad \text{and} \quad x_{q2} = \frac{A}{2} - \sqrt{\frac{A^2}{4} - B} \mod q.$$

Four roots X_1, X_2, X_3, and X_4 of the congruence (1) are computed as solutions of the following four systems of linear congruences:

$$\begin{cases} X_1 \equiv x_{p1} \mod p \\ X_1 \equiv x_{q1} \mod q, \end{cases} \quad \begin{cases} X_2 \equiv x_{p1} \mod p \\ X_2 \equiv x_{q2} \mod q, \end{cases}$$

$$\begin{cases} X_3 \equiv x_{p2} \mod p \\ X_3 \equiv x_{q1} \mod q, \end{cases} \quad \begin{cases} X_4 \equiv x_{p2} \mod p \\ X_4 \equiv x_{q2} \mod q. \end{cases}$$

In accordance with the Chinese remainder theorem these systems have the following four solutions, respectively:

$$X_1 = (x_{p1}q(q^{-1} \mod p) + x_{q1}p(p^{-1} \mod q)) \mod n,$$
$$X_2 = (x_{p1}q(q^{-1} \mod p) + x_{q2}p(p^{-1} \mod q)) \mod n,$$
$$X_3 = (x_{p2}q(q^{-1} \mod p) + x_{q1}p(p^{-1} \mod q)) \mod n,$$
$$X_4 = (x_{p2}q(q^{-1} \mod p) + x_{q2}p(p^{-1} \mod q)) \mod n.$$

Evidently, two of the four roots X_1, X_2, X_3, and X_4 are equal to values Z and R. Two other roots represent random values that are to be discarded.

3 Deniable encryption and associated probabilistic ciphering

The encryption scheme described in the previous section can be used as deniable encryption method. For this purpose the sender of the secret message T is previously to generate a fake message (the message planned for presenting to the coercive attacker) and then, using the public key n of the receiver, to compute the pseudorandom value $R = T^2 \mod n$ and to encrypt simultaneously the values M and R. In such encryption method it is supposed that only the receiver has possibility to compute the value T from the value R using his private key (p, q). It is so if the secret message T is sufficiently large, for example $n \mod 2^{128} < T < n$. In the case of short secret messages, for example $n^{1/2} < T < 2^{20}n^{1/2}$ finding T from R without knowing the secret key (p, q) takes about 2^{40} multiplication operations. To provide possibility to encrypt short secret messages one can propose the following modified formula for computing the value R from the value $T : R = (n - T)^2 \mod n$. In its turn to provide more secure encryption of the message M it is defined the second root of the considered quadratic congruence in the form $Z = (R - M) \mod n$.

Thus, we have come to the following deniable encryption protocol with using receiver's public key n which includes the following steps:

1. It is computed the pseudorandom value $R = (n - T)^2 \mod n$.
2. It is formed the fake message M.
3. It is computed the value $A = (2R - M) \mod n$.
4. It is computed the value $B = R(R - M) \mod n$.
5. The ciphertext $C = (A, B)$ is sent to the receiver via a public communication channel.

The receiver performs the decryption procedure as follows:

1. The receiver using his private key (p, q) computes four roots X_1, X_2, X_3, and X_4 of the congruence $x^2 - Ax + B \equiv 0 \mod n$.
2. He computes the values M_1, M_2, M_3, and M_4 using the formula $M_i = (2X_i - A) \mod n (i = 1, 2, 3, 4)$.
3. He rejects three random messages from the set $\{M_1, M_2, M_3, M_4\}$. Let the fourth (sensible) message to represent the message M.
4. Then the receiver using his private key (p, q) computes the value $R = (M + A)/2 \mod n$ and four quadratic roots S_1, S_2, S_3, and S_4 from R modulo n.
5. He discloses the secret message computing the values $T_i = (n -$

$S_i) \mod n$, where $i = 1, 2, 3, 4$, and selecting the sensible message T from the set $\{T_1, T_2, T_3, T_4\}$.

Distinguishing the pseudorandom value R from a random choice is computationally infeasible without knowing the private key of the receiver of the message. Therefore the sender, while being coerced, can reasonably invoke to the use of the following probabilistic public-key encryption algorithm for enciphering the fake message M:

1. Generate random value $R_\phi < n$.
2. Compute the value $A = (2R_\phi - M) \mod n$.
3. Compute the value $B = R_\phi(R_\phi - M) \mod n$.

The last algorithm (that can be called the associated probabilistic public-key encryption algorithm) outputs the same ciphertext $C = (A, B)$ as the described above deniable encryption algorithm, if $R_\phi = R$ is send to the receiver via a public communication channel.

4 Discussion and conclusion

Since the value R is computed as squaring the value $n-T$ modulo n, it looks like a random choice. Therefore the sender of the cryptogram $C = (A, B)$ can claim that the cryptogram is the result of the probabilistic public-key encryption of the message M while using the public key n and the random value R. When being coerced the sender provides to the attacker the message M and value R. The coercer encrypts the message M using receiver's public key and value R as random choice. The encryption procedure produces the cryptogram C that has been intercepted previously by the coercer. To argue that the value R is not a random choice, the coercive attacker has to compute at least one quadratic root (modulo n) from the value R. The last problem is computationally infeasible since finding quadratic roots modulo n is as difficult as factoring the value n [6]. Thus, the attacker has no possibility to detect that the value R is not random, and to disclose the lie of the sender is as difficult as factoring problem. The proposed sender-deniable encryption protocol is provably deniable.

One can consider the proposed protocol and the associated public-key encryption algorithm as some extensions of the Rabin public-key encryption system [6] in which the public key represents the pair of integers n and $b < n$. The private key is the pair of primes p and q such that $n = pq$. Using the public key (n, b) the message $M < n$ can be encrypted by some sender as computing the cryptogram $C = M(M+b) \mod n$. Using the private key (p, q) decryption of the cryptogram C is performed by the receiver (owner of the public key) as solving the quadratic congruence $x^2 + bx - C \equiv 0 \mod n$. From four solutions the receiver selects the sensible one as message M. In paper [6] it is proven that the Rabin encryption algorithm is as secure as factoring the composite value n. In the Rabin public-key encryption system the cryptogram represents one of coefficients of the quadratic congruence (the second coefficient is equal to number b that is a part of the public key) whereas in the public encryption described in Section 2 the cryptogram represents two coefficients and the public key is the number n.

In paper [7] there is described a class of provably secure public-key cryptosystems in one of which the public-key encryption of the message M is performed as computing the cryptogam $C = M^3 \mod n$, where $n = pq$ and private primes p and q are such that number 3 divides p and does not dived q. Decryption of the cryptogram C in that cryptoscheme consists in finding three cubic roots modulo n from C and selecting one of them as message M. Using the results of paper [7] one can assume possibility of designing a provably secure method for simultaneous encryption of three independent messages interpreting them as roots of cubic congruence modulo n. In this case the cryptogram will represent three coefficients of some cubic congruence selected from the following eight variants: $x^3 \pm Ax^2 \pm Bx \pm D \equiv 0 \mod n$. Existence of the algebraic formulas for computing roots of the cubic equations provides potential possibility of implementing the procedure for decryption of the cryptogram $C = (A, B, D)$.

Detailed consideration of constructing the deniable encryption scheme based on computing cubic roots represents an individual research topic as well as extension to the case of simultaineous encryption of four messages interpreted as roots of the fourth-degree congruence.

The public-key encryption method described in Section 2 can be also implemented using any of the following four variants of the quadratic congruences: $x^2 \pm Ax \pm B \equiv 0 \mod n$.

As a topic directly connected with the proposed deniable encryption method, one can indicate development of the receiver-deniable and bi-deniable encryption protocols based on the algorithm of simultaneous public-key encryption of two messages.

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Properties of figures and properties of spaces

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Abstract

Right triangle equations as well as sine and cosine laws in elliptic, Euclidean and hyperbolic geometries have similar form. This paper analyzes what is exact meaning of “similarity”. Namely, what properties of figures are common among different geometries and what properties are geometry–specific.

This research has impact on volume theory in non–linear spaces.

Keywords: space invariant equations, space as parameter.

1 Notations and definitions

Let $k \in \{-1, 0, 1\}$. Define functions:

$$C(x) = C(x,k) = \sum_{n=0}^{\infty} (-k)^n \frac{x^{2n}}{(2n)!} = \begin{cases} \cos x, & k = 1, \\ 1, & k = 0, \\ \cosh x, & k = -1; \end{cases} \tag{1}$$

$$S(x) = S(x,k) = \sum_{n=0}^{\infty} (-k)^n \frac{x^{2n+1}}{(2n+1)!} = \begin{cases} \sin x, & k = 1, \\ x, & k = 0, \\ \sinh x, & k = -1; \end{cases} \tag{2}$$

$$T(x) = T(x,k) = \frac{S(x,k)}{C(x,k)} = \begin{cases} \tan x, & k = 1, \\ x, & k = 0, \\ \tanh x, & k = -1. \end{cases} \tag{3}$$

Parameter k is named [3] *characteristic*. It depends on space geometry and is the property of geometrical meaning of argument.

2 Universal form of triangle equations

In these notations, all equations of right triangle, sine and cosine laws have universal form among different geometries.

Table 1. Universal form of right triangle equations

Elliptic ($k = 1$)	Euclidean ($k = 0$)	Hyperbolic ($k = -1$)	Universal
$\tan b = \tan c \cos \alpha$	$b = c \cos \alpha$	$\tanh b = \tanh c \cos \alpha$	$T(b) = T(c) \cos \alpha$
$\tan a = \tan c \cos \beta$	$a = c \cos \beta$	$\tanh a = \tanh c \cos \beta$	$T(a) = T(c) \cos \beta$
$\sin a = \sin c \sin \alpha$	$a = c \sin \alpha$	$\sinh a = \sinh c \sin \alpha$	$S(a) = S(c) \sin \alpha$
$\sin b = \sin c \sin \beta$	$b = c \sin \beta$	$\sinh b = \sinh c \sin \beta$	$S(b) = S(c) \sin \beta$
$\tan a = \sin b \tan \alpha$	$a = b \tan \alpha$	$\tanh a = \sinh b \tan \alpha$	$T(a) = S(b) \tan \alpha$
$\tan b = \sin a \tan \beta$	$b = a \tan \beta$	$\tanh b = \sinh a \tan \beta$	$T(b) = S(a) \tan \beta$
$\cos \alpha = \cos a \sin \beta$	$\cos \alpha = 1 \cdot \sin \beta$	$\cos \alpha = \cosh a \sin \beta$	$\cos \alpha = C(a) \sin \beta$
$\cos \beta = \cos b \sin \alpha$	$cos \beta = 1 \cdot \sin \alpha$	$\cos \beta = \cosh b \sin \alpha$	$\cos \beta = C(b) \sin \alpha$
$\cos c \tan \alpha \tan \beta = 1$	$1 \cdot \tan \alpha \tan \beta = 1$	$\cosh c \tan \alpha \tan \beta = 1$	$C(c) \tan \alpha \tan \beta = 1$
$\tan^2 c = \tan^2 a$		$\tanh^2 c = \tanh^2 a$	$T^2(c) = T^2(a)$
$+ \tan^2 b$	$c^2 = a^2 + b^2$	$+ \tanh^2 b$	$+ T^2(b)$
$+ \tan^2 a \tan^2 b$		$- \tanh^2 a \tanh^2 b$	$+ kT^2(a)T^2(b)$

Table 2. Universal form of sine and cosine laws (for one side and angle)

$$\frac{S(a)}{\sin \alpha} = \frac{S(b)}{\sin \beta} = \frac{S(c)}{\sin \gamma}$$

$$T^2(c) = \frac{T^2(a) + T^2(b) - 2T(a)T(b) \cos \gamma + kT^2(a)T^2(b) \sin^2 \gamma}{(1 + kT(a)T(b) \cos \gamma)^2}$$

$$\cos \gamma = - \cos \alpha \cos \beta + \sin \alpha \sin \beta C(c)$$

Remark. Presented in Tables 1 and 2 equations are particular form of even more general equations for homogeneous spaces [1], that beside length parameter k_1 use also angular parameter k_2. For elliptic, Euclidean and hyperbolic planes the angular parameter $k_2 = 1$.

3 Distinction between properties of figures and properties of spaces

Theorem 1 [3]. *For any right–lined figure Ω in homogeneous space, if there exists equation:*

$$F(p_1, ..., p_n) = 0$$

that relates elements $p_1, ..., p_n$ of this figure, then it is possible to find its form:

$$H(Tr(p_1), ..., Tr(p_n)) = 0$$

which consists of:

- *function H that is algebraic and space invariant;*
- *function Tr is any one from these three: $C(p_i)$, $\sqrt{k_i}S(p_i)$, $\sqrt{k_i}T(p_i)$, which depends on characteristic k_i of its argument p_i.*

Function H describes properties of figure Ω regardless of space. Functions Tr describe properties of space that are not figure–specific.

4 Application in non–linear volume theory

Although Teorem 1 is proven for length and angle parameters $p_1, ..., p_n$, it is also true for areas. Area s of right triangle with catheti a, b may be presented as:

$$T(s) = \frac{S(a)S(b)}{C(a) + C(b)} \tag{4}$$

where characteristics of area k_s and of catheti k_a, k_b is related as:

$$k_s = k_a k_b. \tag{5}$$

The Lobachevsky function $\Lambda(\theta)$ has important role in volume theory of hyperbolic space [2]. It is not known to be expressed in terms of elementary functions:

$$\Lambda(\theta) = -\int_0^\theta log|2\sin t|dt. \tag{6}$$

The volume v of hyperbolic simplex with all vertices in absolute can be obtained from dihedral angles α, β, γ (the opposite angles are equal):

$$v = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma). \tag{7}$$

If Theorem 1 is true also for volume parameter v, then there exists the following form of equation (7):

$$H(Tr(v), Tr(\alpha), Tr(\beta), Tr(\gamma)) = 0 \tag{8}$$

that can be solved with respect to v as:

$$Tr(v) = h(Tr(\alpha), Tr(\beta), Tr(\gamma)),$$
$$v = Tr^{-1}(h(Tr(\alpha), Tr(\beta), Tr(\gamma))),$$

where h is solution of *algebraic* equation (8) with respect to $Tr(v)$. In this case, volume can be expressed as elementary function of angles.

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On locally compact rings of continuous endomorphisms of LCA groups

Valeriu Popa

Abstract

We characterise certain types of LCA groups with the property that their rings of continuous endomorphisms are locally compact in the compact-open topology.

Keywords: LCA groups, rings of continuous endomorphisms, compact-open topology.

Let $\mathcal{L}$ be the class of locally compact abelian groups. For $X \in \mathcal{L}$, let $c(X)$, $d(X)$, $k(X)$, $m(X)$, X^*, and $E(X)$ denote respectively the connected component of X, the maximal divisible subgroup of X, the subgroup of compact elements of X, the smallest closed subgroup K of X such that the quotient group X/K is torsion-free, the character group of X, and the ring of continuous endomorphisms of X taken with the compact-open topology. If X is totally disconnected, then X_p stands for the topological p-primary component of X corresponding to the prime p, and $S(X) = \{p \mid X_p \text{ is non-zero}\}$. We also use the group of rationals $\mathbb{Q}$, the quasi-cyclic groups $\mathbb{Z}(p^\infty)$, the groups of p-adic integers $\mathbb{Z}_p$, and the cyclic groups $\mathbb{Z}(p^n)$, where p is a prime and n is a positive integer.

M. Levin in [1], O. Mel'nikov in [2], and P. Plaumann in [3] have investigated various types of LCA groups with the property that their group of topological automorphisms is locally compact in the Birkhoff topology. By analogy, one may ask for a description of groups $X \in \mathcal{L}$ with the property that the ring $E(X)$ is locally compact. We present here some results answering this question.

Definition 1 ([1]). *Let $X \in \mathcal{L}$. A subgroup L of X is called a lattice in X if L is discrete and X/L is compact. If there exists such a subgroup L in X, then X is said to contain a lattice. If X decomposes as a topological direct sum of a discrete subgroup and a compact one, then it is said to contain a lattice trivially. If X contains a lattice but cannot be decomposed as a topological direct sum of a discrete group and a compact one, it is said to contain a lattice non-trivially.*

Theorem 1. *Let X be a group in $\mathcal{L}$ containing a lattice non-trivially. The ring $E(X)$ is locally compact if and only if the following conditions hold:*

(i) *$c(X)$ has finite dimension.*

(ii) *For each prime number p, $\big(k(X)/c(X) \cap k(X)\big)_p$ has finite rank.*

(iii) *$X/k(X)$ has finite rank.*

Theorem 2. *If X contains a lattice trivially, say $X = L \oplus K$ with L discrete and K compact, then $E(X)$ is locally compact if and only if $E(L)$ and $E(K)$ are both locally compact.*

Since $E(K)$ and $E(K^*)$ are topologically anti-isomorphic, this reduces the problem of the local compactness of $E(X)$ in the case when X contains a lattice trivially to the case when X is discrete.

Theorem 3. *Let X be a discrete group in $\mathcal{L}$. If $E(X)$ is locally compact, then either*

(i) *$X/k(X)$ has finite rank and for each prime number p, X_p has finite rank, or else*

(ii) *$X/k(X)$ has infinite rank and for each prime number p, X_p is finite.*

Moreover, case (i) is conclusive: if X satisfies (i), then $E(X)$ is locally compact. However, case (ii) is inconclusive: for some discrete groups $X \in \mathcal{L}$ satisfying (ii) $E(X)$ is locally compact and for some it is not.

Theorem 4. *Let $X \in \mathcal{L}$ be discrete and non-reduced. The ring $E(X)$ is locally compact if and only if the following conditions hold:*

(i) $d(X) \cong \mathbb{Q}^{r_0} \times \bigoplus_{p \in S(X)} \mathbb{Z}(p^\infty)^{r_p}$*, where* $r_0 \in \mathbb{N}$ *and the* r_p*'s are positive integers.*

(ii) There exists a finitely generated subgroup G of X such that X/G is torsion and, for each prime p, either $(X/G)_p$ is divisible or $(X/d(X))_p \cong \prod_{i=0}^{l(p)} \mathbb{Z}(p^{n_i(p)})$, where $l(p) \in \mathbb{N}$ and $n_0(p), \ldots, n_{l(p)}(p)$ are positive integers.

Theorem 5. *Let X be a compact group in $\mathcal{L}$ with $m(X) \neq X$. The ring $E(X)$ is locally compact if and only if the following conditions hold:*

(i) $X/m(X) \cong (\mathbb{Q}^*)^{r_0} \times \prod_{p \in S(X)} \mathbb{Z}_p^{r_p}$*, where* $r_0 \in \mathbb{N}$ *and the* r_p*'s are positive integers.*

(ii) There exists a closed totally disconnected subgroup Γ of X such that X/Γ is elementary and, for each prime p, either Γ_p is torsion-free or $(X/c(X))_p \cong \prod_{i=0}^{l(p)} \mathbb{Z}(p^{n_i(p)})$, where $l(p) \in \mathbb{N}$ and $n_0(p), \ldots, n_{l(p)}(p)$ are positive integers.

Theorem 6. *Let $X \in \mathcal{L}$ be either discrete and torsion or compact and totally disconnected. The following statements are equivalent:*

(i) $E(X)$ is compact.

(ii) $E(X)$ is locally compact.

(iii) X is topologically isomorphic either with the discrete group $\bigoplus_{p \in S(X)} \big(\mathbb{Z}(p^\infty)^{r_p} \times \prod_{i=0}^{l(p)} \mathbb{Z}(p^{n_i(p)})\big)$ or with the compact group $\prod_{p \in S(X)} \big(\mathbb{Z}_p^{r_p} \times \prod_{i=0}^{l(p)} \mathbb{Z}(p^{n_i(p)})\big)$, where $l(p) \in \mathbb{N}$ and $r_p, n_0(p), \ldots, n_{l(p)}(p)$ are positive integers.

Theorem 7. *Let $X \in \mathcal{L}$ be either discrete and torsion-free or compact and connected. The ring $E(X)$ is locally compact if and only if it is discrete.*

Theorem 8. *Let X be a discrete torsion-free group in $\mathcal{L}$ containing a subgroup A such that $A = \bigoplus_{i \in I} A_i$, where the A_i's are subgroups of finite rank of X and X/A is of bounded order. The following statements are equivalent:*

(i) $E(X)$ is locally compact.

(ii) I is finite, i.e. A is of finite rank.

Theorem 9. *Let X be a compact and connected group in $\mathcal{L}$ containing a closed subgroup of bounded order B such that $X/B = \prod_{i \in I} C_i$, where the C_i's are groups of finite dimension. The following statements are equivalent:*

(i) $E(X)$ is locally compact.

(ii) I is finite, i.e. X/B has finite dimension.

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On near-totally conjugate orthogonal quasigroups

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Abstract

In the article the near-totally conjugate orthogonal quasigroups (near-$totCO$-quasigroups), i.e., quasigroups for which there exist five (but there are no six) pairwise orthogonal conjugates, are studied. We proved that for any integer $n \geq 7$ that is relatively prime to 2, 3 and 5 there exist near-$totCO$-quasigroups of order n, and characterized graphs of conjugate orthogonality, connected with these quasigroups.

Keywords: quasigroup, T-quasigroup, conjugate orthogonal quasigroup, Latin square, graph of conjugate orthogonality.

1 Introduction

A quasigroup is an ordered pair (Q, A), where Q is a set and A is a binary operation, defined on Q, such that each of the equations $A(a, y) = b$ and $A(x, a) = b$ is uniquely solvable for any pair of elements a, b in Q. It is known that the multiplication table of a finite quasigroup defines a Latin square and six (not necessarily distinct) conjugates (or parastrophes) are associated with each quasigroup (Q, A) (Latin square): $A = {}^1A$, ${}^rA$, ${}^lA$, ${}^{rl}A$, ${}^{lr}A$, ${}^sA$, where ${}^rA(x, y) = z \Leftrightarrow A(x, z) = y$, ${}^lA(x, y) = z \Leftrightarrow A(z, y) = x$ and ${}^sA(x, y) = A(y, x)$ which are quasigroups (${}^{rl}A ={}^r ({}^lA)$).

Two quasigroups (Q, A) and (Q, B) are orthogonal ($A \perp B$) if the system of equations $\{A(x, y) = a,\ B(x, y) = b\}$ is uniquely solvable for all $a, b \in Q$.

A set $\Sigma = \{A_1, A_2, ..., A_n\}$ of quasigroups, defined on the same set, is orthogonal if any two quasigroups of it are orthogonal.

The notion of orthogonality plays the important role in the theory of Latin squares, also in quasigroup theory and in distinct applications, in particular, in coding theory and cryptography. In addition, quasigroups that are orthogonal to some of their conjugates or two conjugates of that are orthogonal (known as conjugate orthogonal or parastrophic-orthogonal quasigroups) have the significant interest.

Many articles were devoted to the investigation of various aspects of conjugate orthogonal quasigroups.

In [1], the quasigroups all six conjugates of which form an orthogonal set were investigated. In this paper we study the quasigroups for that there exist five (but there are no six) pairwise orthogonal conjugates. We call such quasigroups by near-totally conjugate orthogonal quasigroups (shortly, near-*totCO*-quasigroups) and give an information about the spectrum of these quasigroups and characterize the graphs, connected with them.

2 Totally and near-totally conjugate-orthogonal quasigroups

It is known that the number of distinct conjugates of a quasigroup can be 1,2,3 or 6 [2].

A quasigroup (Q, A) is called *a totally conjugate-orthogonal or totCO-quasigroup* if all six its conjugates are pairwise orthogonal [1]. In this case the system of six conjugates is an orthogonal set of quasigroups. Any conjugate of a *totCO*-quasigroup is also a *totCO*-quasigroup.

A quasigroup (Q, A) is called *a T-quasigroup* if there exists an abelian group $(Q, +)$, its automorphisms φ , ψ and an element $a \in Q$ such that $A(x, y) = \varphi x + \psi y + a$.

In [1], the following criterion for a *totCO*-T-quasigroup was established.

Theorem 1 [1]. *A T-quasigroup (Q, A): $A(x, y) = \varphi x + \psi y + a$ is a totCO-quasigroup if and only if all the following mappings $\varphi+\varepsilon$, $\varphi-\varepsilon$, $\psi+\varepsilon$, $\psi-\varepsilon$, $\varphi^2+\psi$, $\psi^2+\varphi$, $\varphi-\psi$, $\varphi+\psi$, $\psi\varphi-\varepsilon$ are permutations.*

In [3], it was proved that there exist infinite *totCO*-quasigroups. For finite quasigroups it is valid the following:

Theorem 2 [1]. *For any $n = p_1^{k_1} p_2^{k_2} ... p_s^{k_s}$, where p_i, $i = 1, 2, ..., s$, are prime numbers not equal to 2, 3, 5, 7, $k_i \geq 1$, there exists a totCO-quasigroup of order n.*

We consider quasigroups, that are no *totCO*-quasigroups, five conjugates of which form an orthogonal set and study the spectrum of these quasigroups.

Definition 1. *A quasigroup (Q, A) is called near-totally conjugate orthogonal (a near-totCO-quasigroup) if it is not a totCO-quasigroup and there exist five its pairwise orthogonal conjugates.*

Theorem 3. *A $T-$quasigroup $(Q,\, A)$, $A(x,\, y) = \varphi x + \psi y + a$ is a near-totCO$-$quasigroup if and only if from all mappings of Theorem 1 only the single mapping $\varphi - \varepsilon$ ($\psi - \varepsilon$ or $\varphi + \psi$) is not permutation.*

Theorem 4. *For any integer $n \geq 7$ that is prime with 2, 3 5, there exists a near-totCO-quasigroup of order n.*

Let (Q, A) be a quasigroup with six distinct conjugates, then there are six different subsets of the conjugate set:

$\Sigma_1 = \{l,\, r,\, rl,\, lr,\, s\}$, $\Sigma_s = \{1,\, l,\, r,\, lr,\, rl\}$ $\Sigma_l = \{1,\, r,\, rl,\, lr,\, s\}$, $\Sigma_{lr} = \{1,\, l,\, r,\, rl,\, s\}$ $\Sigma_r = \{1,\, l,\, rl,\, lr,\, s\}$, $\Sigma_{rl} = \{1,\, l,\, r,\, lr,\, s\}$.

Say that a near-*totCO*-quasigroup (Q, A) has *the type* $\Sigma_1 (\Sigma_s, \Sigma_l, \Sigma_{lr}, \Sigma_r, \Sigma_{rl})$, if the corresponding set of conjugates is orthogonal.

Proposition 1. *A near-totCO-quasigroup of the type Σ_1 is a near-totCO-quasigroup of the type Σ_s.*

A near-totCO-quasigroup of the type Σ_l is a near-totCO-quasigroup of the type Σ_{lr}.

A near-totCO-quasigroup of the type Σ_r is a near-totCO-quasigroup of the type Σ_{rl}.

In the article [4], it was considered *a graph of conjugate orthogonality* of a Latin square (a finite quasigroup), i. e., the graph the vertices of which are six conjugates of a Latin square and two vertices are connected if and only if the corresponding pair of conjugates is orthogonal. It is evident that the complete graph K_6 of conjugate orthogonality corresponds to a $totCO$-quasigroup.

We call the graph of conjugate orthogonality of a quasigroup *near-complete* if its complement (with respect to the complete graph K_6) contains a single edge. Such graph includes exactly 14 edges.

Theorem 5. *A near-$totCO$-quasigroup of the type Σ_1 (Σ_s) corresponds to the near-complete graph of conjugate orthogonality without the edge $(1, s)$.*

A near-$totCO$-quasigroup of the type Σ_l (Σ_{lr}) corresponds to the near-complete graph of conjugate orthogonality without the edge (l, lr).

A near-$totCO$-quasigroup of the type Σ_r (Σ_{rl}) corresponds to the near-complete graph of conjugate orthogonality without the edge (r, rl).

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Groupoids with Schröder Identity of Generalised Associativity

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Abstract

We study groupoids that are close to quasigroups with the following Schröder identity $(x \cdot y) \cdot (y \cdot (z \cdot x)) = z$ of generalised associativity. It is proved that if a left cancellation (left division) groupoid $(Q, \cdot, \backslash)$ satisfies this identity and the identity $x \backslash x = y \backslash y$, then $(Q, \cdot)$ is a group of exponent two. The similar results are proved for the right case.

Keywords: groupoid, left (right) division groupoid, left (right) cancellative groupoid, Schröder identity, quasigroup, abelian group

1 Introduction

In this paper we continue the study of left (right) division (cancellation) groupoids with various identities of generalized associativity [4].

Ernst Schröder (a German mathematician mainly known for his work on algebraic logic) introduced and studied the following identity of generalized associativity [5, 2, 3] on a quasigroup $(Q, \cdot, \backslash, /)$:

$$(x \cdot y) \backslash z = y \cdot zx. \tag{1}$$

It is easy to see that in the quasigroup case the identity (1) is equivalent with the following identity

$$(x \cdot y) \cdot (y \cdot zx) = z. \tag{2}$$

Often various variants of associative identity, which are true in a quasigroup, guarantee that this quasigroup is a loop. The following example demonstrates that this is not so in the case with identity (2).

$*$	0	1	2	3	4	5	6	7
0	1	4	7	0	6	5	2	3
1	5	2	3	6	0	1	4	7
2	0	7	4	1	5	6	3	2
3	6	3	2	5	1	0	7	4
4	4	1	0	7	3	2	5	6
5	3	6	5	2	4	7	0	1
6	7	0	1	4	2	3	6	5
7	2	5	6	3	7	4	1	0

Definition 1. *A groupoid $(Q,\cdot)$ is called:*
a left cancellation groupoid, if $ax = ay \Longrightarrow x = y \quad \forall\, a, x, y \in Q$,
a right cancellation groupoid, if $xa = ya \Longrightarrow x = y \quad \forall\, a, x, y \in Q$,
a left division groupoid if the mapping $L_a : a \to ax$ is surjective for every $a \in Q$,
a right division groupoid if the mapping $R_a : a \to ax$ is surjective for every $a \in Q$.

T. Evans [1] defined a binary quasigroup as an algebra $(Q,\cdot,/,\backslash)$ with three binary operations satisfying the following four identities:

$$x \cdot (x\backslash y) = y, \tag{3}$$

$$(y/x) \cdot x = y, \tag{4}$$

$$x\backslash(x \cdot y) = y, \tag{5}$$

$$(y \cdot x)/x = y. \tag{6}$$

We shall use the following
Theorem 1. *A groupoid $(Q,\cdot)$ is:*

1) *a left cancellation groupoid if and only if there exists a left division groupoid $(Q,\backslash)$ such that the algebra $(Q,\cdot,\backslash)$ satisfies* (5)*;*

2) *a left division groupoid if and only if there exists a left cancellation groupoid* $(Q, \backslash)$ *such that the algebra* $(Q, \cdot, \backslash)$ *satisfies* (3)*;*

3) *a right cancellation groupoid if and only if there exists a right division groupoid* $(Q, /)$ *such that the algebra* $(Q, \cdot, /)$ *satisfies* (6)*;*

4) *a right division groupoid if and only if there exists a right cancellation groupoid* $(Q, /)$ *such that the algebra* $(Q, \cdot, /)$ *satisfies* (4) *[6, 7].*

2 Left (right) cancellation (division) groupoids

We study left cancellation and left division e-groupoids (i.e. equational groupoids) $(Q, \cdot, /)$ with the following identity:

$$x \backslash x = y \backslash y. \tag{7}$$

Recall, if in a primitive quasigroup $(Q, \cdot, /, \backslash)$ the identity (7) is true, then this quasigroup has the right identity element. The following theorems are proved.

Theorem 2. *If a left cancellation groupoid* $(Q, \cdot, \backslash)$ *satisfies identities (2) and (7), then* $(Q, \cdot)$ *is a group of exponent two.*

If a left division groupoid $(Q, \cdot, \backslash)$ *satisfies identities (2) and (7), then* $(Q, \cdot)$ *is a group of exponent two.*

We study right cancellation and right division e-groupoids with the following additional identity:

$$x/x = y/y. \tag{8}$$

Theorem 3. *If a right cancellation groupoid* $(Q, \cdot, /)$ *satisfies identities (2) and (8), then* $(Q, \cdot)$ *is an abelian group of exponent two.*

If a right division groupoid $(Q, \cdot, /)$ *satisfies identities (2) and (8), then* $(Q, \cdot)$ *is a group of exponent two.*

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About one cryptoalgorithm

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Abstract

We give a generalization of one well known cryptoalgorithm which is based on a finite binary quasigroup.

Keywords: cryptology, algorithm, quasigroup, translation

1 Introduction

In [3, 4] it is proposed to use quasigroups for secure encoding. A quasigroup $(Q, \cdot)$ and its (23)-parastrophe $(Q, \backslash)$ satisfy the following identities $x\backslash(x \cdot y) = y$, $x \cdot (x\backslash y) = y$. It is proposed to use this quasigroup property to construct the following stream cipher.

Algorithm 1. *Let A be a non-empty alphabet, k be a natural number, $u_i, v_i \in A$, $i \in \{1, ..., k\}$. Define a quasigroup $(A, \cdot)$. It is clear that the quasigroup $(A, \backslash)$ is defined in a unique way. Take a fixed element l ($l \in A$), which is called a leader.*

Let $u_1u_2...u_k$ be a k-tuple of letters from A. The authors propose the following ciphering procedure $v_1 = l \cdot u_1, v_i = v_{i-1} \cdot u_i$, $i = 2, ..., k$. Therefore we obtain the following cipher-text $v_1v_2 \dots v_k$. The enciphering algorithm is constructed in the following way: $u_1 = l\backslash v_1, u_i = v_{i-1}\backslash v_i, i = 2, ..., k$ [3, 4].

It is easy to see that in Algorithm 1 n-ary quasigroups and their parastrophes [1] can be used. This fact is mentioned in [7, 8]. Ternary and quaternary generalisations of Algorithm 1 are described in [5]. The weakness of Algorithm 1 and its ternary analogue relatively some cryptographical attacks is pointed out in [10, 2].

Here we present further generalizations of Algorithm 1. See [2, 9].

2 Binary generalization of Algorithm 1

We start with rewriting Algorithm 1 in slightly other form.

Algorithm 1^*. *Let Q be a non-empty finite alphabet. Define a quasigroup $(Q, \cdot)$. It is clear that the quasigroup $(Q, \overset{(23)}{\cdot})$ is defined in a unique way. Take a fixed element l ($l \in Q$), which is called a leader. Let $u_1u_2...u_k$ be a k-tuple of letters from Q.*

It is proposed the following ciphering procedure

$v_1 = l \cdot u_1 = L_l u_1$,

$v_2 = v_1 \cdot u_2 = L_{v_1} u_2$,

$v_i = v_{i-1} \cdot u_i = L_{v_{i-1}} u_i$, $i = 3, ..., k$.

Therefore we obtain the following cipher-text $v_1 v_2 \dots v_k$.

The deciphering algorithm is constructed in the following way. We have the following cipher-text: $v_1 v_2 \dots v_k$. Recall $L_a^{\overset{(23)}{\cdot}} = (L_a^{\cdot})^{-1}$ for any $a \in Q$ [6]. Below we shall denote translation $L_a^{\overset{(23)}{\cdot}}$ as L_a^*, translation $L_a^{\cdot}$ as L_a for any $a \in Q$. Then

$$u_1 = l \overset{(23)}{\cdot} v_1 = L_l^* (v_1) = L_l^* (L_l u_1) = L_l^{-1} (L_l u_1) = u_1;$$

$$u_i = v_{i-1} \overset{(23)}{\cdot} v_i = L^*_{v_{i-1}} (v_i) = L^*_{v_{i-1}} \left(L_{v_{i-1}} u_i\right) = L^{-1}_{v_{i-1}} \left(L_{v_{i-1}} u_i\right) = u_i$$

for all $i \in \overline{2, k}$.

From this form of Algorithm 1^* we can obtain the following generalization. Instead of translations L_x, $x \in Q$, we propose to use in the enciphering part of this algorithm powers of these translations, i.e., to use permutations of the form L_x^k, $k \in \mathbb{Z}$, instead of permutations of the form L_x.

The proposed modification forces us to use permutations of the form L_x^k, $k \in \mathbb{Z}$, also in the decryption procedure.

Algorithm 2. *Let Q be a non-empty finite alphabet. Define a quasigroup $(Q, \cdot)$. It is clear that the quasigroup $(Q, \overset{(23)}{\cdot})$ is defined in a unique way.*

Take a fixed element l ($l \in Q$), which is called a leader. Let $u_1u_2...u_k$ be a k-tuple of letters from Q.

It is proposed the following ciphering procedure

$$
\begin{aligned}
v_1 &= L_l^a u_1, a \in \mathbb{Z}, \\
v_2 &= L_{v_1}^b u_2, b \in \mathbb{Z}, \\
v_i &= L_{v_{i-1}}^c u_i, i \in \overline{3, k}, c \in \mathbb{Z}.
\end{aligned} \tag{1}
$$

Therefore we obtain the following cipher-text $v_1 v_2 \dots v_k$. *The deciphering algorithm is constructed in the following way. We use notations of Algorithm* 1^*. *Recall* $(L_x^*)^a = L_x^{-a}$ *for all* $x \in Q$. *Then*

$$
\begin{aligned}
&(L_l^*)^a (v_1) = (L_l^*)^a (L_l^a u_1) = u_1, \\
&(L_{v_1}^*)^b (v_2) = (L_{v_1}^*)^b \left(L_{v_1}^b u_2\right) = u_2, \\
&(L_{v_{i-1}}^*)^c (v_i) = (L_{v_{i-1}}^*)^c (L_{v_{i-1}}^c u_i) = u_i, i \in \overline{3, k}.
\end{aligned}
$$

Notice, the elements a, b, c in equalities (1) should vary from step to step in order to protect this algorithm against chosen plain-text and chosen cipher-text attack. It is clear that the right and middle translations [6] can also be used in Algorithm 2 instead of the left translations. Information about n-ary generalization of Algorithm 2 is given in [9].

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About top-quasigroups

Fedir M. Sokhatsky, Ievgen Pirus

Abstract

An n-ary quasigroup is an asymmetric top-quasigroup, if all its parastrophes are pairwise different and n-wise orthogonal. Let p be the least prime factor of its order. An n-ary asymmetric top-quasigroup, which is linear on a cyclic group, exists if and only if $n = 2, p > 7$ or $n = 3, p > 19$.

Keywords: parastrophes, Latin cubes, Latin hyper cubes, quasigroup, top-quasigroup.

1 Introduction

Quasigroups and its combinatorial analogue Latin squares, cubes and hyper-cubes have wide application in many areas of sciences: design of experiments, coding theory, geometry, automata theory and others. The main mathematical problem, which is under consideration, is to find methods for constructing orthogonal quasigroup operations.

Some parastrophes of the same quasigroup operation can coincide, but all different parastrophes can be n-wise orthogonal. In this case, it is called a top-quasigroup. Here, we investigate linear top-quasigroups.

2 Top-quasigroups

Let Q be a set. An n-ary operation f defined on Q is called a *quasigroup*, if each of the equations

$$f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) = a_i, \quad i = 1, \ldots, n$$

has a unique solution for all elements $a_1, \ldots, a_n \in Q$. The relationships

$$^{\sigma}f(x_{1\sigma}, \ldots, x_{n\sigma}) = x_{(n+1)\sigma} \;\Leftrightarrow\; f(x_1, \ldots, x_n) = x_{n+1}, \quad \sigma \in S_{n+1}$$

define $(n+1)!$ quasigroup operations $^{\sigma}f$, which are called *parastrophes* of f. A quasigroup is said to be *totally parastrophically orthogonal (top-quasigroup)*, if all its different parastrophes are orthogonal.

General problem: *Describe all finite top-quasigroups up to parastrophy-isomorphy relation.*

The notion of top-quasigroup has four main parameters: an *arity* of the quasigroup, its *order*, a number of its *different parastrophes* and a *type of orthogonality*.

There are many types of orthogonality for multiary quasigroups. Here, n-ary operations $f_1, \ldots, f_n$ are called *orthogonal*, if every system $\{f_i(x_1, \ldots, x_n) = a_i|_1^n$ has a unique solution for arbitrary element $a_1, \ldots, a_n$ from its carrier Q.

A mapping $(\sigma; f) \mapsto {}^{\sigma}f$ is an action of the symmetric group S_{n+1} on the set of all n-ary quasigroup operations defined on Q. A stabilizer $\mathrm{Sym}(f) := \{\sigma \mid {}^{\sigma}f = f\}$ of the action is called a *symmetry group of f*. Symmetry groups of parastrophes are conjugate, namely $\mathrm{Sym}({}^{\sigma}f) = \sigma\mathrm{Sym}(f)\sigma^{-1}$. The number of all different parastrophes of f is equal to $(n+1)!/|\mathrm{Sym}(f)|$. In particular, all parastrophes of f are different if and only if its symmetry group is trivial, i.e., the quasigroup is *asymmetric*.

A quasigroup $(Q; f)$ is called *linear over a group* $(Q; +)$ if there exist automorphisms $\varphi_1, \varphi_2, \ldots, \varphi_n$ of $(Q; +)$ such that

$$f(x_1, x_2, \ldots, x_n) = \varphi_1 x_1 + \varphi_2 x_2 + \cdots + \varphi_n x_n + a.$$

Subproblem: Describe all finite linear top-quasigroups up to parastrophy-isomorphy relation.

Theorem 1. *Linear n-ary top-quasigroups do not exist if $n > 3$.*

In binary case the number of pairwise unconjugated subgroups of the symmetric group S_3 is 4. There exist four parastrophically closed classes of top-quasigroups: *asymmetric*, i.e., their symmetry groups are trivial and such a quasigroup has six different parastrophes; *one-side symmetric*, i.e., their symmetry groups contain two elements and such a quasigroup has three different parastrophes; *skew symmetric*, i.e., their symmetry groups contain three elements and such a quasigroup has two different parastrophes; and totally symmetric quasigroups, their symmetry groups is S_3 and all parastrophes of each quasigroup coincide. So, binary top-quasigroups can be divided into three parastrophically closed classes: asymmetric, one-side symmetric and skew symmetric.

Theorem 3. **[1]** *A binary linear asymmetric top-quasigroup over an m-order cyclic group exists if and only if the least prime factor of m is greater than 7.*

Ternary case. The group S_4 has 30 subgroups, which are divided into 11 conjugated classes. Pairwise unconjugated subgroups are:

$$E := \{\iota\},\ S_2 := \{\iota, (12)\},\ S_{22} := \{\iota, (12)(34)\},\ A_3 := \{\iota, (123), (132)\},$$

$$\mathbb{Z}_4 := \{\iota, (12)(34), (1423), (1324)\},\ K_4 := \{\iota, (12)(34), (13)(24), (14)(23)\},$$

$$C_4 := \{\iota, (12), (34), (12)(34)\}, \quad S_3 := \{\iota, (12), (13), (23), (123), (132)\},$$

$$D_8 := \{\iota, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}, \quad S_4,$$

$$A_4 := \{\iota, (123), (132), (134)(143), (124), (142), (234), (243), \\ (13)(24), (12)(34), (14)(23)\}.$$

The numbers of different parastrophes of f are given in

Sym$f$	E	S_2	S_{22}	A_3	$\mathbb{Z}_4$	K_4	C_4	S_3	D_8	A_4	S_4
number of different parastrophes	24	12	12	8	6	6	6	4	3	2	1

Since we consider triple-wise orthogonality, there exist nine classes of the top-quasigroups. Here, we give results only for asymmetric quasigroups. Each of them has 24 triple-wise orthogonal parastrophes.

Theorem 4. *If the least prime factor of m is greater than 107, then* $(\mathbb{Z}_m; f)$, *where* $f(x; y; z) := 2x + 8y + 11z$, *is an asymmetric top-quasigroup of the order m.*

Lemma 5. *If* $23 \leqslant p \leqslant 107$, *then* $(\mathbb{Z}_p; g)$ *defined by* $g(x;y;z) := \alpha x + \beta y + \gamma z$ *is an asymmetric top-quasigroup of the order* p, *where* p *and* $\{\alpha, \beta, \gamma\}$ *are taken from*

p	$\{\alpha, \beta, \gamma\}$	p	$\{\alpha, \beta, \gamma\}$
$23, 103$	$\{12, 14, 21\}$	$41, 67$	$\{7, 14, 21\}$
$29, 83, 97$	$\{6, 8, 16\}$	$43, 53, 107$	$\{21, 31, 35\}$
31	$\{3, 4, 25\}$	47	$\{2, 5, 41\}$
$37, 61$	$\{2, 7, 29\}$	$59, 61, 71, 79, 89, 101$	$\{2, 8, 11\}$

Theorem 6. *A ternary linear asymmetric top-quasigroup over an* m*-order cyclic group exists if and only if the least prime factor of* m *is greater than 19.*

3 Conclusion

If $n > 3$, then linear n-ary asymmetric top-quasigroups do not exist. An n-ary asymmetric top-quasigroup, being linear on a cyclic m-order group, exists if and only if the least prime factor of m is greater then: 7, if $n = 2$ and 19, if $n = 3$. A binary asymmetric top-quasigroup has six pair-wise orthogonal parastrophes, but a ternary one has 24 triple-wise orthogonal parastrophes.

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About one special inversion matrix of 3-ary and 4-ary IP-loops

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Abstract

It is known that an n-IP-quasigroup can have more than one inversion matrix. We prove that one of these matrices for a 3-ary IP-loop and for a 4-ary IP-loop is the matrix of permutations every of which fixes identity of a loop and has order two. It is a good matrix allowing to investigate n-IP-loops in more detail.

Keywords: 3-IP-loop, 4-IP-loop, inversion system, inversion matrix

1 Introduction

The definitions of an n-IP-quasigroup (an n-IP-loop), $n \geq 2$, and of its inversion matrix one can find in [1]. It is known that an n-IP-quasigroup can have more than one inversion matrix and one of these matrices for an n-IP-loop with an identity e can be a matrix of the special form $[I_{ij}]_e$ which facilitates the study of n-IP-loops. V.D.Belousov has assumed that every n-IP-loop has the matrix $[I_{ij}]_e$ as an inversion matrix.

In [2], the example of a 3-IP-loop for which one of inversion matrices is the matrix $[I_{ij}]_e$ was given. Later in [3], it was proved that the matrix $[I_{ij}]_e$ exists for any n-IP-group with an identity e, for any symmetric n-IP-loop and for any n-IP-loop with one inversion parameter. The existence of the matrix $[I_{ij}]_e$ for any nonsymmetric n-IP-loops at present is not proved.

In this article we establish that the matrix $[I_{ij}]_e$ exists for any nonsymmetric 3-ary IP-loop and for any nonsymmetric 4-ary IP-loop with an identity e.

2 Preliminaries

A ternary operation $Q(\,)$, defined on a set Q, is called *a 3-ary quasigroup with the invertible property (shortly, a 3-IP-quasigroup)* if on Q there exist permutations υ_{ij}, $i = 1, 2, 3$ (or $i \in \overline{1,3}$), $j \in \overline{1,4}$, where $\upsilon_{ii} = \upsilon_{i4} = \varepsilon$ (ε is the identity permutation) such that the following equalities hold: $((x_1^3), \upsilon_{12}x_2, \upsilon_{13}x_3) = x_1$, $(\upsilon_{21}x_1, (x_1^3), \upsilon_{23}x_3) = x_2$, $(\upsilon_{31}x_1, \upsilon_{32}x_2, (x_1^3)) = x_3$ for any $x_1^3 \in Q^3$.

The matrix

$$\left[\upsilon_{ij}\right] = \begin{bmatrix} \varepsilon & \upsilon_{12} & \upsilon_{13} & \varepsilon \\ \upsilon_{21} & \varepsilon & \upsilon_{23} & \varepsilon \\ \upsilon_{31} & \upsilon_{32} & \varepsilon & \varepsilon \end{bmatrix}$$

is called *an inversion matrix* for a 3-IP-quasigroup, the permutations $\upsilon_{i,j}$ are called *inversion permutations.* Any row of an inversion matrix is called *an inversion system* for a 3-IP-quasigroup.

All these definitions are analogous for any n-IP-quasigroups.

An n-IP-quasigroup is called *symmetric or an n-TS-quasigroup* if $\upsilon_{ij} = \varepsilon$ for all $i, j \in \overline{1,n}$.

The least common multiple of the orders of all permutations of the i-th inversion system is called *the order of the i-th inversion system.*

The least common multiple of the orders of all inversion systems is called *the order of an inversion matrix.*

An element $e \in Q$ is called an identity for an n-quasigroup $Q(\,)$ if $(x, e, e) = (e, x, e) = (e, e, x) = x$ for any $x \in Q$. *An n-loop* is an n-quasigroup with an identity.

The permutations I_{ij} on a set Q for an n-IP-loop with an identity e are defined as follows: $(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}) = e$ for any $x \in Q$ and any $i, j \in \overline{1, n}$ [1]. The matrix $[I_{ij}]_e$ for a 3-IP-loop has the form:

$$[I_{ij}] = \begin{bmatrix} \varepsilon & I_{12} & I_{13} & \varepsilon \\ I_{21} & \varepsilon & I_{23} & \varepsilon \\ I_{31} & I_{32} & \varepsilon & \varepsilon \end{bmatrix}.$$

If $(\varepsilon, \upsilon_{i2}, \upsilon_{i3}, \varepsilon)$ is the i-th inversion system of a 3-IP-loop, $i \in \overline{1, 3}$, then $(\varepsilon, \upsilon_{i2}^{2n-1}, \upsilon_{i3}^{2n-1}, \varepsilon)$ is also the i-th inversion system of some inversion matrix of this 3-IP-loop, and $(\varepsilon, \upsilon_{i2}^{2n}, \upsilon_{i3}^{2n}, \varepsilon)$ is an autotopy of this 3-IP-loop [2]. The main definitions and results for a 3-IP-loop are true for a 4-IP-loop as well.

3 Permutations of inversion systems and matrices of 3-IP-loops

The obtained results relative to nonsymmetric 3-IP-loops (to non-3-TS-loops).

Proposition 1. *If $Q()$ is a 3-IP-loop with an inversion matrix $[\upsilon_{ij}]$, $i \in \overline{1, 3}$, $j \in \overline{1, 4}$ and with an identity e, then any non-identity inversion permutation in even power of any inversion system does not leave fixed the identity e.*

Corollary 1. *If $Q()$ is a 3-IP-loop with an inversion matrix $[\upsilon_{ij}]$, $i \in \overline{1, 3}$, $j \in \overline{1, 4}$, and with an identity e, then any non-identity inversion permutation of any inversion system only in odd power leaves fixed the identity e.*

It means that for any non-identity inversion permutation υ_{ij}, $i, j \in \overline{1, 3}$, of an inversion matrix of a 3-IP-loop there exists odd number $2n+1, n \in N$, such that $\upsilon_{ij}^{2n+1} e = e$, i.e., the identity of a loop is in a cycle of odd length in this inversion permutation.

Theorem 1. *The matrix $[I_{ij}]_e$ is an inversion matrix for any 3-IP-loop with an identity e.*

4 Permutations of inversion systems and matrices of 4-IP-loops

The obtained results relative to 4-IP-loops that are not 4-TS-loops and, in contrast to the ternary case. Another approach is required for the proof of analogous results.

Proposition 2. *If $Q()$ is a 4-IP-loop with an inversion matrix $[\upsilon_{ij}]$, $i \in \overline{1,4}$, $j \in \overline{1,5}$ and with an identity e, then any non-identity inversion permutation in even power of any inversion system does not leave fixed the identity e.*

Corollary 2. *If $Q()$ is a 4-IP-loop with an inversion matrix υ_{ij}, $i \in \overline{1,4}$, $j \in \overline{1,5}$, and with an identity e, then any non-identity inversion permutation of any inversion system only in odd power leaves fixed the identity e.*

These results are used in the proof of the following

Theorem 2. *The matrix $[I_{ij}]_e$ is an inversion matrix for any 4-IP-loop with an identity e.*

The question about existence of the matrix $[I_{ij}]_e$ for any n-IP-loop, $n > 4$, is still opened.

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On isohedral tilings by 12-gons for hyperbolic group of genus two

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Abstract

For hyperbolic translation group of genus 2 isohedral tilings of the hyperbolic plane with 12-gons are obtained.

Keywords: isohedral tilings, hyperbolic plane, group of translations, genus two.

On the Euclidean plane the group of translations $p1$ is one of 17 crystallographic plane groups. As it is well known there are 2 types (and 2 Delone classes) of isohedral tilings of the Euclidean plane with disks for the group $p1$, those disks being parallelograms and center-symmetric hexagons, respectively. The orbifold symbol by Conway of this group is $\circ$ (a circle). The analog of this group on the hyperbolic plane, i. e. a discrete group of translations with a compact (bounded) fundamental domain, is characterized with its genus, which is the genus of the quotient of the hyperbolic plane by the group action. The orbifold symbol by Conway of such a group is $\circ \circ \cdots \circ$ with the number of circles being equal to its genus. The smallest genus of a hyperbolic translation group is 2, so the hyperbolic group of genus 2 is the simplest hyperbolic group of translations. Thus I investigate isohedral tilings for the hyperbolic group of genus two.

Definition 1. *Let W be a tiling of the hyperbolic plane with disks, G be a discrete isometry group of the hyperbolic plane with a bounded fundamental domain. The tiling W is called isohedral with respect to the group G if the group G acts transitively on the set of all disks of the tiling.*

Definition 2. *Consider all possible pairs (W, G) where W is a tiling of the hyperbolic plane with disks which is isohedral with respect to a discrete isometry group G of the hyperbolic plane with a bounded fundamental domain. Two pairs (W, G) and (W', G') are said to belong to the same Delone class if there exists a homeomorphic transformation φ of the plane which maps the tiling W onto the tiling W' and the relation $G = \varphi^{-1}G'\varphi$ holds.*

The classification of isohedral tilings of the hyperbolic plane is done using Delone classes.

A translation group contains no non-trivial isometry with invariant points, so it admits only fundamental Delone classes of tilings, i. e. tilings of the plane with fundamental domains. I am going to find these tilings similarly as the adjacency symbols were applied by B. Delone [1, 2] to obtain all the fundamental isohedral tilings of the Euclidean plane (see also [3]).

First we work out an equation that relates the number k of edges and the valencies α_1, α_2, …, α_k of any tile in the tiling and write:

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_k} = \frac{k}{2} - 3. \tag{1}$$

From the inequality $\alpha_j \geq 3$ and the equation (1) we derive the inequality $k \leq 18$. As all the isometries of the group are translations, the number k is even. Because the group orbifold has no singular points, any vertex valency α_j occurs a multiple of α_j times.

Using the above conditions we solve this Diophantine equation in integers. The obtained solutions are $k = 8$, 10, 12, 14, 16, 18. In the present communication I investigate the case $k = 12$, i. e. tilings with 12-gons. So for $k = 12$ we obtain sets of numbers which correspond to possible vertex valencies.

To the obtained ordered sets of valencies (we call them cycles) we apply the adjacency symbols method. All the edges of a tile are labelled consecutively with the letters a, b, … . In a symbol the first position is occupied by the letter that labels the first chosen edge, the second position is occupied by the letter that labels the edge adjacent to the first edge, then the lower index is the vertex valency of the first edge, after that we pass to the second consecutive edge, and so on. We generate possible adjacency symbols. For each candidate in adjacency symbol we check if the condition of transition around a vertex is satisfied (for every vertex equivalence class). Using adjacency diagrams we determine whether two adjacency symbols correspond to the same Delone class.

The solutions of the equation for $k = 12$ are three sets of numbers. For the set containing six '3' and six '6' there are several Delone classes of isohedral tilings of the hyperbolic plane. One of them, with the adjacency symbol

$$(aj_3be_6cl_3dg_6eb_3fi_6gd_3hk_6if_3ja_6kh_3lc_6),$$

as the quotient gives rise to the second richest Riemann surface of genus two with the automorphism group G^*_{24} (see Fig. 1, b in [4] or [5]). For the set containing twelve '4' there are several Delone classes of isohedral tilings of the hyperbolic plane, too. One of Delone classes is given by the adjacency symbol

$$(ac_4be_4ca_4df_4eb_4fd_4gi_4hk_4ig_4jl_4kh_4lj_4).$$

And the most 'exotic' isohedral tilings of the hyperbolic plane correspond to the set containing three '3', four '4' and five '5'. Among them one of Delone classes is given by the adjacency symbol

$$(ad_3be_4cf_3da_4eb_3fc_4gl_5hj_5ik_5jh_5ki_5lg_4).$$

Remark that the known method of cutting orbifold by Z. Lučić and E. Molnár permits to find fundamental isohedral tilings of the hyperbolic plane if the isometry group is given. That method gives quite satisfactory results if applied to some isometry groups with rotations and reflections. For a hyperbolic translation group Z. Lučić and E. Molnár propose to use the known canonical cutting of handles of the manifold combined with adding some new vertices. However some of the new isohedral tilings of the hyperbolic plane cannot be obtained by the method of cutting orbifold.

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Section 2

Mathematical Analysis and Differential Equations

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The $GL(2,\mathbb{R})$-invariant center conditions for the cubic differential systems with degenerate infinity

Iurie Calin, Valeriu Baltag

Abstract

In this paper the $GL(2,\mathbb{R})$-invariant center conditions for the origin of the coordinates of the phase plane of the cubic differential systems with degenerate infinity were established.

Keywords: differential system, center, $GL(2,\mathbb{R})$-invariant, transvectant.

Let us consider the cubic system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= P_1(x,y) + P_2(x,y) + P_3(x,y) = P(x,y), \\ \frac{dy}{dt} &= Q_1(x,y) + Q_2(x,y) + Q_3(x,y) = Q(x,y), \end{aligned} \tag{1}$$

where $P_i(x,\ y)$, $Q_i(x,\ y)$ are homogeneous polynomials of degree i in x and y with real coefficients. The $GL(2,\mathbb{R})$-comitants [1] of the first degree with respect to the coefficients of system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ i = 1,2,3.$$

From the classical invariant theory [2] the definition of the transvectant of two polynomials is well known.

Definition 1. *Let $f(x,y)$ and $\varphi(x,y)$ be homogeneous polynomials in x and y with real coefficients of the degrees $\rho \in \mathbb{N}^*$ and $\theta \in \mathbb{N}^*$, respectively, and $k \in \mathbb{N}^*$. The polynomial*

$$(f,\varphi)^{(k)} = \frac{(\rho-k)!(\theta-k)!}{\rho!\theta!}\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial x^{k-h}\partial y^h}\frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}}$$

is called the transvectant of the index k of polynomials f and φ.

Remark 1. *If the polynomials f and φ are $GL(2,\mathbb{R})$-comitants of the degrees $\rho \in \mathbb{N}^*$ and $\theta \in \mathbb{N}^*$, respectively, for the system (1), then the transvectant of the index $k \leq \min(\rho,\theta)$ is a $GL(2,\mathbb{R})$-comitant of the degree $\rho+\theta-2k$ for the system (1). If $k > \min(\rho,\theta)$, then $(f,\varphi)^{(k)} = 0$.*

In repeated using of the transvectants a set of the parenthesis of the type $((\ldots($ will be replaced by a single parenthesis of the form $[\![$. By using the transvectants, the following $GL(2,\mathbb{R})$-invariants were constructed for the system (1):

$$H_1 = 3(R_2,R_1)^{(2)},\quad K_1 = 3(R_2,S_3)^{(2)},\quad I_1 = S_1,$$
$$I_2 = (R_1,R_1)^{(2)},\quad I_3 = [\![R_2,R_1)^{(2)},S_2)^{(1)},\quad I_4 = [\![R_1,S_2)^{(1)},S_2)^{(1)},$$
$$I_5 = [\![R_2,R_2)^{(2)},R_1)^{(2)},\quad I_6 = [\![R_2,R_1)^{(2)},R_1)^{(1)},S_2)^{(1)},$$
$$I_7 = [\![R_2,R_2)^{(2)},S_2)^{(1)},S_2)^{(1)},\quad I_8 = [\![R_2,R_2)^{(2)},R_2)^{(1)},R_2)^{(3)},$$
$$I_9 = [\![R_2,S_2)^{(1)},S_2)^{(1)},S_2)^{(1)},\quad I_{10} = [\![R_2,R_1)^{(2)},R_1)^{(1)},H_1)^{(1)},$$
$$I_{11} = [\![R_2,R_2)^{(2)},R_2)^{(1)},R_1)^{(2)},S_2)^{(1)},$$
$$I_{12} = [\![R_2,R_2)^{(2)},R_1)^{(1)},S_2)^{(1)},S_2)^{(1)},$$
$$I_{13} = [\![R_2,R_1)^{(1)},S_2)^{(1)},S_2)^{(1)},S_2)^{(1)},$$
$$I_{14} = [\![R_2,R_2)^{(2)},R_2)^{(1)},R_1)^{(2)},R_1)^{(1)},S_2)^{(1)},$$
$$J_1 = (R_1,S_3)^{(2)},\quad J_2 = (S_3,S_3)^{(2)},\quad J_3 = [\![R_2,R_2)^{(2)},S_3)^{(2)},$$
$$J_4 = [\![R_2,S_3)^{(2)},S_2)^{(1)},\quad J_5 = [\![S_3,S_2)^{(1)},S_2)^{(1)},$$
$$J_6 = [\![R_2,R_2)^{(2)},R_1)^{(1)},S_3)^{(2)},\quad J_7 = [\![R_2,R_1)^{(2)},S_3)^{(1)},S_2)^{(1)},$$
$$J_8 = [\![R_2,S_3)^{(2)},R_1)^{(1)},S_2)^{(1)},\quad J_9 = [\![R_1,S_3)^{(1)},S_2)^{(1)},S_2)^{(1)},$$
$$J_{10} = [\![R_2,S_3)^{(2)},S_3)^{(1)},S_2)^{(1)},\quad J_{11} = [\![R_2,R_1)^{(2)},R_1)^{(1)},K_1)^{(1)},$$
$$J_{14} = [\![R_2,R_1)^{(2)},R_1)^{(1)},S_3)^{(1)},S_2)^{(1)},$$

$$J_{19} = [\![R_2, R_1)^{(2)}, R_1)^{(1)}, S_3)^{(1)}, H_1)^{(1)},$$
$$J_{20} = [\![R_2, R_1)^{(1)}, S_3)^{(2)}, S_3)^{(1)}, H_1)^{(1)},$$

$$\begin{aligned}
L_1 = {} & 12I_{12} + 8I_{13} + 3(12I_3 + I_4)J_1 - 2I_2(9J_3 + 6J_4 - 2J_5) + 36J_{14},\\
L_2 = {} & (165I_3 - 914I_4 - 1755I_5)I_{10}+\\
& 6I_2^2(93I_7 - 270I_8 + 4I_9 + 81J_6 + 288J_7 + 126J_8 - 135J_9)-\\
& 3I_2(489I_3^2 - 40I_3I_4 + 507I_3I_5 + 268I_4I_5 - 1458I_{14} + 243J_{19}),\\
L_3 = {} & 2I_2(48I_3 - 16I_4 - 9I_5) + 45I_{10},\\
G_1 = {} & 4I_6 + 3I_2J_1,\\
G_2 = {} & 4L_1 - 180I_{11} + 120I_{12} - 45(I_3 - 6I_5)J_1+\\
& 5I_2(9J_3 + 24J_4 + 4J_5) + 90J_{11},\\
G_3 = {} & -2L_1(2I_2I_3 + 9I_2I_5 - 7I_{10}) - 18I_2^2(58I_3 - 61I_4 - 24I_5)J_3-\\
& 6I_2^2(23I_3 - 60I_4 - 48I_5)J_4 + 4I_2^2(43I_3 + 16I_4 + 48I_5)J_5-\\
& 3I_2^2J_1(186I_7 + 88I_9 - 378J_7 + 207J_8 + 6J_9)-\\
& 27I_2^2(3I_2J_1J_2 - 8I_2J_{10} + 8J_{20}),\\
G_4 = {} & L_1(16(105I_3 - 286I_4 - 945I_5)I_{10} + 9I_2^2(-16I_7 - 945I_8-\\
& 64I_9 + 27I_2J_2 + 396J_6 + 1296J_7 + 720J_8 - 624J_9)-\\
& 3I_2(1776I_3^2 - 512I_3I_4 + 8268I_3I_5 - 1696I_4I_5 - 4725I_5^2-\\
& 6552I_{14} + 1602J_{19})),\\
G_5 = {} & L_1L_2L_3.
\end{aligned}$$

By using the comitants R_i and S_i $(i = 1, 2, 3)$ the system (1) can be written in the form

$$\frac{dx}{dt} = \frac{1}{2}\frac{\partial R_1}{\partial y} + \frac{1}{2}S_1x + \frac{1}{3}\frac{\partial R_2}{\partial y} + \frac{2}{3}S_2x + \frac{1}{4}\frac{\partial R_3}{\partial y} + \frac{3}{4}S_3x,$$

$$\frac{dy}{dt} = -\frac{1}{2}\frac{\partial R_1}{\partial x} + \frac{1}{2}S_1y - \frac{1}{3}\frac{\partial R_2}{\partial x} + \frac{2}{3}S_2y - \frac{1}{4}\frac{\partial R_3}{\partial x} + \frac{3}{4}S_3y.$$

In [3] for the cubic systems (1) with $I_1 = 0$, $I_2 > 0$, $R_3 \equiv 0$ the conditions for the singular point $(0, 0)$ to be a center were constructed in the terms of the coefficients of the normal form of the system. In this paper the $GL(2,\mathbb{R})$-invariant conditions for the singular point $(0,0)$ to be a center for the cubic differential systems (1) with $I_1 = 0$, $I_2 > 0$, $R_3 \equiv 0$ were constructed.

Theorem 1. *A system* (1) *with the conditions* $I_1 = 0$*,* $I_2 > 0$*,* $R_3 \equiv 0$ *has the center in the origin of the coordinates if and only if the following conditions are fulfilled:*

$$G_1 = G_2 = G_3 = G_4 = G_5 = 0.$$

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The rational bases of $GL(2,\mathbb{R})$-comitants and of invariants for the bidimensional system of differential equations with nonlinearities of the fourth degree

Iurie Calin, Stanislav Ciubotaru

Abstract

In this paper minimal rational bases of $GL(2,\mathbb{R})$-comitants and a rational basis of $GL(2,\mathbb{R})$-invariants for the bidimensional system of differential equations with nonlinearities of the fourth degree are presented.

Keywords: differential system, comitant, invariant, rational basis.

Let us consider the system of differential equations with nonlinearities of the fourth degree

$$\frac{dx}{dt} = P_1(x,y) + P_4(x,y), \quad \frac{dy}{dt} = Q_1(x,y) + Q_4(x,y), \qquad (1)$$

where $P_i(x,y),\ Q_i(x,y)$ are homogeneous polynomials of degrees i in x and y with real coefficients. We denote by A the 14-dimensional coefficient space of the system (1), by $q \in Q \subseteq Aff(2,\mathbb{R})$ a nondegenerate linear transformation of the phase plane of (1) and by $r_q(a)$ its linear representation in the space A.

Definition 1. [1] *A polynomial $K(a,\overline{x})$ in the coefficients of the system* (1) *and the coordinates of the vector $\overline{x} = (x,y) \in \mathbb{R}^2$ is called a comitant of the system* (1) *with respect to the group Q if there exists a function $\lambda : Q \to \mathbb{R}$ such that*

$$K(r_q(a), q\cdot\overline{x}) \equiv \lambda(q)K(a,\overline{x})$$

for every $q \in Q,\ a \in A$ *and* $\overline{x} \in \mathbb{R}^2$.

If Q is the group $GL(2,\mathbb{R})$, then the comitant is called $GL(2,\mathbb{R})$-comitant or centroaffine comitant.

If a comitant does not depend on the coordinates of the vector $\overline{x}$, then it is called *invariant*

Definition 2. *The set S of the comitants (invariants) is called a rational basis on $M \subseteq A$ of the comitants (invariants) for the system* (1) *with respect to the group Q if any comitant (invariant) of the system* (1) *with respect to the group Q can be expressed as a rational function of elements of the set S.*

Definition 3. *A rational basis on $M \subseteq A$ of the comitants (invariants) for the system ((1)) with respect to the group Q is called minimal if by the removal from it of any comitant (invariant) it ceases to be a rational basis.*

Definition 4. [2] *Let f and φ be the polynomials in the coordinates of the vector $(x,y) \in \mathbb{R}^2$ of the degrees r and ρ, respectively. The polynomial*

$$(f,\varphi)^{(k)} = \frac{(r-k)!(\rho-k)!}{r!\rho!}\sum_{h=0}^{k}(-1)^h\binom{k}{h}\frac{\partial^k f}{\partial(x)^{k-h}\partial(y)^h}\frac{\partial^k \varphi}{\partial(x)^h\partial(y)^{k-h}}$$

is called the transvectant of index k of polynomials f and φ.

If the polynomials f and φ are $GL(2,\mathbb{R})$-comitants of the system (1), then the transvectant of the index $k \le \min(r,\rho)$ is also a $GL(2,\mathbb{R})$-comitant of the system (1) [3].

The $GL(2,\mathbb{R})$-comitants [1] of the first degree with respect to the coefficients of the system (1) have the form

$$R_i = P_i(x,y)y - Q_i(x,y)x,\ S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right),\ i = \overline{1,4}.$$

In repeated using of the transvectants a set of the parenthesis of the type $((\ldots($ will be replaced by a single parenthesis of the form $[\![$. By using the comitants R_i and S_i $(i = \overline{1,4})$, and the notion of transvectant the following GL-comitants and invariants of the system (1) were constructed:

$$K_1 = R_4, \quad K_2 = S_4, \quad K_3 = (R_4, R_4)^{(4)}, \quad K_4 = (R_4, R_4)^{(2)},$$
$$K_5 = (R_4, S_4)^{(3)}, \quad K_6 = (R_4, S_4)^{(2)}, \quad K_7 = (R_4, S_4)^{(1)},$$
$$K_8 = (S_4, S_4)^{(2)}, \quad K_{10} = [\![R_4, R_4)^{(4)}, R_4)^{(1)},$$
$$K_{13} = [\![R_4, R_4)^{(2)}, R_4)^{(1)}, \quad K_{17} = [\![R_4, S_4)^{(3)}, S_4)^{(2)},$$
$$K_{18} = [\![R_4, S_4)^{(3)}, S_4)^{(1)}, \quad K_{21} = [\![S_4, S_4)^{(2)}, S_4)^{(1)}, \quad Q_1 = R_1,$$
$$Q_2 = S_1, \quad Q_3 = (R_4, R_1)^{(2)}, \quad Q_4 = (R_4, R_1)^{(1)}, \quad Q_5 = (S_4, R_1)^{(2)},$$
$$Q_6 = (S_4, R_1)^{(1)}, \quad Q_{10} = [\![R_4, R_1)^{(2)}, R_1)^{(2)},$$
$$H_1 = (Q_{10}, Q_5)^{(1)}, \quad H_2 = [\![R_1, Q_5)^{(1)}, Q_5)^{(1)},$$
$$H_3 = [\![R_1, Q_5)^{(1)}, Q_{10})^{(1)}, \quad H_4 = [\![R_1, Q_{10})^{(1)}, Q_{10})^{(1)},$$
$$H_5 = [\![R_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$H_6 = [\![R_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_{10})^{(1)},$$
$$H_7 = [\![S_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)},$$
$$H_8 = [\![R_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_5)^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)},$$
$$H_9 = [\![S_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_{10})^{(1)},$$
$$H_{10} = [\![R_4, Q_5)^{(1)}, Q_5)^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)},$$
$$H_{11} = [\![S_4, Q_5)^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)},$$
$$H_{12} = [\![R_4, Q_5)^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)},$$
$$H_{13} = [\![S_4, Q_{10})^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)},$$
$$H_{14} = [\![R_4, Q_{10})^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)}, Q_{10})^{(1)}.$$

Theorem 1. *The set of $GL(2,\mathbb{R})$-comitants*

$$\{K_1,\ K_2,\ K_3,\ K_4,\ K_5,\ K_6,\ K_7,\ K_{10},\ K_{13},\ Q_1,\ Q_2,\ Q_3,\ Q_4\}$$

is a minimal rational basis of the $GL(2,\mathbb{R})$-comitants for the system (1) *of differential equations with nonlinearities of the fourth degree on* $M = \{\, a \in A \mid K_1 \not\equiv 0 \}$.

Theorem 2. *The set of $GL(2,\mathbb{R})$-comitants*

$$\{K_1,\ K_2,\ K_5,\ K_6,\ K_7,\ K_8,\ K_{17},\ K_{18},\ K_{21},\ Q_1,\ Q_2,\ Q_5,\ Q_6\}$$

is a minimal rational basis of the $GL(2,\mathbb{R})$-comitants for the system (1) *of differential equations with nonlinearities of the fourth degree on* $M = \{\, a \in A \mid\ K_2 \not\equiv 0\}\,.$

Theorem 3. *The set of $GL(2,\mathbb{R})$-invariatns*

$$\{Q_2, H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8, H_9, H_{10}, H_{11}, H_{12}, H_{13}, H_{14}\}$$

is a rational basis of the $GL(2,\mathbb{R})$-invariants for the system (1) *of differential equations with nonlinearities of the fourth degree on* $M = \{\, a \in A \mid\ H_1 \not\equiv 0\}\,.$

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Bifurcation diagrams and quotient topological spaces under the action of the affine group on a subclass of quadratic vector fields

Oxana Diaconescu, Dana Schlomiuk, Nicolae Vulpe

Abstract

In this article we consider the class $\mathbf{QSL_4^{p^c+q^c+r,\infty}}$ of all real quadratic differential systems $\dfrac{dx}{dt} = p(x,y)$, $\dfrac{dy}{dt} = q(x,y)$ with $\gcd(p,q) = 1$, having invariant lines of total multiplicity four and one real and two complex infinite singularities. We construct compactified canonical forms for this class and bifurcation diagrams for these compactified canonical forms. These diagrams contain many repetitions of phase portraits due to symmetries under the action of the group of affine transformations and time homotheties. We construct the orbit spaces under this action and the corresponding bifurcation diagrams in these orbit spaces. These diagrams retain only the essence of the dynamics and thus make transparent their content.

Keywords: Quadratic differential system, topological equivalence, group action, phase portrait, bifurcation diagram.

1 Introduction

We consider here real planar differential systems of the form

$$(S) \qquad \frac{dx}{dt} = p(x,y), \qquad \frac{dy}{dt} = q(x,y), \tag{1}$$

where p, $q \in \mathbb{R}[x,y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$, their associated vector fields $\tilde{D} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$ and differential equations $q(x,y)dx - p(x,y)dy = 0$. We call *degree* of a system (1) (or of a vector field (1) or of a differential equation (1)) the integer $n = \deg(S) = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $n = 2$.

A system (1) is said to be integrable on an open set U of $\mathbb{R}^2$ if there exists a C^1 function $F(x,y)$ defined on U which is a first integral of the system, i.e. such that $\tilde{D}F(x,y) = 0$ on U and which is nonconstant on any open subset of U. The cases of integrable systems are rare but as Arnold said in [1, p. 405] "...these integrable cases allow us to collect a large amount of information about the motion in more important systems...".

In [2] Darboux gave a method of integration of planar polynomial differential equations in terms of invariant algebraic curves. Darboux showed that if an equation (1) (or a system (1) or a vector field (1)) possesses a sufficient number of such invariant algebraic solutions $f_i(x,y) = 0$, $f_i \in \mathbb{C}[x,y]$, $i = 1, 2, \ldots, s$ then the equation has a first integral of the form $F = f_1(x,y)^{\lambda_1} \cdots f_s(x,y)^{\lambda_s}$, $\lambda_i \in \mathbb{C}$.

The simplest class of integrable quadratic systems due to the presence of invariant algebraic curves is the class of integrable quadratic systems due to the presence of invariant lines. The study of this class was initiated in articles [3–7].

An important tool in the classification of quadratic systems possessing invariant lines is the notion of *configuration of invariant lines* of a polynomial differential system. This concept splits the class we study in disjoint subsets according to their distinct configurations making the respective classification more easily accessible.

Definition 1.1. *[3] We call configuration of invariant lines of a system* (1)*, the set of all its (complex) invariant lines, each endowed with its own multiplicity and together with all the real singular points of this system located on these lines, each one endowed with its own multiplicity.*

In classifying planar differential systems the topological equivalence plays an important role. In this work we say that two planar differential systems (S_1) and (S_2) are *topologically equivalent* if and only if there exists an homeomorphism of the plane carrying orbits to orbits and preserving or reversing globally their orientation. A finer equivalence relation is the following: two systems are equivalent if they are in the same orbit under the action of the group $\mathcal{G} = Aff(2, R) \times R^*$ of affine transformations and time homotheties. This finer equivalence relation allows us to choose convenient normal forms depending on fewer than the twelve parameters, the coefficients of a general quadratic system.

2 Description of main results

Whenever in mathematics we encounter an equivalence relation R on a structured object A it is customary to construct its quotient object A/R, i.e. the set of equivalence classes of A and to inquire about its structure.

The equivalence relation on QS induced by the group $\mathcal{G}$ action yields a quotient object $QS/\mathcal{G}$, i.e. the set of all orbits of the group action, which is a five-dimensional topological space. In this work
– we construct compactified canonical forms and compactified bifurcation diagrams for the class $\mathbf{QSL_4^{p^c+q^c+r,\infty}}$;
– we show that the class $\mathbf{QSL_4^{p^c+q^c+r,\infty}}$ splits into a finite set of orbits, a finite set of families of orbits each with parameter space $\mathbb{R}$ and a set of families of orbits each with parameter space $\mathbb{R}^2$;
– we show that there are distinct systems in the bifurcation diagrams lying on the same orbit of the group $\mathcal{G}$ action.
– we construct quotient spaces with respect to the action of the group $\mathcal{G}$ and the induced bifurcation diagram on these quotient spaces.

The resulting diagrams are much simpler, retaining only a unique representatives for each orbit, capturing the essence of the dynamics when parameters vary and also when we allow the group to act.

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On Admissible Limits of Holomorphic Functions of Several Complex Variables

Peter V. Dovbush

Abstract

The aim of the present article is to establish the connection between the existence of the limit along the normal and the admissible limit at a fixed boundary point for holomorphic functions of several complex variables.

Keywords: holomorphic function, boundary behavior, admissible limit.

1 Introduction and Main Result

The connection between the existence of a radial limit and an angular limit for a holomorphic function defined on the unit disc is described by Lehto and Virtanen [3, Theorem 5] in terms of the growth of the spherical derivative.

Let D be a bounded domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary ∂D, then at each $\xi \in \partial D$ the tangent space $T^c_\xi(\partial D)$ and the unit outward normal vector ν_ξ are well-defined. We denote by $T^c_\xi(\partial D)$ and $N^c_\xi(\partial D)$ the complex tangent space and the complex normal space, respectively. The complex tangent space at ξ is defined as the $(n-1)$ dimensional complex subspace of $T_\xi(\partial D)$ and given by $T^c_\xi(\partial D) = \{z \in \mathbb{C}^n : (z, w) = 0, \forall w \in N^c_\xi(\partial D)\}$, where $(\cdot, \cdot)$ denotes canonical Hermitian product of $\mathbb{C}^n$. Let $p(z, T_\xi(\partial D))$ is the Euclidean distance from z to the real tangent plane $T_\xi(\partial D)$.

An admissible domain $\mathcal{A}_\alpha(\xi)$ with vertex $\xi \in \partial D$ and aperture $\alpha > 0$ is defined as follows [5]:

$$\mathcal{A}_\alpha(\xi) = \{\, z \in D \ : |(z-\xi, \nu_\xi)| < (1+\alpha) r_\xi(z), |z-\xi|^2 < \alpha r_\xi(z)\,\},$$

where $r_\xi(z) = \min\{r(z), p(z, T_\xi(\partial D))\}$.

The existence of admissible limits (in Fatou's theorem in the space $\mathbb{C}^n$, $n > 1$) was discovered by Koranyi [1] and Stein [2]; the complex geometrical nature of this phenomenon has been investigated by Chirka [3].

The function f defined in a domain D in $\mathbb{C}^n$ has a limit L, $L \in \overline{\mathbb{C}}$, along the normal ν_ξ to ∂D at the point ξ iff $\lim_{t\to 0} f(\xi - t\nu_\xi) = L$; f has an *admissible limit* L, at $\xi \in \partial D$ iff

$$\lim_{\mathcal{A}_\alpha(\xi)\ni z\to\xi} f(z) = L$$

for every $\alpha > 0$; f is admissible bounded at ξ if $\sup_{z\in\mathcal{A}_\alpha(\xi)} |f(z)| < \infty$ for every $\alpha > 0$.

We call

$$\frac{\left|\sum_{j=1}^n \frac{\partial f}{\partial z_j}(z) v_j\right|}{1+|f(z)|^2}$$

the spherical derivative of $f(z)$ in the direction of the vector $v \in C^n$.

We say that the vector $v \in C^n$ has normal (resp. complex tangential) direction if $v \in z + N^c_\xi(\partial D)$ (resp. $v \in z + T^c_\xi(\partial D)$).

We can now state our main result:

Theorem 1. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. If $f \in \mathcal{O}(D)$ has a limit L along the normal to ∂D at the point ξ, then at the point $\xi \in \partial D$ the function f has an admissible limit L if and only if in every admissible domain with vertex ξ the spherical derivative of f in the normal and complex tangent directions increases like $o(1/r_\xi(z))$ and $o(1/\sqrt{r_\xi(z)})$, respectively.*

2 An extension of the Lindelöf principle

The following refinement of Lindelöf's principle holds.

Theorem 2. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. If a function f in D has a limit L, $L \in \overline{\mathbb{C}}$, along the normal ν_ξ at a point $\xi \in \partial D$, and in every admissible domain with vertex ξ the function f is holomorphic, L is his omitted value and the spherical derivative of f in the normal and complex tangent directions grows no faster than $K/d(z)$ and $K/\sqrt{d(z)}$, respectively, then f has an admissible limit L at ξ.*

Also we have the following theorem.

Theorem 3. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. Let in every admissible domain with vertex ξ the function f be holomorphic and its spherical derivative in the normal and complex tangent directions grows no faster than $K/d(z)$ and $K/\sqrt{d(z)}$, respectively. If*

$$\lim_{A_\beta(\xi) \ni z \to \xi} f(z) = L \text{ for some } \beta > 0,$$

then f has an admissible limit at ξ.

For bounded holomorphic functions this theorem appears in Chirka's paper [1], with the proof sketched there relying on certain estimates on harmonic measures. A proof based on a different method was given by Ramey [4, Theorem 2].

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The Monge equation and topology of solutions of the second order ODE's

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Abstract

Properties of the Monge equation $\Phi\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right) = 0$ associated with the partial differential of equation $F_x \dot{x} + F_y \dot{y} = 0$ to the first integral $F(x, y) = C$ of the system ODE $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ are studied. Examples of solutions of the equation $P\mu_x + Q\mu_y + (P_x + Q_y)\mu = 0$ to the integrating multiplier $\mu = \mu(x, y)$ of the first order ODE $Qdx - Pdy = 0$, where $Q(x, y)$ and $P(x, y)$ are the polynomial on x, y, are constructed. Topological properties of the relations $F(x, y, a, b) = 0$ which generate dual second order ODE's $y'' = f(x, y, y')$ and $b'' = \phi(a, b, b')$ are considered.

Keywords: The Monge equation, integrating multiplier, duality, ODE.

1 Introduction

Finding integrating multiplier $\mu(x, y)$ for the equation $y' = f(x, y) = \frac{Q(x,y)}{P(x,y}$ leads to the search for solutions of linear p.d.e. $P\mu_x + Q\mu_y + (P_x + Q_y)\mu = 0$, which is a problem equivalent to solution of the equation $y' = f(x, y)$. In this paper we study general properties of the equation for the integrating multiplier using associated equations of Monge. Another problem considered here is the study of the properties of the relations $F(x, y, a, b) = 0$ which generate the second order ODE $y'' = h(x, y, y')$ and the equation $b'' = \phi(a, b, b')$ dual to it.

2 The Monge equation

Monge equation for the first order p.d.e. $H(x, y, z, z_x, z_y) = 0$ or $H(x, y, z, p, q) = 0$ is a result of an exclusion of the variables $p = z_x$ and $q = z_y$ from the system of equations $H(x, y, z, p, q) = 0$, $\frac{dx}{H_p} = \frac{dy}{H_q} = \frac{dz}{pH_p + qH_q}$. To construct the Monge equation for the p.d.e. $P_2 F_x + Q_2 F_y = 0$, which determines the first integral $F(x, y) = C$ of the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ we transform it into the form $a_0\, \omega_x + a_1\, x\omega_x + a_2\, \omega_t\omega_x + a_{11}\, x^2\omega_x + a_{12}\, x\omega_t\omega_x + a_{22}\, \omega_t^2\omega_x - b_0\, t - x b_1\, t - \omega_t b_2\, t - x^2 b_{11}\, t - \omega_t x b_{12}\, t - \omega_t^2 b_{22}\, t = 0$, with the help of change of the variables $F(x, y) = u(x, t)$, $y = v(x, t)$, $F_x = u_x - \frac{u_t}{v_t} v_x$, $F_y = \frac{u_t}{v_t}$, where $u(x, t) = t\omega_t - \omega$, $v(x, t) = \omega_t$, $\omega = \omega(x, t)$ and t is the parameter. To obtain Monge equation associated with this equation we restrict ourselves to the special case when the parameters are $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_{11} = 0$, $a_{12} = 1$, $a_{22} = 0$. Remark that full quadratic system of equations is reduced to the form $(1 + xy)y' = Q_2(x, y)$ by means of linear change of variables.

Theorem 1. *The Monge equation for the p.d.e. which is associated with ODE of the form* $(1 + xy)dy - Q_2(x, y)dx = 0$ *is*

$$\begin{gathered} x^2 (z_x)^2 + \left(4\, y b_{22} + 2\, x y_x - 2\, b_{12}\, x^2 y - 2\, b_2\, yx\right) z_x + \\ + (y_x)^2 + \left(-4\, x^2 b_1\, y - 4\, x^3 b_{11}\, y + 2\, b_2\, y - 4\, b_0\, yx + 2\, b_{12}\, xy\right) y_x + \\ + x^2 y^2 {b_{12}}^2 - 4\, b_0\, y^2 b_{22} - 4\, x b_1\, y^2 b_{22} - 4\, x^2 b_{11}\, y^2 b_{22} + 2\, x y^2 b_2\, b_{12} + \\ + y^2 {b_2}^2 = 0, \end{gathered} \tag{1}$$

where $z = z(x)$ *and* $y = y(x)$.

Integral curves of the Monge equation envelope characteristic curves of the p.d.e. and so they can be useful to study properties of solutions of corresponding ODE. Properties of solutions of equation (1) as function on parameters can be studied geometrically after their consideration as quadratic integral of geodesics of 3-dimensional space with the Riemann metric $ds^2 = x^2 dz^2 + \left(2\, dy\, x + \left(-2\, b_{12}\, x^2 y - 2\, b_2\, yx + 4\, y b_{22}\right) dx\right) dz + dy^2 + \left(-4\, x^2 b_1\, y - 4\, x^3 b_{11}\, y + 2\, b_2\, y - 4\, b_0\, yx + 2\, b_{12}\, xy\right) dx\, dy + + y^2 \left(-4\, x^2 b_{11}\, b_{22} + x^2 {b_{12}}^2 + 2\, x b_2\, b_{12} - 4\, x b_1\, b_{22} + {b_2}^2 - 4\, b_0\, b_{22}\right) dx^2$.

2.1 Integrating multiplier

The first order of p.d.e. for integrating multiplier $\mu(x,y) = \exp(h(x,y)$ for the equation $P_2 dy - Q_2 dx = 0$ has the form $Q_2 h_y + P_2 h_x + P_{2x} + Q_{2y} = 0$. To obtain the equation of Monge associated with this equation we transform it into another form with the help of Ampere transformation $x = \xi,\ y = \rho_\eta,\ z = \eta\rho_\eta - \rho(\xi,\eta),\ h_x = -\rho_\xi,\ h_y = \eta$. In result we get the equation $(\eta\, b_{22} - a_{22}\,\rho_\xi)\,(\rho_\eta)^2 + (a_{12} + 2\,b_{22} + (-a_2 - \xi\,a_{12})\,\rho_\xi + \eta\,b_2 + \eta\,b_{12}\,\xi)\,\rho_\eta +$
$+ \left(-a_{11}\,\xi^2 - a_0 - \xi\,a_1\right)\rho_\xi + b_2 + 2\,a_{11}\,\xi + b_{12}\,\xi + \eta\,b_{11}\,\xi^2 +$
$+ a_1 + \eta\,b_1\,\xi + \eta\,b_0 = 0$, from which in particular case $a_0 = 1, a_1 = 0, a_2 = 0, a_{11} = 0, a_{12} = 1, a_{22} = 0$ follows the Monge equation

$$-(z_x)^2 x^2 + \left(2\,y b_{12}\,x^2 - 2\,y_x x + 2\,y x b_2 + 2\,x - 4\,b_{22}\,y + 4\,x b_{22}\right) z_x -$$

$$-(y_x)^2 + 2\left(y(-b_{12}\,x + 2b_0\,x + 2b_1\,x^2 - b_2\right) y_x +$$

$$+2\left(-2\,b_{22} + 2\,b_{12}\,x^2 - 1 + 2\,x b_2 + 2\,y b_{11}\,x^3\right) y_x +$$

$$+4\,y^2 b_0\,b_{22} - 1 - y^2 {b_2}^2 - y^2 {b_{12}}^2 x^2 - 4\,b_{22} - 2\,y^2 b_{12}\,b_2\,x +$$

$$+4\,y^2 b_{22}\,b_{11}\,x^2 + 4\,y^2 b_{22}\,b_1\,x - 4\,{b_{22}}^2 - 2\,y b_{12}\,x - 2\,b_2\,y = 0.$$

3 On topology of General Integral

Definition. *The equations $y'' = f(x,y,y')$ and $b'' = \phi(a,b,b')$ form a dual pair if each of them is obtained by excluding variables (a,b) or (x,y) from the relation $F(x,y(x),a,b(a)) = 0$, after its double differentiation on the corresponding variable x or a* [2]. The relation $F(x,y,a,b) = 0$ in this case is a common General Integral for both equations and it can have nontrivial topological properties.

Theorem 1. *Differentiating twice the relation $F(x,y,a,b) = y(x)^2 b + x^3 + axb^2 + nb^3 = 0$ with respect to x and excluding the variables (b,a) from the obtained equalities, we get the differential equation*

$$2\,x^2\,(y(x))^3\,(y'')^3 + \left(6\,x^2\,(y')^2\,y(x)^2 + 3\,y(x)^4 - 6\,xy(x)^3 y'\right)(y'')^2 +$$

$$+\left(6\,y(x)^3(y')^2+6\,x^2(y')^4y(x)-12\,xy(x)^2(y')^3\right)y''+3\,y(x)^2(y')^4+$$
$$+27\,nx^2+2\,x^2(y')^6-6\,xy(x)(y')^5=0.$$

By analogy the dual equation is obtained

$$b(a)^6(b'')^3+\left(-9\,b(a)^5(b')^2-3\,(b(a)^4a(b')^3)\right)(b'')^2+$$
$$+\left(12\,b(a)^3(b')^5a+24\,b(a)^4(b')^4\right)b''+12\,b(a)(b')^8a^2-108\,(b')^9n^2+$$
$$+4\,a^3(b')^9-16\,b(a)^3(b')^6=0$$

Corollary 1 *The relation* $F(x,y,a,b)=y^2b+x^3+axb^2+nb^3=0$ *in variables* (x,y) *is a family of cubic curves of genus* $g=1$ *dependent on parameters* (a,b) *and arbitrary* $n=n(a)$*. But this relation, considered as family of algebraic curves in the variables* (a,b)*, can have arbitrary genus* $g=N, N=0,1,2,3...$*, for example, (*$g=0$ *if* $n=a^3$*, and* $g=2$ *when* $n=a^4$*). Appropriate classes of dual equations differ and this property can be used in the theory of* $H(x,y,y',y'')=0$*, as well as the first order theory of ODE* $h(x,y,y')=0$*.*

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A Bifurcation Scenario of Chaos Transition for the Classical Lorenz System

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Abstract

Developing a bifurcational geometric approach to the qualitative analysis of polynomial dynamical systems and using some numerical results, we present a new bifurcation scenario of chaos transition for the classical Lorenz system.

Keywords: Lorenz system, bifurcation, chaos, limit cycle.

1 Introduction

Consider a three-dimensional polynomial dynamical system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = x(r - z) - y, \quad \dot{z} = xy - bz \tag{1}$$

known as the Lorenz system. Historically, (1) was the first dynamical system for which the existence of an irregular attractor (chaos) was proved for $\sigma = 10$, $b = 8/3$, and $24{,}06 < r < 28$. For many years, the Lorenz system has been the subject of study by numerous authors (see, [3]–[7]). However, until now the structure of the Lorenz attractor is not yet completely clear, and the most important question at present is to understand the bifurcation scenario of chaos transition in the system (1), which is related to *Smale's Fourteenth Problem* [6]. In this paper, we present a new bifurcation scenario for system (1), where $\sigma = 10$, $b = 8/3$, and $r > 0$, using numerical results of [4] and a bifurcational geometric approach to the global qualitative analysis of three-dimensional dynamical systems, which was applied earlier in the two-dimensional case [1, 2].

2 A bifurcation scenario of chaos transition

1. The Lorenz system (1) is dissipative and symmetric with respect to the z-axis. The origin $O(0,0,0)$ is a singular point of system (1) for any σ, b, and r. It is a stable node for $r < 1$. For $r = 1$, the origin becomes a triple singular point, and then, for $r > 1$, there are two more singular points in the system: $O_1(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$ and $O_2(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$, which are stable up to the parameter value $r_a = \sigma(\sigma + b + 3)/(\sigma - b - 1)$ ($r_a \approx 24{,}74$ for $\sigma = 10$ and $b = 8/3$). For all $r > 1$, the point O is a saddle-node. It has a two-dimensional stable manifold W^s and a one-dimensional unstable manifold W^u. If $1 < r < r_l \approx 13{,}9$, then the separatrices Γ_1 and Γ_2 issuing from the point O along its one-dimensional unstable manifold W^u are attracted by their nearest stable points O_1 and O_2, respectively.

2. If $r = r_l$, then each of the separatrices Γ_1 and Γ_2 becomes a closed homoclinic loop. In this case, two unstable homoclinic loops, C_0^+ and C_0^-, are formed around the points O_1 and O_2, respectively. They are tangent to each other and the z-axis at the point O, and form together a homoclinic butterfly.

3. If $r_l < r < r_a \approx 24{,}74$, then, unfortunately, neither the classical scenario (see, e. g., [7]) nor the scenario of [4] can be realized. The reason is that, in both cases, trajectories of system (1) should intersect the two-dimensional stable manifold W^s of the point O. Since this is impossible, the only way to overcome the contradiction is to suppose that a cascade of period-doubling bifurcations [4] will begin immediately in each of the half-spaces with respect to the manifold W^s, when $r > r_l$. In this case, each of the homoclinic loops C_0^+ and C_0^- generates an unstable limit cycle of period 2 and a stable limit cycle of period 1 lying between the coils of the cycle of period 2 in the corresponding half-spaces containing the points O_1 and O_2, respectively. With further growth of r, each of the cycles of period 2 generates an unstable limit cycle of period 4 with a stable limit cycle of period 3 inside of it and each of the cycles of period 1 generates a stable limit cycle of period 2 with an unstable limit cycle of period 1 inside of it. Then, after next

doubling, we will have in each of the half-spaces an unstable limit cycle of period 8 with an inserted stable limit cycle of period 7 and a stable limit cycle of period 6 with an inserted unstable limit cycle of period 5, and a stable limit cycle of period 4 with an inserted unstable limit cycle of period 3, and an unstable limit cycle of period 2 with an inserted stable limit cycle of period 1. Continuing this process further, we will obtain limit cycles of all periods from one to infinity, and the space between these cycles will be filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as $t \to +\infty$. These cycles are inserted into each other, they make various combinations of rotation around the points O_1 and O_2 in the corresponding half-spaces containing these points and form geometric constructions (limit periodic sets), which look globally like very flat truncated cones described in the chaos transition scenario of [4].

4. For $r = r_a$, the biggest unstable limit cycles of infinite period disappear through the Andronov–Shilnikov bifurcation [3, 5] in each of the half-spaces containing the points O_1 and O_2 (the cone vertices are at these points), and these points become unstable saddle-foci generating two small stable limit cycles lying on two-dimensional focus manifolds of O_1 and O_2.

5. If $r_a < r < +\infty$, then a cascade of period-halving bifurcations [4] occurs in each of the half-spaces with respect to the manifold W^s. We have got again two symmetric with respect to the z-axis limit periodic sets consisting of limit cycles of all periods, which are inserted into each other and make various combinations of rotation around the points O_1 and O_2 in the corresponding half-spaces containing these points, and the space between the cycles is filled by spirals issuing from unstable limit cycles and tending to stable limit cycles as $t \to +\infty$. With further growth of r, the period-halving process makes the limit periodic sets more and more flat. The obtained geometric constructions are the only stable limit sets of system (1). The spirals of the unstable saddle-foci O_1 and O_2 and the trajectories issuing from infinity tend to these limit periodic sets (more precisely, to their stable limit cycles) as $t \to +\infty$. Just these stable limit periodic sets form two symmetric parts of the

so-called Lorenz attractor, and this really looks very chaotic.

6. If $r \to +\infty$ (numerically, when $r > 313$), then the period-halving process will be finishing and system (1) will have two stable limit cycles lying on the two-dimensional focus manifolds of the unstable saddle-foci O_1 and O_2 in two phase half-spaces of (1) containing these points. This completes our scenario of chaos transition in the Lorenz system (1).

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On compact viable sets in affine IFS with coherent hyperbolic structure

Vasile Glavan, Valeriu Guţu

Abstract

Viable sets in multi-valued dynamics, generated by finite families of mappings, are considered. In the case of affine Iterated Function Systems with coherent hyperbolic structure the existence of compact viable sets is stated, and some topological characteristics of these sets are given.

Keywords: Set-valued dynamics, iterated function systems, viable set, cone condition.

1 Introduction

Contractive Iterated Function Systems (IFS) are very well studied, many classical fractals are their attractors and various algorithms for constructing these fractals are elaborated (see, e.g., [1]). Generalizations for (weakly) contracting relations have been purposed in [4], [5], where existence of the attractors has been stated and the dynamics of relations near these attractors have been studied. The attractor is (forwardly) invariant and the dynamics on it characterizes the dynamics on the whole phase space. If the IFS is not contractive the dynamic itself is much more complex and proper invariant subsets may not exist. In control theory a weaker notion is considered – viability. In our paper we are concerned with viable compact sets in affine IFS. We adapt the well known concept of cone condition from ordinary dynamics to the set-valued case and give necessary and sufficient conditions for a compact viable set to exist. We also prove that under these conditions the compact viable set is the closure of periodic chains (trajectories).

2 Viable sets in IFS

In this section we introduce the notion of viable set under an IFS, and characterize it as a surjective subset for a relation in terms of E. Akin (see [2]). We also give some topological properties of viable subsets.

Given the complete metric space (X, d), consider the Iterated Function System $\mathcal{F} = \{X; f_1, f_2, \dots, f_m\}$, consisting of pairwise distinct continuous functions $f_j : X \to X$ $(1 \leq j \leq m)$. Let $T = \mathbb{Z}$ or $T = \mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \geq 0\}$, or $T = \mathbb{Z}_- = \{n \in \mathbb{Z} \mid n \leq 0\}$.

Let $\alpha, \beta \in \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}$, $\alpha < \beta$, and $S = (\alpha, \beta)$. A sequence $(x_n)_{n \in S}$ in X is called a *chain on* S of the IFS $\mathcal{F}$ if for every $n \in S$ there exists $j_n \in \{1, 2, \dots, m\}$ such that $x_{n+1} = f_{j_n}(x_n)$, provided $(n+1) \in S$ as well.

Given $\alpha \in \mathbb{Z}_- \cup \{-\infty\}$ and $\beta \in \mathbb{Z}_+ \cup \{+\infty\}$, $\alpha < \beta$, the nonempty subset $A \subseteq X$ is called *viable* on $S = (\alpha, \beta)$ if for every $a \in A$ there exists at least one chain $(x_n)_{n \in S}$ such that $x_0 = a$ and $x_n \in A$ for all $n \in S$.

Obviously, every (completely) invariant set is viable on $(\mathbb{Z})$ $\mathbb{Z}_+$, as well as each fixed point or periodic chain does.

Given the IFS $\{X; f_1, \dots, f_m\}$ denote by $F_* : \mathcal{P}(X) \to \mathcal{P}(X)$ the corresponding *Nadler-Hutchinson operator* defined as follows: $F_*(C) = \bigcup\limits_{j=1}^{m} f_j[C]$, where $f_j[C] = \bigcup\limits_{x \in C} f_j(x)$. (Here $\mathcal{P}(X)$ denotes the space of all nonempty compact subsets of X endowed with the Pompeiu-Hausdorff metric.)

Let X_F be the set of all chains $(x_n)_{n \in \mathbb{Z}}$ in X endowed with compact-open topology (see [2]).

Denote by $\pi_0 : X_F \to X_F$, $\pi_0 : (x_n)_{n \in \mathbb{Z}} \mapsto x_0$, the projection on zero-coordinate.

Lemma [2]. $\bigcap\limits_{i \in \mathbb{Z}} F_*^i(X) = \pi_0(X_F)$.

Recall that a subset A is said to be *completely invariant*, if $F_*^{-1}(A) = A = F_*(A)$. The following theorem gives a similar characteristic for viable sets.

Theorem 1. *A set A is viable on $\mathbb{Z}$ if and only if $A \subset F_*(A) \cap F_*^{-1}(A)$.*

Theorem 2. *Let $\mathcal{F} = \{X; f_1, f_2, \dots, f_m\}$ be an IFS, consisting of homeomorphisms. If the set A is viable on $S \subset \mathbb{Z}$, then its closure $\bar{A}$ is also viable on S.*

3 Families of invariant cones

In what follows we are concerned with a class of expanding affine IFS' with "coherent hyperbolic structure".

Following [3], let $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ and let $x = (u, v) \in \mathbb{R}^n$. Given a real positive number γ, we call *standard horizontal γ-cone* at the origin of coordinates the subset

$$H_\gamma = \{(u, v) \in \mathbb{R}^n : \|v\| \leq \gamma \|u\|\}.$$

The corresponding *standard vertical cone* is defined analogously.

We say that a family of cones is definite on the metric space Y, provided that to any point $y \in Y$ a cone is attributed; the inward condition $A(H) \Subset (H')$ means that the (closed) cone H is mapped by A into the interior of the cone H'.

Theorem 3. *Let $\mathcal{H}, \mathcal{V} \subset \Sigma_m \times \mathbb{R}^k$ be two families of cones such that:*

1. *For any $t = (\dots, t_{-1}, t_0, t_1, \dots) \in \Sigma_m$ one has $A_{t_0} \mathcal{H}_t \Subset \mathcal{H}_{\sigma(t)}$ and $A_t^{-1} \mathcal{V}_t \Subset \mathcal{V}_{\sigma^{-1}(t)}$;*
2. *There exists $\mu > 1$ such that $\|A_{t_0} \xi\| \geq \mu \|\xi\|$ for any $\xi \in \mathcal{H}_t$, $t \in \Sigma_m$, and $\|A_{t_0}^{-1} \eta\| \geq \mu \|\eta\|$ for any $\eta \in \mathcal{V}_t$, $t \in \Sigma_m$.*

Then there exist two families of linear subspaces $E_t^s \subset \mathcal{H}_t$ and $E_t^u \subset \mathcal{V}_t$ such that

$$E_t^s \oplus E_t^u = \mathbb{R}^k,$$

$$A_{t_0}(E_t^s) = E_{\sigma(t)}^s, \quad A_{t_0}(E_t^u) = E_{\sigma(t)}^u, \quad A_{t_0}^{-1}(E_{\sigma(t)}^u) = E_t^u,$$

$$\|A_{t_0} \xi\| \leq \frac{1}{\mu} \|\xi\| \ (\xi \in E_t^s), \quad \|A_{t_0}^{-1} \eta\| \leq \frac{1}{\mu} \|\eta\| \ (\eta \in E_{\sigma(t)}^u).$$

Moreover, the vectorial sets E_t^s and E_t^u depend continuously on t.

Theorem 4. *Assume that the linear part of the affine IFS F satisfies the generalized cone conditions 1 and 2 from Theorem 3. Let Λ denote the subset of points $x \in \mathbb{R}^n$, for which there exists at least one bounded chain $(x_n)_{n\in\mathbb{Z}}$ with $x_0 = x$. Then:*

a) The subset Λ is nonempty compact and viable on $\mathbb{Z}$. Moreover, Λ is the maximal closed subset with these properties.

b) The set Λ is the smallest closed and viable on $\mathbb{Z}$ subset containing $Per(F)$, thus $\Lambda = \overline{Per(F)}$.

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Limits of solutions to the singularly perturbed abstract hyperbolic-parabolic systems

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Abstract

We study the behavior of solutions to the problem

$$\begin{cases} \varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + A(t)u_\varepsilon(t) = f_\varepsilon(t), & t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases}$$

in the Hilbert space H as $\varepsilon \to 0$, where $A(t), t \in (0,\infty)$, is a family of linear self-adjoint operators.
Keywords: singular perturbation, abstract second order Cauchy problem, boundary layer function, a priori estimate.

Let H be a real Hilbert space endowed with the scalar product $(\cdot,\cdot)$ and the norm $|\cdot|$, V be a real Hilbert space endowed with the norm $||\cdot||$. Let $A(t) : V \subset H \to H$, $t \in [0,\infty)$, be a family of linear self-adjoint operators. Consider the following Cauchy problem:

$$\begin{cases} \varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + A(t)u_\varepsilon(t) = f_\varepsilon(t), & t \in (0,T), \\ u_\varepsilon(0) = u_{0\varepsilon}, \quad u_\varepsilon'(0) = u_{1\varepsilon}, \end{cases} \tag{P_ε}$$

where $\varepsilon > 0$ is a small parameter($\varepsilon \ll 1$), $u_\varepsilon, f_\varepsilon : [0,T) \to H$.

We investigate the behavior of solutions u_ε to the problems (P_ε) when $u_{0\varepsilon} \to u_0$, $f_\varepsilon \to f$ as $\varepsilon \to 0$. We establish a relationship between solutions to the problems (P_ε) and the corresponding solution to the following unperturbed problem:

$$\begin{cases} v'(t) + A(t)v(t) = f(t), & t \in (0,T), \\ v(0) = u_0. \end{cases} \tag{P_0}$$

The framework of our studying is determined by the following conditions:

(H1) *V is separable and $V \subset H$ continuously and densely* i.e.

$$|u|^2 \leq \gamma ||u||^2, \quad \forall u \in V;$$

(H2) *For each $u, v \in V$ the function $t \mapsto (A(t)u, v)$ is continuously differentiable on $(0, \infty)$ and*

$$\big|(A'(t)u, v)\big| \leq a_0 |u||v|, \quad \forall u, v \in V, \quad \forall t \in [0, \infty);$$

(H3) *The operators $A(t) : V \subset H \to H, t \in [0, \infty)$ are linear, self-adjoint and positive definite, i.e. there exists $\omega > 0$ such that*

$$(A(t)u, u) \geq \omega \, ||u||^2, \quad \forall u \in V, \quad \forall t \in [0, \infty);$$

(H4) *For each $u, v \in V$ the function $t \mapsto (A(t)u, v)$ is twice continuously differentiable on $(0, \infty)$ and*

$$\big|(A''(t)u, v)\big| \leq a_1 |u||v|, \quad \forall u, v \in V, \quad \forall t \in [0, \infty).$$

In [1] the existence and uniqueness of solutions to the problems (P_ε) and (P_0) is proved.

If conditions **(H1)-(H3)** are fulfilled, $u_{0\varepsilon} \in V$, $u_{1\varepsilon} \in H$ and $f_\varepsilon \in L^2(0, T; H)$, then there exists the unique function $u_\varepsilon \in W^{2,2}(0, T; H) \bigcap L^2(0, T; V)$, $A(\cdot)u_\varepsilon \in L^2(0, T; H)$ (strong solution), which satisfies the equation a. e. on $(0, T)$ and the initial conditions from (P_ε). If, in addition, $u_{1\varepsilon} \in V$, $f_\varepsilon(0) - A(0)u_{0\varepsilon} \in V$, $f_\varepsilon \in W^{2,1}(0, T; H)$, then $A(\cdot)u_\varepsilon \in W^{1,2}(0, T; H)$ and $u_\varepsilon \in W^{3,2}(0, T; H) \bigcap W^{1,2}(0, T; H)$.

If conditions **(H1)-(H3)** are fulfilled, $u_{0\varepsilon} \in H$, and $f_\varepsilon \in L^2(0, T; H)$, then there exists the unique function $u_\varepsilon \in W^{2,2}(0, T; H) \bigcap L^2(0, T; V)$, which satisfies the equation a. e. on $(0, T)$ and the initial conditions from (P_0).

Theorem 1. *Let $T > 0$. Let us assume that conditions* **(H1) - (H3)** *are fulfilled. If u_0, $u_{0\varepsilon}\, u_{1\varepsilon} \in V$ and f, $f_\varepsilon \in W^{1,2}(0, T; H)$, then there* *exist constants* $C = C(T, \gamma, a_0, \omega) > 0$*,* $\varepsilon_0 = \varepsilon_0(\gamma, a_0, \omega)$*,* $\varepsilon_0 \in (0, 1)$*, such that*

$$||u_\varepsilon - v||_{C([0,T];H)}$$
$$\leq C\left(M\,\varepsilon^{1/4} + |u_{0\varepsilon} - u_0| + ||f_\varepsilon - f||_{L^2(0,T;H)}\right), \forall \varepsilon \in (0, \varepsilon_0),$$

where u_ε *and* v *are strong solutions to problems* (P_ε) *and* (P_0)*, respectively,*

$$M = M(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = |A(0)u_{0\varepsilon}| + |A^{1/2}(0)u_{1\varepsilon}| + ||f_\varepsilon||_{W^{1,2}(0,T;H)}.$$

Theorem 2. *Let* $T > 0$*. Let us assume that conditions* **(H1) - (H4)** *are fulfilled. If* $u_0, u_{0\varepsilon}, A(0)u_{0\varepsilon}\, u_{1\varepsilon}, f_\varepsilon(0) \in V$ *and* $f, f_\varepsilon \in W^{2,2}(0, T; H)$*, then there exist constants* $C = C(T, \omega, \gamma, a_0, a_1) > 0$*,* $\varepsilon_0 = \varepsilon_0(\omega, \gamma, a_0, a_1)$*,* $\varepsilon_0 \in (0, 1)$*, such that*

$$||u'_\varepsilon - v' + h_\varepsilon e^{-t/\varepsilon}||_{C([0,\,T]\,;H)} \leq C\left(M_1, \varepsilon^{1/4} + D_\varepsilon\right), \forall \varepsilon \in (0, \varepsilon_0),$$

where u_ε *and* v *are strong solutions to problems* (P_ε) *and* (P_0) *respectively,* $h_\varepsilon = f_\varepsilon(0) - u_{1\varepsilon} - A(0)u_{0\varepsilon}$*,*

$$M_1(T, u_{0\varepsilon}, u_{1\varepsilon}, f_\varepsilon) = |A^{1/2}(0)f_\varepsilon(0)| + |A^{3/2}(0)u_{0\varepsilon}|$$
$$+|A^{1/2}(0)u_{1\varepsilon}| + ||A(t)h_\varepsilon||_{L^2(0,\infty;H)} + ||f_\varepsilon||_{W^{2,2}(0,\infty;H)},$$
$$D_\varepsilon = ||f_\varepsilon - f||_{W^{1,\,2}(0,\,T;\,H)} + |A_0(u_{0\varepsilon} - u_0)|.$$

Sketch of proof: Our approach is based on two key points. The first one is the relationship between solutions to the problems (P_ε) and (P_0). This relationship was established in the work [2].

The second key point based on *a priori* estimates of solutions to the unperturbed problem, which are uniform with respect to small parameter ε. Namely, if conditions **(H1)-(H3)** are fulfilled, $u_{0\varepsilon} \in V$, $u_{1\varepsilon} \in H$ and $f_\varepsilon \in L^2(0, \infty; H)$, then there exist constants $C = C(\gamma, a_0, \omega) > 0$ such that for every solution u_ε to the problem (P_ε) the estimate

$$||u_\varepsilon||_{C([0,t];H)} + ||A^{1/2}(\cdot)u_\varepsilon||_{L^2([0,t];H)} \leq CM_{0\varepsilon}, \forall \varepsilon \in (0, \varepsilon_0), \forall t \geq 0,$$

is valid, where

$$M_{0\varepsilon} = |A^{1/2}(0)u_{0\varepsilon}| + \varepsilon|u_{1\varepsilon}| + ||f_\varepsilon||_{L^2(0,\infty;H)}, \quad \varepsilon_0 = \min\left\{1, \frac{\omega}{2\gamma\, a_0}\right\}.$$

If, in addition, $u_{1\varepsilon} \in V$ and $f_\varepsilon \in W^{1,2}(0,\infty;H)$, then

$$||u_\varepsilon'||_{C([0,t];H)} + ||A^{1/2}(\cdot)u_\varepsilon'||_{L^2([0,t];H)} \le C\ M_\varepsilon, \forall\varepsilon \in (0,\varepsilon_0), \forall t \ge 0,$$

$$M_\varepsilon = |A(0)u_{0\varepsilon}| + |A^{1/2}(0)u_{1\varepsilon}| + ||f_\varepsilon||_{W^{1,2}(0,\infty;H)}.$$

Let u_ε be the strong solution to the problem (P_ε) and let us denote

$$z_\varepsilon(t) = u_\varepsilon'(t) + h_\varepsilon e^{-t/\varepsilon}, \quad h_\varepsilon = f_\varepsilon(0) - u_{1\varepsilon} - A(0)u_{0\varepsilon}. \tag{1}$$

If conditions **(H1)-(H4)** are fulfilled, $f_\varepsilon(0) - A(0)u_{0\varepsilon}$, $u_{1\varepsilon}, \in V$ and $f_\varepsilon \in W^{1,2}(0,\infty;H)$, then there exist constants $C = C(\gamma,\omega,a_0,a_1) > 0$ and $\varepsilon_0 = \varepsilon_0(\gamma,\omega,a_0,a_1) \in (0;1)$ such that for z_ε, defined by (1), the estimate

$$\big|\big|A^{1/2}(\cdot)z_\varepsilon\big|\big|_{C(0,t;H)} + \big|\big|z_\varepsilon'\big|\big|_{L^2(0,t;H)} \le C\, M_{1\varepsilon}, \forall\varepsilon \in (0,\varepsilon_0), \forall t \ge 0,$$

is valid, where $M_{1\varepsilon} = |A^{1/2}(0)\big(f_\varepsilon(0) - A(0)u_{0\varepsilon}\big)| + |A^{1/2}(0)u_{1\varepsilon}| + ||A(t)h_\varepsilon||_{L^2(0,\infty;H)} + ||f_\varepsilon||_{W^{2,2}(0,\infty;H)}$.

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Complete normal forms for germs of vector field with quadratic leading part

Ewa Stróżyna

We consider the complex plane vector field with zero linear part

$$\dot{x} = ax^2 + bxy + cy^2 + \ldots, \qquad \dot{y} = dx^2 + exy + fy^2 + \ldots$$

We divide the family of such differential equations into a few cases depending on the number of invariant lines of the system. As the first case we investigate the case with the polynomial first integral. We find the non-orbital normal form as the product of the orbital normal form and the orbital factor. We apply the generalization of the method used to solve completely the problem of formal classification of the vector fields with Bogdanov–Takens singularity.

The general situation is such that we have a vector field V (which is polynomial and 'good') and a perturbation W (which is of high order). We want to reduce W in $V + W$ to some normal form by application of a diffeomorphism $\exp Z$, i.e. the time 1 flow generated by a vector field Z (also of high order). Recall that linear in Z part of the action of $\exp Z$ on V equals the commutator $[Z, V] = -\mathrm{ad}_V Z$ (the homological operator).

We 'divide' the perturbations W into two parts: transversal to V and tangential to V; also the vector fields Z are subject to such division. Following our previous paper (where the formal orbital normal forms for initial vector fields are obtained) we measure the 'transversal' to V part by the bivector fields $V \wedge W = h(x, y) \cdot \partial_x \wedge \partial_y$, i.e. by one function h. The tangential to V part is of the form $g(x, y)V$, hence it is also measured by one function g.

There exist homological operators, analogues to ad_V, which act on the transversal and tangential parts:

$$\begin{aligned} f &\longmapsto C(V)f := V(f) \quad \text{(tangential)}, \\ f &\longmapsto D(V)f := V(f) - \mathrm{div} V \cdot f \quad \text{(transversal)}. \end{aligned}$$

The kernel of the operator $C(V)$ consists of first integrals of V and the kernel of $D(V)$ consists of so-called inverse integrating factors for V. Note that we have not fixed the vector field V; it is fixed at given step of the reduction to normal form.

At the first step we take $V = V_0$ as quadratic homogeneous. It is easy to find $\ker C(V_0)$ and $\ker D(V_0)$. Moreover, using so-called Schwarz–Christoffel integrals one can find complementary subspaces to $\mathrm{Im} C(V_0)$ and to $\mathrm{Im} D(V_0)$.

In the second step one takes $V = V_0 + V_1$, where V_1 is the 'first' non-trivial term from the first order normal form. Here the operator $C(V)$ (respectively $D(V)$) acts on functions of the form $H + f$ (respectively $M + f$), where $H \in \ker C(V_0)$ (respectively $M \in \ker D(V_0)$).

The third (and the last) step appears when the vector field $V = V_0 + V_1$ (used in the second step) has a homogeneous inverse integrating multiplier M or homogeneous first integral H. Then those functions M, H are used in homological operators associated with vector field $V = V_0 + V_1 + V_2$.

Let us pass to more precise description of our method (with details given in [2]). The quadratic part of the initial system defines a homogeneous vector field which, after transforming an invariant line to $x = 0$, takes the following form:

$$V_0 = (Ax^2 + Bxy)\partial_x + (Cx^2 + Dxy + Ey^2)\partial_y.$$

After the change of coordinates $(x, u) = (x, y/x)$, one gets a simple system which has first integrals of the form

$$F(x, u) = \left\{ x \exp \int \frac{A + Bu}{\beta u^2 + \gamma u + \delta} du \right\}^\tau, \quad \tau \in \mathbb{C},$$

for some constants β, γ, δ. We consider the case when the first integral for V_0 is of the form H_0^σ, where

$$H_0 = x^p y^q (y-x)^r$$

is a polynomial with $\gcd(p,q,r) = 1$. We say that V_0 is of type $Q^{p,q,r}$.

Let

$$\begin{array}{rcl} \mathcal{F}_d &=& \{f \in \mathbb{C}[x,y] : \deg f = d\}, \\ \mathcal{F}_{\geq d} &=& \{f \in \mathbb{C}[[x,y]] : \deg f \geq d\}, \\ \mathcal{Z}_d &=& \{Z \in \mathcal{Z} : \deg Z = d\}, \\ \mathcal{Z}_{\geq d} &=& \{Z \in \mathcal{Z} : \deg Z \geq d\} \end{array}$$

denote spaces of homogeneous polynomials, of formal power series, of homogeneous vector fields and of formal vector fields respectively. For $V = V_0 + \ldots$ as above we introduce the operators

$$\begin{array}{rcl} A(V)f &=& f \cdot V, \\ B(V)Z &=& V \wedge Z / \partial_x \wedge \partial_y, \\ C(V)f &=& V(f) = \partial f / \partial V, \\ D(V)f &=& V(f) - \operatorname{div}(V) \cdot f. \end{array}$$

In the second equation we encounter so-called bivector fields; they measure the part of the vector field Z transversal to V. The operators $C(V)$, ad_V and $D(V)$ are called the homological operators. It turns out that the kernel of $C(V)$ (respectively of $D(V)$) consists of first integrals (respectively of inverse integrating multipliers) of the vector field V.

Consider the following diagram, whose rows form the so-called Koszul complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_{\geq d-1} & \xrightarrow{A(V)} & \mathcal{Z}_{\geq d} & \xrightarrow{B(V)} & \mathcal{F}_{\geq d+3} & \longrightarrow & 0 \\ & & \downarrow C(V) & & \downarrow \mathrm{ad}_V & & \downarrow D(V) & & \\ 0 & \longrightarrow & \mathcal{F}_{\geq d} & \xrightarrow{A(V)} & \mathcal{Z}_{\geq d+1} & \xrightarrow{B(V)} & \mathcal{F}_{\geq d+4} & \longrightarrow & 0. \end{array}$$

In the first level analysis we assume that $V = V_0$ and consider the following operators

$$\begin{array}{c} C_d(V_0) : \mathcal{F}_d \longmapsto \mathcal{F}_{d+1}, \\ D_d(V_0) : \mathcal{F}_d \longmapsto \mathcal{F}_{d+1}, \end{array}$$

i.e. restrictions of $C(V)$ and $D(V)$ to corresponding subspaces. We have $\mathcal{F}_d \approx \mathbb{C}^{d+1}$ and therefore the operators $C_d(V_0)$ and $D_d(V_0)$ act between spaces of dimensions $d+1$ and $d+2$. In the below propositions we describe the kernels and cokernels to these two operators.

Proposition 1 *For the vector field V_0 of type $Q^{p,q,r}$ we have:*

(i) $\ker C_d(V_0) = \mathbb{C} \cdot H_0^l$, *if* $d = ls = l(p+q+r)$, *and* $\ker C_d(V_0) = 0$ *otherwise.*

(ii) $\ker C_d(V_0) = \mathbb{C} \cdot H_0^l xy(y-x)$, *if* $d = ls+3$, *and* $\ker C_d(V_0) = 0$ *otherwise.*

Proposition 2 *For the vector field V_0 of type $Q^{p,q,r}$ we have:*

(i) A subspace complementary to $\mathrm{Im} C_d(V_0)$ *can be chosen in the form* $\mathcal{N}_{\mathcal{C}} = \{\mathbb{C} \cdot x^{d+1} + \mathbb{C} \cdot y^{d+1}\}$, *if* $d = ls$, *and* $\mathcal{N}_{\mathcal{C}} = \{\mathbb{C} \cdot x^{d+1}\}$ *otherwise;*

(ii) A subspace complementary to $\mathrm{Im} D_d(V_0)$ *can be chosen in the form* $\mathcal{N}_{\mathcal{C}} = \{\mathbb{C} \cdot x^{d-1} y(y-x) + \mathbb{C} \cdot xy^{d-1}(y-x)\}$, *if* $d = ls+3$, *and* $\mathcal{N}_{\mathcal{C}} = \{\mathbb{C} \cdot x^{d-1} y(y-x)\}$ *otherwise. This corresponds, via the operator* $B(V_0)$, *to the vector fields* $\mathrm{const} \cdot x^{d-2} E + \mathrm{const} \cdot y^{d-2} E$ *and* $\mathrm{const} \cdot x^{d-2} E$ *respectively, where* $E = x\partial_x + y\partial_y$ *is homogeneous Euler vector field.*

In the second level we study the homological operators ad_V, $C(V)$ and $D(V)$ associated with vector fields of the form $V = V_0 + V_1$, where V_1 is a homogeneous vector field which was not reduced in the first order analysis. We have to consider several cases. In the third level we repeat the procedure of reducing the normal form obtained on the previous level.

We obtained the following result which is proved in [1].

Theorem 3 *The first order formal non-orbital form is the following*

$$\{V_0 + [\varphi_1(x) + \varphi_2(y)] \cdot E\} \cdot \{1 + \psi_1(x) + \psi_2(y)\},$$

where

$$\begin{array}{ll} \varphi_1(x) = \sum_{j \in \mathbb{Z}_+} a_j^{(1)} x^j, & \varphi_2(y) = \sum_{j \in I_2} a_j^{(2)} y^j \\ \psi_1(x) = \sum_{j \in \mathbb{Z}_+} b_j^{(1)} x^j, & \psi_2(y) = \sum_{j \in I_{-1}} b_j^{(2)} y^j \end{array}$$

are formal power series and $I_l = \{j \in \mathbb{Z}_+ : j + l = 0 (\mathrm{mod}\, s)\}$.

For the second level normal form in the case with V_1 *of the type* $a_0^{(1)} x^R \cdot E$ *is as above, but with*

$$\varphi_1(x) = x^R \left[a_0^{(1)} + \sum_{\mathbb{Z}_+ \setminus I_0 \setminus I_{-1}} a_j^{(1)} x^j \right]$$

or

$$\varphi_1(x) = x^R \left[a_0^{(1)} + \sum_{\mathbb{Z}_+ \setminus I_0 \setminus I_{-1} \cup \{k_0 s\}} a_j^{(1)} x^j \right],$$

where $R = k_0 s + 1$ *for some* $k_0 \in \mathbb{N}$ *and*

$$\psi_1(x) = \sum_{j \in \mathbb{Z}_+ \setminus I_0} b_j^{(1)} x^j.$$

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The double eventual periodicity of the asynchronous flows

Serban E. Vlad

Abstract

Let $\Phi : \{0,1\}^n \to \{0,1\}^n$ be a function whose coordinates $\Phi_i, i \in \{1,...,n\}$ are iterated independently on each other, in discrete time or real time. The resulted flows, called asynchronous, model the asynchronous circuits from the digital electrical circuits. The concept of double eventual periodicity refers to two eventually periodic simultaneous phenomena, one of the function (so called computation function) indicating when and how Φ is iterated and the other one of the flow itself. The paper introduces the double eventual periodicity of the asynchronous flows and gives two important results on them.

Keywords: computation function, asynchronous flows, circuits.

1 Preliminaries

We denote with $\mathbf{B} = \{0,1\}$ the binary Boole algebra, together with the usual laws ' $\overline{}$', ' $\cdot$', ' $\cup$', ' $\oplus$'. These laws induce laws that are denoted with the same symbols on $\mathbf{B}^n, n \geq 1$. Both sets $\mathbf{B}$ and $\mathbf{B}^n$ are organized as topological spaces by the discrete topology. $\mathbf{N}_{\_} = \mathbf{N} \cup \{-1\}$ is the notation of the discrete time set. $\chi_A : \mathbf{R} \to \mathbf{B}$ is the notation of the characteristic function of the set $A \subset \mathbf{R}$:

$$\forall t \in \mathbf{R}, \chi_A(t) = \begin{cases} 1, if\ t \in A, \\ 0, otherwise \end{cases}.$$

We denote with Seq the set of the real, strictly increasing sequences (t_k) which are unbounded from above.

Definition 1. *The* ***discrete time signals*** *are by definition the functions* $\widehat{x}:\mathbf{N}\_\to\mathbf{B}^n$. *Their set is denoted with* $\widehat{S}^{(n)}$. *The* ***continuous time signals*** *are the functions* $x:\mathbf{R}\to\mathbf{B}^n$ *of the form* $\forall t\in\mathbf{R}$, $x(t)=\mu\cdot\chi_{(-\infty,t_0)}(t)\oplus x(t_0)\cdot\chi_{[t_0,t_1)}(t)\oplus...\oplus x(t_k)\cdot\chi_{[t_k,t_{k+1})}(t)\oplus...$ *where* $\mu\in\mathbf{B}^n$ *and* $(t_k)\in Seq$. *We denote their set with* $S^{(n)}$.

Remark. The signals are the 'nice functions' that model the electrical signals from the digital electrical engineering.

Definition 2. *The* ***discrete time computation functions*** *are by definition the sequences* $\alpha:\mathbf{N}\to\mathbf{B}^n$. *Their set is denoted by* $\widehat{\Pi}'_n$. *In general, we write* α^k *instead of* $\alpha(k),k\in\mathbf{N}$. *The* ***real time computation functions*** $\rho:\mathbf{R}\to\mathbf{B}^n$ *are by definition the functions of the form* $\rho(t)=\alpha^0\cdot\chi_{\{t_0\}}(t)\oplus\alpha^1\cdot\chi_{\{t_1\}}(t)\oplus...\oplus\alpha^k\cdot\chi_{\{t_k\}}(t)\oplus...$, *where* $\alpha\in\widehat{\Pi}'_n$ *and* $(t_k)\in Seq$. *Their set is denoted with* Π'_n.

Definition 3. *The discrete time computation function* $\alpha\in\widehat{\Pi}'_n$ *is called* ***progressive*** *if* $\forall i\in\{1,...,n\}$,*the set* $\{k|k\in\mathbf{N},\alpha_i^k=1\}$ *is infinite. The set of the discrete time progressive computation functions is denoted by* $\widehat{\Pi}_n$. *The real time computation function* $\rho\in\Pi'_n$ *is called* ***progressive*** *if* $\forall i\in\{1,...,n\}$,*the set* $\{t|t\in\mathbf{R},\rho_i(t)=1\}$ *is infinite. The set of the real time progressive computation functions is denoted by* Π_n.

Definition 4. *Let* $\widehat{x}\in\widehat{S}^{(n)},\alpha\in\widehat{\Pi}_n$. *For* $p\geq 1,p'\geq 1$ *and* $k'\in\mathbf{N}\_$, $k''\in\mathbf{N}$, *if* $\forall k\geq k',\widehat{x}(k)=\widehat{x}(k+p),\quad\forall k\geq k'',\alpha^k=\alpha^{k+p'}$, *we say that* $\widehat{x},\alpha$ *are* ***eventually periodic*** *with the* ***periods*** p,p' *and the* ***limits of periodicity*** k',k''. *We consider* $x\in S^{(n)},\rho\in\Pi_n,T>0,T'>0$, $t'\in\mathbf{R}$,$t''\in\mathbf{R}$. *If* $\forall t\geq t',x(t)=x(t+T),\quad\forall t\geq t'',\rho(t)=\rho(t+T')$, *we use to say that* x,ρ *are* ***eventually periodic*** *with the* ***periods*** T,T' *and the* ***limits of periodicity*** t',t''.

Definition 5. *For the function* $\Phi:\mathbf{B}^n\to\mathbf{B}^n$ *and* $\lambda\in\mathbf{B}^n$, *we define* $\Phi^\lambda:\mathbf{B}^n\to\mathbf{B}^n$ *by* $\forall\mu\in\mathbf{B}^n,\Phi^\lambda(\mu)=(\overline{\lambda_1}\cdot\mu_1\oplus\lambda_1\cdot\Phi_1(\mu),...,\overline{\lambda_n}\cdot\mu_n\oplus\lambda_n\cdot\Phi_n(\mu))$.

Definition 6. *Let* $\alpha^0,...,\alpha^k,\alpha^{k+1}\in\mathbf{B}^n,k\geq 0$. *We define iteratively the function* $\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}:\mathbf{B}^n\to\mathbf{B}^n$ *by* $\forall\mu\in\mathbf{B}^n,\Phi^{\alpha^0...\alpha^k\alpha^{k+1}}(\mu)=\Phi^{\alpha^{k+1}}(\Phi^{\alpha^0...\alpha^k}(\mu))$.

Definition 7. *The function* $\mathbf{B}^n\times\mathbf{N}\_\times\widehat{\Pi}_n\ni(\mu,k,\alpha)\longmapsto\widehat{\Phi}^\alpha(\mu,k)\in$

$\mathbf{B}^n$ *defined by* $\forall k \in \mathbf{N}_{\_}$, $\widehat{\Phi}^\alpha(\mu,k) = \begin{cases} \mu, if\ k=-1, \\ \Phi^{\alpha^0...\alpha^k}(\mu), if\ k \geq 0 \end{cases}$ *is called (discrete time)* ***evolution function****. We define the function* $\mathbf{B}^n \times \mathbf{R} \times \Pi_n \ni (\mu,t,\rho) \longmapsto \Phi^\rho(\mu,t) \in \mathbf{B}^n$ *in the following way. Let* $\forall t \in \mathbf{R}$, $\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t_1\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_k\}}(t) \oplus ...$, *where* $\alpha \in \widehat{\Pi}_n$ *and* $(t_k) \in Seq$. *Then* $\Phi^\rho(\mu,t) = \widehat{\Phi}^\alpha(\mu,-1) \cdot \chi_{(-\infty,t_0)}(t) \oplus \widehat{\Phi}^\alpha(\mu,0) \cdot \chi_{[t_0,t_1)}(t) \oplus ... \oplus \widehat{\Phi}^\alpha(\mu,k) \cdot \chi_{[t_k,t_{k+1})}(t) \oplus ...$ *is called (real time)* ***evolution function****.*

Definition 8. *We fix in the argument of the discrete time evolution function* $\mu \in \mathbf{B}^n$ *and* $\alpha \in \widehat{\Pi}_n$. *The signal* $\widehat{\Phi}^\alpha(\mu,\cdot) \in \widehat{S}^{(n)}$ *is called (discrete time)* ***flow (through*** μ, ***under*** α*). We fix in the argument of the real time evolution function* $\mu \in \mathbf{B}^n$ *and* $\rho \in \Pi_n$. *The signal* $\Phi^\rho(\mu,\cdot) \in S^{(n)}$ *is called (real time)* ***flow (through*** μ, ***under*** ρ*).*

Remark. The function Φ applied to the argument μ is computed on all its coordinates: $\Phi(\mu) = (\Phi_1(\mu), \Phi_2(\mu), ..., \Phi_n(\mu))$. The function Φ^λ applied to μ computes those coordinates Φ_i of Φ for which $\lambda_i = 1$ and it does not compute those coordinates Φ_i for which $\lambda_i = 0 : \forall i \in \{1,2,...,n\}$, $\Phi_i^\lambda(\mu) = \begin{cases} \Phi_i(\mu), \lambda_i = 1, \\ \mu_i, \lambda_i = 0 \end{cases}$. Unlike the usual computations from the dynamical systems theory that take place synchronously on all the coordinates: $\Phi(\mu)$, $(\Phi \circ \Phi)(\mu)$, $(\Phi \circ \Phi \circ \Phi)(\mu), ...$ here things happen on some coordinates only. The asynchronous flows represent a generalization of the computations from the dynamical systems theory, since the constant sequence $\alpha^k = (1,...,1) \in \mathbf{B}^n, k \in \mathbf{N}$ belongs to $\widehat{\Pi}_n$, and it gives for any $\mu \in \mathbf{B}^n$ that $\Phi^{\alpha^0}(\mu) = \Phi(\mu)$, $\Phi^{\alpha^0\alpha^1}(\mu) = (\Phi \circ \Phi)(\mu)$, $\Phi^{\alpha^0\alpha^1\alpha^2}(\mu) = (\Phi \circ \Phi \circ \Phi)(\mu), ...$ So, the functions $\alpha \in \widehat{\Pi}_n, \rho \in \Pi_n$ show when and how the coordinates $\Phi_i, i = \overline{1,n}$ are computed.

Remark. The progressiveness of α, ρ means that $\widehat{\Phi}^\alpha(\mu,\cdot)$, $\Phi^\rho(\mu,\cdot)$ compute each coordinate $\Phi_i, i = \overline{1,n}$ infinitely many times as $k \to \infty, t \to \infty$. In electrical engineering, this corresponds to the so called **unbounded delay model** of computation of the Boolean functions, stating basically that each coordinate i of Φ is computed independently on the other coordinates, in finite time.

2 Double eventual periodicity

Definition 9. *Let* $\mu \in \mathbf{B}^n, \alpha \in \widehat{\Pi}_n$ *and* $\rho \in \Pi_n$. *If* $p \geq 1, p' \geq 1$ *and* $k' \in \mathbf{N}$ *exist such that*

(1) $\forall k \geq k', \alpha^k = \alpha^{k+p}$ *and* **(2)** $\forall k \geq k', \widehat{\Phi}^{\alpha}(\mu, k) = \widehat{\Phi}^{\alpha}(\mu, k+p')$ *are true, then* $\widehat{\Phi}^{\alpha}(\mu, \cdot)$ *is called* ***double eventually periodic***. *And if* $T > 0, T' > 0, t' \in \mathbf{R}$ *exist with*

(3) $\forall t \geq t', \rho(t) = \rho(t+T)$ *and* **(4)** $\forall t \geq t', \Phi^{\rho}(\mu, t) = \Phi^{\rho}(\mu, t+T')$ *fulfilled, then* $\Phi^{\rho}(\mu, \cdot)$ *is called* ***double eventually periodic***.

Theorem 10. *Let* $\alpha \in \widehat{\Pi}_n$. *We ask that* $t_0 \in \mathbf{R}$ *and* $h > 0$ *exist such that*

(5) $\rho(t) = \alpha^0 \cdot \chi_{\{t_0\}}(t) \oplus \alpha^1 \cdot \chi_{\{t_0+h\}}(t) \oplus ... \oplus \alpha^k \cdot \chi_{\{t_0+kh\}}(t) \oplus ...$

Then the equivalence ((**1**) *and* (**2**))$\Longleftrightarrow$ ((**3**) *and* (**4**)) *is true.*

Theorem 11. *Let* $\mu \in \mathbf{B}^n$. *a) If* $\alpha \in \widehat{\Pi}_n$ *is eventually periodic, then* $\widehat{\Phi}^{\alpha}(\mu, \cdot)$ *is double eventually periodic. b) If* $\rho \in \Pi_n$ *is eventually periodic, then* $\Phi^{\rho}(\mu, \cdot)$ *is double eventually periodic.*

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Subsection 2.

Minisimposia

“Dynamical systems and their applications”

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Construction of a Pfaff System of Fuchs Type with the Reducible Residue Matrices

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Abstract

We construct a multidimensional Pfaff system of Fuchs type of a second-order in the case of the reducible residue matrices.

Keywords: reducible residue matrices, Pfaff system of Fuchs type, pencil of surfaces, complete integrability of system.

First of all we recall that a set of the square matrices $U_j, j = \overline{1,n}$, is called reducible one if all matrices U_j can be simultaneously reduced by a nonsingular transformation to an upper-triangular form. Now let

$$\overline{M_j} = \{x \in \mathbb{CP}^n \mid P_j(x) = 0\}, j = \overline{1,n}, x = (x_1, x_2, ..., x_{n+1}),$$

where $x_1, x_2, ..., x_{n+1}$ are the homogeneous coordinates in $\mathbb{CP}^n$, be irreducible nonsingular algebraic varieties of codimension 1 on $\mathbb{CP}^n$. On the open set $M = \mathbb{CP}^n \backslash \overline{M}$, where $\overline{M} = \bigcup\limits_{j=1}^{4} \overline{M_j}$, we consider the completely integrable linear Pfaff system of Fuchs type

$$dY = \omega(x)Y, \tag{1}$$

where Y is a second-order square matrix and $\omega(x)$ is a differential 1-form,

$$\omega(x) = \sum_{j=1}^{4} U_j \frac{dP_j(x)}{P_j(x)}.$$

In the last formula the reducible residue U_j are constant square (2x2) matrices.

We consider the case when the algebraic surfaces $\overline{M_j}, j = \overline{1,n}$, form a pencil, and there exist homogeneous polynomials $R(x)$ and $Q(x)$ of equal degree in $x \in \mathbb{CP}^n$ such that $P_j(x) = \alpha_j R(x) + \beta_j Q(x)$, where $\alpha_j, \beta_j \in \mathbb{C}$ are not equal to zero.

The change of variables $z = Q(x)/R(x)$ reduces the Pfaff system (1) to the linear differential equation with Fuchs singularities on $\mathbb{CP}^n$

$$dY = \left(\sum_{j=1}^{4} \frac{U_j}{z - a_j}\right) Y dz, \tag{2}$$

where $a_j = -\alpha_j/\beta_j$ and the matrices U_j are considered as the matrices with a condition $\sum_{j=1}^{4} U_j = 0$.

Theorem 1. *The change of variables*

$$\tau = (\alpha z + \beta)/(\gamma z + \delta), \ \ \alpha\delta - \gamma\beta \neq 0, \tag{3}$$

where $\alpha = a_2 - a_3, \beta = a_1(a_3 - a_2), \gamma = a_3 + a_2 - 2a_1, \delta = a_1(a_2 + a_3) - 2a_2a_3$, *reduces the system (2) to the system*

$$dY = \left(\frac{U_1}{\tau} + \frac{U_2}{\tau - 1} + \frac{U_3}{\tau + 1} + \frac{U_4}{\tau - a}\right) Y d\tau, \tag{4}$$

where

$$a = \frac{(a_2 - a_3)(a_4 - a_1)}{(a_2 + a_3)(a_1 + a_4) - 2(a_2a_3 + a_1a_4)}.$$

Proof. It follows from relation (3), that

$$z = (-\delta\tau + \beta)/(\gamma\tau - \alpha) \tag{5}$$

and, consequently,

$$dz = \frac{\alpha\delta - \gamma\beta}{(\gamma\tau - \alpha)^2} d\tau, \frac{U_j dz}{z - a_j} = \frac{U_j d\tau}{\tau - b_j}, \tag{6}$$

where

$$b_j = \frac{\beta + \alpha a_j}{\delta + \gamma a_j} = \frac{a_1(a_3 - a_2) + (a_2 - a_3)a_j}{a_1(a_2 + a_3) - 2a_2a_3 + (a_3 + a_2 - 2a_)a_j}. \tag{7}$$

Assuming $a_j = a_1$ in (7) we obtain $b_1 = 0$. We have $b_2 = 1$ if $a_j = a_2$ and $b_3 = -1$ if $a_j = a_3$. Finally, when $a_j = a_4$ we obtain $b_4 = a$. A reference to relations (5), (6) and values of the $b_j, j = \overline{1,4}$, completes the proof of the theorem.

Returning to the solution of the posed problem we construct the residue matrices $U_j, j = \overline{1,3}$, of system (1) ((4)), without loss of generality, in the form of

$$U_j = \begin{pmatrix} \theta_j & (\xi_j - \theta_j)/h \\ h\theta_j & \xi_j - \theta_j \end{pmatrix},$$

where $h \neq 0$ is a real or complex constant, θ_j are the unknown parameters and $\xi_j, 0$ are the eigenvalues such that $\xi_1 + \xi_2 + \xi_3 = -1$.

The system (1) is completely integrable if and only if $d\omega = \omega \wedge \omega$. The last relation means [1, p. 182] that

$$dU_1 = [U_4, U_1] d\ln(-a), dU_2 = [U_4, U_2] d\ln(1-a),$$

$$dU_3 = [U_4, U_3] d\ln(-1-a),$$

$$dU_4 = [U_1, U_4] d\ln a + [U_2, U_4] d\ln(a-1) + [U_3, U_4] d\ln(1+a), \tag{8}$$

where

$$[U_j, U_k] = U_j U_k - U_k U_j = (\xi_j \theta_k - \xi_k \theta_j) \begin{pmatrix} 1 & -1/h \\ h & -1 \end{pmatrix}. \tag{9}$$

In accordance with (9) we can rewrite system (8) in the scalar form

$$d\theta_1 = (\xi_4\theta_1 - \xi_1\theta_4) d\ln(-a), d\theta_2 = (\xi_4\theta_2 - \xi_2\theta_4) d\ln(1-a),$$

$$d\theta_3 = (\xi_4\theta_3 - \xi_3\theta_4) d\ln(-1-a), d\theta_4 = (\xi_1\theta_4 - \xi_4\theta_1) d\ln a +$$

$$+(\xi_2\theta_4 - \xi_4\theta_2) d\ln(a-1) + (\xi_3\theta_4 - \xi_4\theta_3) d\ln(1+a). \tag{10}$$

Theorem 2. *The parameters $\theta_j, j = \overline{1,4}$, such that $\theta_1 = -2a, \theta_2 = a-1, \theta_3 = a+1, \theta_4 = 0$, define the solution of the system (10).*

Thus

$$U_1 = \begin{pmatrix} -2a & (\xi_1 + 2a)/h \\ -2ha & \xi_1 + 2a \end{pmatrix}, U_2 = \begin{pmatrix} a-1 & (\xi_2 - a + 1)/h \\ h(a-1) & \xi_2 - a + 1 \end{pmatrix},$$

$$U_3 = \begin{pmatrix} a+1 & (\xi_3 - a - 1)/h \\ h(a+1) & \xi_3 - a - 1 \end{pmatrix}, U_4 = \begin{pmatrix} 0 & 1/h \\ 0 & 1 \end{pmatrix}.$$

By [2], the fundamental matrix $\Phi_{x_0}(x)$ of system (1) can be represented by the uniformly convergent series

$$\Phi_{x_0}(x) = E + \sum_{\nu=1}^{\infty} \sum_{j_1,\ldots,j_\nu}^{(1,2,3,4)} J_{j_1\ldots j_\nu}(\alpha) U_{j_1} \ldots U_{j_\nu}.$$

The coefficients are iterated integrals $J_{j_1\ldots j_\nu}(\alpha) = \int\limits_\alpha \omega_{j_1}\ldots\omega_{j_\nu}$ over the smooth path $\alpha : [0,1] \to M$ joining the point x_0 with x, where $\omega_j = \frac{dP_j(x)}{P_j(x)}, j = \overline{1,4}$, and $J_1(\alpha) = \int\limits_\alpha \omega_1(\alpha(t), \dot{\alpha}(t))dt$ is the curvilinear integral over the path α, and $J_{j_1\ldots j_\nu}$ is the iterated integral over the path α defined recursively as follows:

$$J_{12\ldots\tau} = \int\limits_\alpha \omega_1\ldots\omega_\tau = \int\limits_0^1 \int\limits_{\alpha^t} (\omega_1\ldots\omega_{\tau-1})\omega_\tau(\alpha(t), \dot{\alpha}(t))dt,$$

where α^t is the restriction of α to the closed interval $[0,t]$.

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Global configurations of singularities for quadratic differential systems with total finite multiplicity three and at most two real singularities

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Abstract

In this work we consider the problem of classifying all configurations of singularities, both finite and infinite of quadratic differential systems, with respect to the *geometric equivalence relation* defined in [2]. This relation is finer than the *topological equivalence relation* which does not distinguish between a focus and a node or between a strong and a weak focus or between foci of different orders. In this article we continue the work initiated in [3] and obtain the *geometric classification* of singularities, finite and infinite, for the subclass of quadratic differential systems possessing finite singularities of total multiplicity three and at most two real singularities.

Keywords: quadratic vector fields, infinite and finite singularities, configuration of singularities, geometric equivalence relation.

1 Introduction

We consider here real planar differential systems of the form

$$\frac{dx}{dt} = p(x, y), \qquad \frac{dy}{dt} = q(x, y), \tag{1}$$

where $p,\ q \in \mathbb{R}[x, y]$, i.e. $p,\ q$ are polynomials in $x,\ y$ over $\mathbb{R}$. We call *degree* of a system (1) the integer $n = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $n = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

The study of the class **QS** has proved to be quite a challenge since hard problems formulated more than a century ago, are still open for this class. It is expected that we have a finite number of phase portraits in **QS**. We have phase portraits for several subclasses of **QS** but the complete list of phase portraits of this class is not known and attempting to topologically classify these systems, which occur rather often in applications, is a very complex task. This is partly due to the elusive nature of limit cycles and partly to the rather large number of parameters involved. This family of systems depends on twelve parameters but due to the group action of real affine transformations and time homotheties, the class ultimately depends on five parameters which is still a rather large number of parameters. For the moment only subclasses depending on at most three parameters were studied globally, including global bifurcation diagrams (for example [1])

The geometric equivalence relation for finite or infinite singularities introduced in [2] and used in [3], [4] and [5] is finer than the *topological equivalence relation*. This last relation does not distinguish:
(i) between a focus and a node; *(ii)* between a strong focus and a weak focus; *(iii)* among foci (or saddles) of different orders; *(iv)* among elementary nodes with different number of characteristic directions (one, two or infinite number); *(v)* among singular points having the same topological phase portraits but different multiplicities. The geometric equivalence relation does detect all these properties.

2 Description of main results

We consider here all geometric configurations of singularities, finite and infinite, of quadratic vector fields having finite singularities of total multiplicity three among which there are at most two real singularities. In other words we examine here the following three subclasses of quadratic differential systems: *(i)* systems with one real and two complex finite singularities; *(ii)* systems with one double and one simple real finite singularities and *(iii)* systems with one triple real finite singularity.

(A) We prove that:

- quadratic systems with one real and two complex finite singularities possess 74 geometrically distinct configurations of singularities;
- quadratic systems with one double and one simple real finite singularities possess 62 geometrically distinct configurations of singularities;
- quadratic systems with one triple real finite singularity possess 19 geometrically distinct configurations of singularities.

(B) Necessary and sufficient conditions for each one of the 155 different geometric equivalence classes are determined in terms of invariant polynomials with respect to the action of the affine group and time rescaling.

(C) We construct the respective diagrams which actually contain the global bifurcation diagram in the 12-dimensional space of parameters, of the global geometric configurations of singularities, finite and infinite, of this family of quadratic differential systems and provides an algorithm for deciding for any given system which is its respective configuration.

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Averaging of a Multifrequency Boundary-value Problem with Delay

Yaroslav Bihun

Abstract

In the work we substantiate the method of averaging for the initial and boundary-value problem with linearly transformed argument and the frequency vector depending on slow variable. The obtained results are used for the investigation of the existence of the method of averaging for some boundary-value problems.

Keywords: method of averaging, boundary value problems, delay, multifrequency systems, resonances.

1 Introduction

Multifrequency systems of ordinary differential equations with the initial, multipoint and integral boundary condition has been thoroughly investigated by averaging methods in the paper [1]. The main problem arising in the study of properties of solutions of the system is the problem of resonance relations between the components of the variable frequency vector. In the works [2], [3] we substantiate the method of averaging for the initial-value problem with linearly transformed argument. For similar systems with constant delay, whose frequency vector depends on slow variables, the method of averaging is substantiated in the work [4].

2 Substantiation of Averaging Method for Initial Problem

Suppose that D is a domain in $\mathbb{R}^n$, $G = [0, L] \times D \times D \times \mathbb{R}^m \times \mathbb{R}^m \times [0, \varepsilon_0]$; λ, θ and σ are certain numbers from $(0, 1)$, $x_\lambda(\tau) = x(\lambda\tau)$, $x_\sigma(\tau) = x(\sigma\tau)$, $\varphi_\theta(\tau) = \varphi(\theta\tau)$. Consider a system of equations with n slow variables x and m $(m \geq 1)$ fast variables φ of the form

$$\frac{dx}{d\tau} = A(\tau, x, x_\lambda, \varphi, \varphi_\theta) + \varepsilon X(\tau, x, x_\lambda, \varphi, \varphi_\theta, \varepsilon),$$

$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau, x, x_\sigma)}{\varepsilon} + Y(\tau, x, x_\lambda, \varphi, \varphi_\theta, \varepsilon), \tag{1}$$

where the vector-functions A, X and Y are 2π-periodic in the variables φ_υ, $\varphi_{\theta\nu}$, $\nu = 1, ..., m$.

Now we construct the averaged system of the first approximation for slow variables:

$$\frac{d\overline{x}}{d\tau} = A_0(\tau, \overline{x}, \overline{x}_\lambda), \tau \in [0, L], \tag{2}$$

where

$$A_0(\tau, x, x_\lambda) = \frac{1}{(2\pi)^{2m}} \int_0^{2\pi} \dots \int_0^{2\pi} A(\tau, x, x_\lambda, \varphi, \varphi_\theta) d\varphi d\varphi_\theta.$$

Suppose that there exists a solution $\overline{x} = \overline{x}(\tau, \overline{y})$ of the system (2) such that $\overline{x}(0, \overline{y}) = \overline{y}$, lying in D together with a certain ρ-neighborhood $D_\rho(\overline{x})$. Let us show that, in this case, there exists a solution of the system (1) such that $x(0, \overline{y}, \overline{\psi}, \varepsilon) = \overline{y}$, $\varphi(0, \overline{y}, \overline{\psi}, \varepsilon) = \overline{\psi}$, and the following inequality:

$$\|x(\tau, \overline{y}, \overline{\psi}, \varepsilon) - \overline{x}(\tau, \overline{y})\| \leq c_1 \varepsilon^{\frac{1-3\beta}{2}} \tag{3}$$

holds for all $\tau \in [0, L]$, $\varepsilon \in (0, \varepsilon^*]$, where $\varepsilon^* \leq \varepsilon_0$, $c_1 > 0$ is independent of ε, and $\beta \in [0, 1/3)$.

3 Boundary-Value Problem

Consider a multifrequency boundary-value problem of the form

$$\frac{dx}{d\tau} = A(\tau, x, x_\lambda, \varphi, \varphi_\theta, \varepsilon),$$
$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau, x, x_\sigma)}{\varepsilon} + X(\tau, x, x_\lambda, \varphi, \varphi_\theta, \varepsilon), \tag{4}$$
$$F(x|_{\tau=\tau_0}, \dots, x|_{\tau=\tau_N}, \varepsilon) + \int\limits_0^L \Phi(\tau, x, x_\lambda, \varphi, \varphi_\theta, \varepsilon) d\tau = 0, \varphi|_{\tau=0} = \psi, \tag{5}$$

where F and Φ are n-dimensional vector functions defined in $D^{N+1} \times (0, \varepsilon_0]$ and G, respectively, the function Φ is 2π-periodic in the variables φ_ν, and $\varphi_{\theta\nu}$; $0 \le \tau_0 < \dots < \tau_N \le L$, and $\psi \in R^m$.

The averaged system corresponding to (4) and (5) takes the form

$$\frac{d\overline{x}}{d\tau} = A_0(\tau, \overline{x}, \overline{x}_\lambda, \varepsilon), \quad \frac{d\overline{\varphi}}{d\tau} = \frac{\omega(\tau, \overline{x}, \overline{x}_\sigma)}{\varepsilon} + Y_0(\tau, \overline{x}, \overline{x}_\lambda, \varepsilon),$$
$$F(\overline{x}|_{\tau=\tau_0}, \dots, \overline{x}|_{\tau=\tau_N}, \varepsilon) + \int\limits_0^L \Phi_0(\tau, \overline{x}, \overline{x}_\lambda, \varepsilon) d\tau = 0, \overline{\varphi}|_{\tau=0} = \psi. \tag{6}$$

Then one can find $\overline{\varepsilon} \in (0, \varepsilon_0]$ and constants $c_2 > 0$ and $c_3 > 0$ such that, for any $\varepsilon \in (0, \overline{\varepsilon}]$ there exists a solution of the boundary-value problem (4), (5), and the following inequality is satisfied:

$$\|x(\tau, \overline{y} + \mu, \psi, \varepsilon) - \overline{x}(\tau, \overline{y}, \varepsilon)\| \le c_2 \varepsilon^{\alpha_1},$$

where $\|\mu\| \le c_3 \varepsilon^{\alpha_1}$, $\alpha_1 = \alpha_0 - \chi$, $\alpha_0 = (1 - 3\beta)/2$, $\chi \in [0, (1 - 3\beta)/4)$.

4 Problem with Constant Delay and Linearly Transformed Argument

The conditions for existence of solution of the system with slow and fast variables are represented in the form of

$$\frac{dx}{d\tau} = A(\tau, x_\Delta, x_\Theta, \varphi_\Delta, \varphi_\Theta),$$

$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + B(\tau, x_\Delta, x_\Theta, \varphi_\Delta, \varphi_\Theta),$$

where $\tau \in [0, L]$, $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$, $x \in D \subset \mathbb{R}^m$, $\varphi \in \mathbb{R}^m$; $x_\Delta = (x_{\delta_1}, \ldots, x_{\delta_r})$, $0 \leq \delta_1 < \cdots < \delta_r$, $x_{\delta_\nu}(\tau) = x(\tau - \varepsilon\delta_\nu)$, $x_\Theta = (x_{\theta_1}, \ldots, x_{\theta_q})$, $0 < \theta_1 < \cdots < \theta_q \leq 1$, $x_{\theta_\nu}(\tau) = x(\theta_\nu \tau)$.

We constructed the averaged system on fast variables in the form:

$$\frac{d\overline{x}}{d\tau} = \overline{A}(\tau, \overline{x}_\Theta), \frac{d\overline{\varphi}}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \overline{B}(\tau, \overline{x}_\Theta).$$

We propose and justify the averaging for systems with delay with multipoint or integral boundary conditions. The average is also applied to integral boundary conditions. For slow and fast variables the averaging method was constructed on the interval $[0, L]$ and the estimate of error $O(\varepsilon^\alpha)$, $\alpha \in (0, (mq)^{-1}]$ was obtained.

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Cubic systems with two distinct infinite singularities and 8 invariant lines

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Abstract

In this article we consider real planar cubic systems which possess eight invariant straight lines, including the line at infinity and including their multiplicities, and in addition they possess two distinct infinite singularities. We prove that these systems could have only 23 distinct configurations of invariant straight lines.

Keywords: cubic differential system, Poincaré compactification, invariant straight line, configuration of invariant straight lines, multiplicity of an invariant straight line.

1 Introduction

It is known that the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic systems is 9. Cubic systems with maximum number of invariant lines are considered in [2] and some families of cubic systems with seven affine invariant lines are investigated in [3,4]. In the articles [5,6] the authors consider cubic systems with exactly six or seven invariant affine straight lines considered with their "parallel" multiplicity.

The goal of this paper is to classify the family of cubic systems with two distinct infinite singularities (real or complex), which possess invariant straight lines of total multiplicity 8, including the line at infinity and taking into account their multiplicities.

Definition 1 [1]. *Consider a planar cubic system. We call configuration of invariant straight lines of this system, the set of (complex)* *invariant straight lines (which may have real coefficients) of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.*

It is well known that the infinite singularities (real and/or complex) of cubic systems are determined by the linear factors of the polynomial $C_3 = yp_3(x,y) - xq_3(x,y)$, where p_3 and q_3 are the cubic homogeneities of these systems. So in the case of two distinct infinite singularities they are determined either by one triple and one simple real or two double real (or complex) factors of the polynomial $C_3(x,y)$.

2 The statement of the Main Theorem

Main Theorem. *Assume that a cubic system possesses invariant straight lines of total multiplicity 8, including the line at infinity with its own multiplicity and, in addition, it has two real infinite singularities. Then this system can not have infinite singularities determined by two double factors of the polynomial C_3. If the infinite singularities are determined by one triple and one simple factors of C_3, then this system has one of the 24 possible configurations Config. 8.23 – Config. 8.46, given in Figure 1.*

Remark. If in a configuration an invariant straight line has multiplicity $k > 1$, then the number k appears near the corresponding straight line and this line is in bold face. Real invariant straight lines are represented by continuous lines, whereas complex invariant straight lines are represented by dashed lines. We indicate next to the real singular points of the system, located on the invariant straight lines, their corresponding multiplicities. By '(a,b)' we denote the maximum number a (respectively b) of infinite (respectively finite) singularities which can be obtained by perturbation of the multiple point.

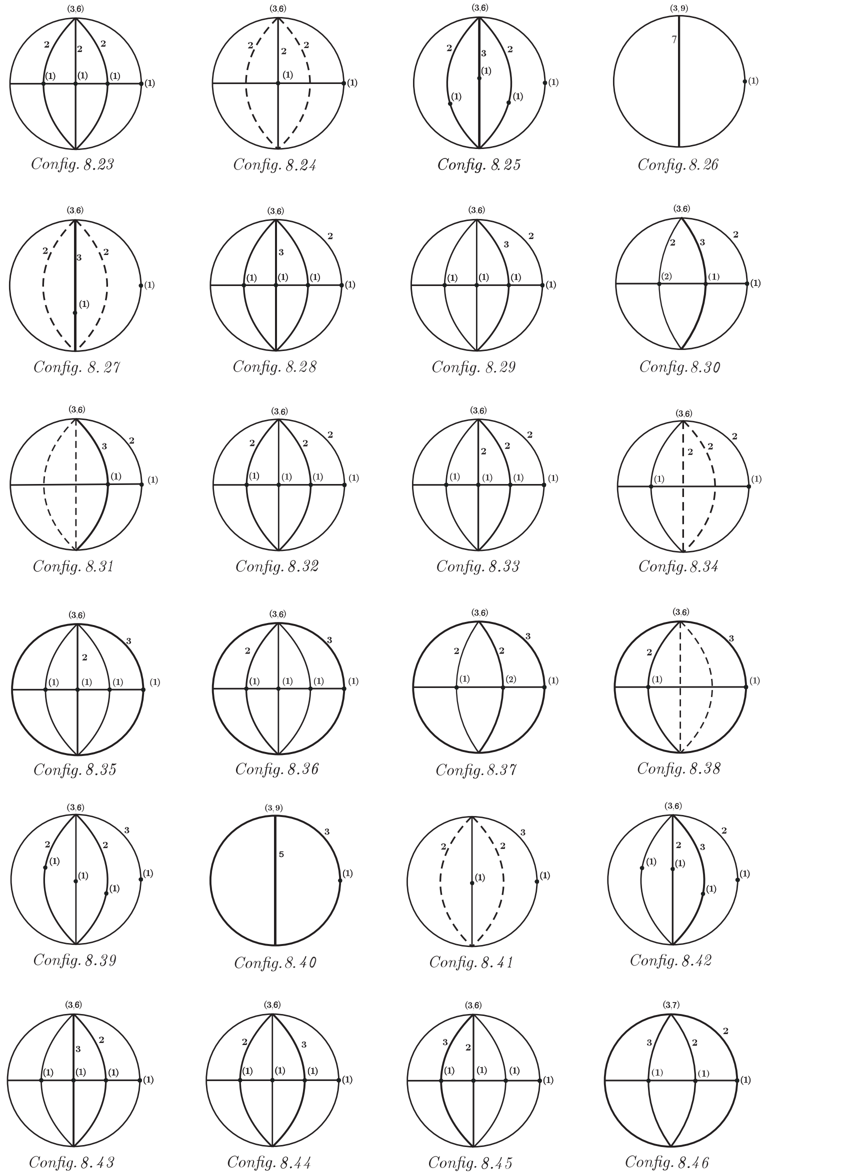

Figure 1. The configurations of invariant lines for cubic systems with two distinct infinite singularities

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A Class of Homogeneous Quadratic Differential Systems without Derivations

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Abstract

In this paper we identify the affine equivalence classes for the class of homogeneous quadratic differential systems, defined on $\mathbb{R}^3$, having no derivation and whose associated commutative algebras are NN-algebras which have at least a 1-dimensional nontrivial ideal.

Keywords: homogeneous quadratic differential system, commutative algebra, derivation, isomorphism.

1 Introduction

Recall that the problem of classification, up to an affine equivalence, of quadratic differential systems is equivalent to the problem of classification, up to an isomorphism, of commutative algebras. The study of any algebra, which is not simultaneously associative and commutative, is realized along two ways depending on the existence or on the absence of a derivation. The existence of a derivation for an algebra significantly facilitates the study of algebra's structure. By using algebraic tools, the classification up to an isomorphism of 3-dimensional commutative algebras, having at least a derivation, was already performed by I. Burdujan. The lack of any derivation for an algebra is dramatically reflected in the fact that the study of such algebra becomes more difficult to be achieved. We exemplify this assertion by presenting a part of efforts needed for classification, up to an isomorphisms, of a subclass of real 3-dimensional NN-algebras having no derivation, namely those algebras having at least an one-dimensional ideal. In fact these results concern a subclass of homogeneous quadratic systems having origin as an isolated critical point.

2 Some classification results for real 3-dimensional commutative NN-algebras having no derivation

The algebras having no nilpotent elements will be named NN-algebras (here NN is the acronym for *no nilpotent*). Such algebras were already studied by Yu. M. Ryabuhin [5], C. Sprengelmeier [6], G.Benkart & M.Osborn [1], I.Burdujan [2], etc. In [3] it was proved that there exist nine families of isomorphism classes of real 3-dimensional commutative NN-algebras having at least a derivation.

Let $A(\cdot)$ be a real 3-dimensional commutative NN-algebra. A theorem due to B.Segre assures that this algebra has at least an idempotent element e. Let $A_1 = \mathbb{R}e$ be the corresponding subalgebra of $A(\cdot)$. Of course, A_1 is or is not an ideal of A. Accordingly, the following two complementary situations must be analyzed: I) there exists an one-dimensional ideal in A, II) there is no one-dimensional ideal in A.

In present paper we classify the real 3-dimensional commutative NN-algebras having no derivation and having at least an one-dimensional ideal, i.e. we deal with Case I, only.

Case I. Let A_1 be a one-dimensional ideal of A and the idempotent e be a basis of it. Then, for each basis $\mathcal{B} = (e, e_1, e_2)$ we have $e^2 = e,\ e \cdot e_1 = \alpha e,\ e \cdot e_2 = \beta e$. It follows that the left multiplication L_e has an eigenvalue equal to 1 and two eigenvalues equal to 0. Moreover, L_e is a semi-simple endomorphism and there exists a basis (e, v, w) such that $e \cdot v = e \cdot w = 0$. Thus $A_2 = Span_{\mathbb{R}}\{v, w\}$ is a vector subspace of A, such that $e \cdot A_2 = 0$, i.e. $A_2 = ker\ L_e$.

The following two possibilities occur:

I1 A_2 is a subalgebra of A,   **I2** A_2 is not a subalgebra of A.

Subcase I1. In this case A_2 is just an ideal of A and $A = A_1 \oplus A_2$ (a direct sum of ideals). Since A_2 is an NN-algebra, we can choose a basis in A_2 such that the corresponding multiplication table has one of the next 12 forms exhibited by Markus in [4]. Therefore, 12 types of nonisomorphic 3-dimensional NN-algebras, which are semisimple, are obtained. By using Markus' classification [4] we get the following 12 families of such algebras

NN1	$e^2 = e \ v^2 = v \ w^2 = w \ ev = 0 \ ew = 0 \ vw = 0$
NN2	$e^2 = e \ \ v^2 = 2v \ w^2 = w \ \ ev = 0 \ \ ew = 0 \ \ vw = w$
NN3	$e^2 = e \ \ v^2 = \alpha v \ \ w^2 = w \ ev = 0 \ \ ew = 0 \ \ vw = w$
with	$0<\alpha<1 \ \ \alpha>2$
NN4	$e^2 = e \ \ v^2 = 2v \ \ w^2 = \beta w \ \ ev = 0 \ \ ew = 0 \ \ vw=v+w$
with	$\beta \neq 0 \ \ \beta \neq 2$
NN5	$e^2 = e \ \ v^2 = \alpha v \ \ w^2 = \beta w \ \ ev = 0 \ \ ew = 0 \ \ vw=v+w$
with	$\alpha \neq 0, \ \alpha \neq 2, \ \beta \neq 0, \ \beta \neq 2, \ \alpha\beta \neq 4, \ \alpha + \beta = \alpha\beta,$
and	$\frac{2}{\alpha-1} \leq \beta \leq \frac{2+\alpha}{\alpha}$ if $\alpha < -1, \alpha > 2$
NN6	$e^2 = e \ \ v^2 = v \ w^2 = v+\beta w \ \ ev = 0 \ \ ew = 0 \ \ vw = 0$
with	$0 \leq \beta < 2$
NN7	$e^2 = e \ \ v^2 = v \ w^2 = \alpha v \ \ ev = 0 \ \ ew = 0 \ \ vw = v+w$
with	$\alpha < -1$
NN8	$e^2 = e \ \ v^2 = v \ \ w^2 = v \ \ ev = 0 \ \ ew = 0 \ \ vw = \frac{1}{2}w$
NN9	$e^2 = e \ \ v^2 = v \ \ w^2 = v + \beta w \ \ ev = 0 \ \ ew = 0 \ \ vw = \alpha w,$
with	$\alpha \neq 0 \ \ \beta \geq 0 \ \ \beta^2 < 4(1-2\alpha)$
NN10	$e^2 = e \ \ v^2 = v \ \ w^2 = -v \ \ ev = 0 \ \ ew = 0 \ \ vw = \frac{1}{2}w$
NN11	$e^2 = e \ \ v^2 = v \ \ w^2 = -v+\beta w \ \ ev = 0 \ \ ew = 0 \ \ vw = \alpha w$
with	$\alpha \neq 0 \ \ \alpha \neq 1 \ \ \beta^2 < 4(1-2\alpha) \ \ \beta^2 \neq 4\alpha^2 \ \ \beta \geq 0$
NN12	$e^2 = e \ \ v^2 = v \ \ w^2 = -v \ \ ev = 0 \ \ ew = 0 \ \ vw = w.$

By a straightforward checking it results that each algebra defined by a Table **NNi** for $i \in \{1, 2, ..., 12\}$ has no derivation.

Theorem 2.1 (i) *Any two algebras of type **NNi** for* $i \in \{3,4,5,6,7,9,11\}$ *corresponding to different values of parameters are non-isomorphic.* (ii) *Any algebra of type **NNi** is not isomorphic to any algebra of type*

***NNj** for $i \neq j,\ i,j \in \{1,2...,12\}$.*
Proof. The proof is realized either by comparing the sets of idempotents or by a direct checking. □

In fact Theorem 2.1 assures that it was obtained the classification, up to an isomorphism, of the class of real 3-dimensional NN-algebras which are semisimple (a direct sum of two simple subalgebras). Accordingly, it was obtained the classification, up to an affinity, of corresponding HQDSs. In fact, each of these HQDSs decouples into two independent subsystems that can be closely solved.

Subcase I2. The algebra $A(\cdot)$ has, in basis (e, v, w), a multiplication table of the form

Table TI2	$e^2 = e$	$v^2 = ae + bv + cw$	$w^2 = a_2e + b_2v + c_2w$
	$ev = 0$	$ew = 0$	$vw = a_1e + b_1v + c_1w.$

with $a^2 + a_1^2 + a_2^2 \neq 0$.

Let us consider the following change of bases $(e, v, w) \to (e, f, g)$ $f = \lambda v + \mu w, \quad g = \theta v + \tau w$ with $\begin{vmatrix} \lambda & \mu \\ \theta & \tau \end{vmatrix} \neq 0$, where $\lambda, \mu, \theta, \tau$ could be chosen such that a maximal number of structure constants becomes either 0 or ± 1. We have

$$f^2 = h_1(f)e + h_2(f)v + h_3(f)w, \quad g^2 = h_1(g)e + h_2(g)v + h_3(g)w,$$
$$f \cdot g = g_1(f,g)e + g_2(f,g)v + g_3(f,g)w,$$

where

$$h_1(f) = h_1(\lambda, \mu) = a\lambda^2 + 2a_1\lambda\mu + a_2\mu^2,$$
$$h_2(f) = h_2(\lambda, \mu) = b\lambda^2 + 2b_1\lambda\mu + b_2\mu^2,$$
$$h_3(f) = h_3(\lambda, \mu) = c\lambda^2 + 2c_1\lambda\mu + c_2\mu^2,$$

while g_i is the polar form for h_i ($i \in \{1,2,3\}$) (i.e. $g_i(f,g) = \frac{1}{2}[h_i(f+g) - h_i(f) - g_i(g)]$). These equations suggest us to consider the polynomials:

$$P_1(x) = ax^2 + 2a_1x + a_2, \ \ P_2(x) = bx^2 + 2b_1x + b_2, \ \ P_3(x) = cx^2 + 2c_1x + c_2.$$

Since *Span* $\{v, w\}$ is not a subalgebra, its results $P_1(x)$ cannot be the null polynomial.

In case $\mu\tau \neq 0$ we put $x_1 = \frac{\lambda}{\mu}$, $x_2 = \frac{\theta}{\tau}$. Then the following equations hold:

$$f^2 = \mu^2[P_1(x_1)e + P_2(x_1)v + P_3(x_1)w],$$

$$g^2 = \tau^2[P_1(x_2)e + P_2(x_2)v + P_3(x_2)w],$$

$$f \cdot g = \mu\tau\left[P_1(x_1) + \frac{1}{2}P_1'(x_1)(x_2 - x_1)\right]e+$$

$$+\mu\tau\left[P_2(x_1) + \frac{1}{2}P_2'(x_1)(x_2-x_1)\right]v+\mu\tau\left[P_3(x_1) + \frac{1}{2}P_3'(x_1)(x_2-x_1)\right]w.$$

Moreover,

$$f \cdot g = \mu\tau[(ap + a_1s + a_2)e + (bp + b_1s + b_2)v + (cp + c_1s + c_2)w],$$

where $s = x_1 + x_2,\ p = x_1x_2$.

Table **TI2** suggests to analyze the vector space $A_2 = Span_{\mathbb{R}}\{v^2, vw, w^2\}$. More exactly, we have to study the following cases:

(i) $dim\ A_2 = 1$, (ii) $dim\ A_2 = 2$, (iii) $dim\ A_2 = 3$.

In order to save the publication space, the complete results for the Case (i) will be presented, only.

Case (i) $dim\ A_2 = 1$

There exists a basis such that the multiplication table gets the form

Table I2	$e_1^2 = e_1$	$e_2^2 = \omega e_1 + e_2$	$e_3^2 = \omega e_1 + e_2$
	$e_1e_2 = 0$	$e_1e_3 = 0$	$e_2e_3 = 0,$

with $\omega \neq 0$.

Let us denote by $A(\omega)$ any algebra having in basis $\mathcal{B}$ the multiplication table **I2**.

Proposition 2.1 *The algebras $A(\omega)$ and $A(\omega')$ are isomorphic if and only if $\omega = \omega$.*

We have to consider the following subclasses of such NN-algebras: **WD.1** $A = A(\omega)$ $(\omega > \frac{1}{4})$, **WD.2** $A = A(\omega)$ $(\omega = \frac{1}{4})$, **WD.3** $A = A(\omega)$ $(\omega < \frac{1}{4})$.

Theorem 2.2 (*i*) *Any two algebras of type* ***WDi*** *for* $i \in \{1, 3\}$ *corresponding to different values of parameters are non-isomorphic.*
(*ii*) *Any algebra of type* ***WDi*** *is not isomorphic to any algebra of type* ***WDj*** *for* $i \neq j,\ i, j \in \{1, 2, 3\}$.

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Birkhoff's center of compact dissipative dynamical systems

David Cheban

Abstract

We introduce the notion of Birkhoff center for arbitrary dynamical systems admitting a compact global attractor. It is shown that Birkhoff center of dynamical system coincides with the closure of the set of all positively Poisson stable points of dynamical system.

Keywords: dynamical system, global attractor, Birkhoff center.

1 Introduction

Let X be a metric space, $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{Z}_+$, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}$ and $(X, \mathbb{T}, \pi)$ be a flow on X and $M \subseteq X$ be a nonempty, compact and positively invariant subset of X. Denote by $\Omega(M) := \{x \in M :$ there exist $\{x_n\} \subset M$ and $\{t_n\} \subset \mathbb{T}$ such that $x_n \to x$, $t_n \to +\infty$ as $n \to \infty$ and $\pi(t_n, x_n) \to x\}$. Recall that the point $x \in X$ is called Poisson stable if $x \in \omega_x \bigcap \alpha_x$, where by ω_x (respectively, α_x) the ω (respectively, α)-limits set of x is denoted.

It is well known the theorem of Birkhoff (see, for example, [1, 3]) for two-sided ($\mathbb{T} = \mathbb{S}$) dynamical systems on the compact metric spaces.
Theorem 1. *The following statements hold:*

1. *there exists a nonempty, compact and invariant subset* $\mathfrak{B}(\pi) \subseteq X$ *with the properties:*

   (i) $\Omega(\mathfrak{B}(\pi)) = \mathfrak{B}(\pi)$;

(ii) $\mathfrak{B}(\pi)$ *is the maximal compact invariant subset of* X *with the property* (i).

2. $\mathfrak{B}(\pi) = \overline{\mathcal{P}(\pi)}$, *i.e. the set of all Poisson stable points* $\mathcal{P}(\pi)$ *of the dynamical system* $(X, \mathbb{R}, \pi)$ *is dense in* $\mathfrak{B}(\pi)$.

Remark 1. *1. The set* $\mathfrak{B}(\pi)$ *is called the Birkhoff center of dynamical system* $(X, \mathbb{R}, \pi)$.

2. Note that Birkhoff theorem remains true also for the discrete dynamical systems $(X, \mathbb{Z}, \pi)$. *This fact was established in the work of V. S. Bondarchuk and V. A. Dobrynsky [1].*

The main result of this paper is the proof of the analogous to the Birkhoff theorem for the one-sided dynamical systems (both ones with continuous and discrete times) with noncompact phase space having a compact global attractor.

2 The set of non-wandering points of dynamical system

Denote by Φ_x the set of all entire trajectories γ_x of $(X, \mathbb{T}, \pi)$ passing through the point x at the initial moment $t = 0$.

Definition 1. *A point* $p \in X$ *is said to be positively (respectively, negatively) Poisson stable, if* $x \in \omega_x$ *(respectively, there exists an entire trajectory* $\gamma_x \in \Phi_x$ *such that* $x \in \alpha_{\gamma_x}$, *where* $\alpha_{\gamma_x} := \{q \in X :$ *there exists* $t_n \to -\infty$ *such that* $\gamma_x(t_n) \to q$ *as* $n \to \infty\}$*).*

Lemma 1. *Let* M *be a nonempty, compact and positively invariant set, then the following statements hold:*

1. *if* $p \in M$ *is positively (negatively) Poisson stable, then* $p \in \Omega(M)$;

2. $\Omega(M)$ *is a nonempty, compact and positively invariant subset of* M;

3. *if* $(X, \mathbb{T}, \pi)$ *is a compactly dissipative dynamical system and* J *is its Levinson center [2, ChI], then the set* $\Omega(M)$ *is nonempty, compact, positively invariant and* $\Omega(M) \subseteq J$.

3 Birkhoff center of compact dissipative dynamical system

Let $(X, \mathbb{T}, \pi)$ be a compact dissipative dynamical system [2, ChI] and J be its Levinson center [2, ChI], and $M \subseteq X$ be a nonempty, closed and positively invariant subset from X. Denote by $M_1 := \Omega(M)$ the set of all non-wandering (with respect to M) points of $(X, \mathbb{T}, \pi)$. By Lemma 1 the set M_1 is a nonempty, compact and positively invariant subset of J. We denote by $M_2 := \Omega(M_1) \subseteq M_1$. Analogously we define the set $M_3 := \Omega(M_2) \subseteq M_2$. We can continue this process and we will obtain $M_n := \Omega(M_{n-1})$ for all $n \in \mathbb{N}$. One has a sequence $\{M_n\}_{n\in\mathbb{N}}$ possessing the following properties:

1. for all $n \in \mathbb{N}$ the set M_n is nonempty, compact and positively invariant;
2. $J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq M_{n+1} \supseteq \ldots$.

Denote $M_\lambda := \bigcap\limits_{n=1}^{\infty} M_n$, then M_λ is a nonempty, compact (since the set J is compact) and invariant subset of J. Now we define the set $M_{\lambda+1} := \Omega(M_\lambda)$ and we can continue this process to obtain the following sequence

$$\begin{gathered} J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq \\ M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq M_{\lambda+1} \supseteq \ldots \supseteq M_{\lambda+k} \supseteq \ldots. \end{gathered}$$

Now we construct the set $M_\mu := \bigcap\limits_{k=1}^{\infty} M_{\mu+k}$ and denote $M_{\mu+1} := \Omega(M_\mu)$ and so on. We obtain a transfinite sequence of nonempty, compact and positively invariant subsets

$$\begin{gathered} J \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots \supseteq M_n \supseteq \\ M_{n+1} \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\lambda \supseteq \ldots \supseteq M_\mu \supseteq \ldots. \end{gathered} \tag{1}$$

Since J is a nonempty compact set, then in the sequence (1) there is at most a countable family of different elements, i.e., there exists ν such that $M_{\nu+1} = M_\nu$.

Definition 2. *The set* $\mathfrak{B}(M) := M_\nu$ *is said to be the center of Birkhoff for the closed and positively invariant set* M. *If* $M = X$, *then the set* $\mathfrak{B}(\pi) := \mathfrak{B}(X)$ *is said to be the Birkhoff center of compact dissipative dynamical system* $(X, \mathbb{T}, \pi)$.

Theorem 2. *Suppose that* $(X, \mathbb{T}, \pi)$ *is a compact dissipative dynamical system and* J *is its Levinson center, then:*

1. $\mathfrak{B}(\pi)$ *is a nonempty, compact and invariant set;*

2. $\mathfrak{B}(\pi)$ *is a maximal compact invariant subset* M *of* X *such that* $\Omega(M) = M$;

3. *if for all* $t > 0$ *the map* $\tilde{\pi}(t, \cdot) := \pi(t, \cdot)\big|_{\mathfrak{B}(\pi)}$ *is open, then the set of all positively Poisson stable points of* $(X, \mathbb{T}, \pi)$ *is dense in* $\mathfrak{B}(\pi)$, *i.e.,* $\mathfrak{B}(\pi) = \overline{P(\pi)}$, *where* $P(\pi) := \{p \in X : \ p \in \omega_p\}$.

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Solving boundary value problems for delay integro-differential equations using spline functions

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Abstract

We consider a boundary value problem for integro-differential equations with delay and obtain sufficient conditions for the existence and uniqueness of its solution. The paper contains an iterative scheme of finding an approximate solution as a sequence of cubic spline functions with defect two.

Keywords: boundary value problems, integro-differential equations, delay, spline functions.

1 Introduction

Dynamic processes in many applied problems are described by differential and integral equations with delay [1]. An analytical solution of such an equation exists only in the simplest cases, so the construction and the study of approximate algorithms for solutions of these equations are important. In the present note we study an approximate method of solving boundary value problems for integro-differential equations with delay based on cubic spline with defect two approximation of the solution. Existence and uniqueness of boundary value problems with delay solution in various functional spaces were considered in [2, 3]. The usage of spline functions for solving differential-difference equations was investigated in [4, 5].

2 Problem statement. Solution existence

Let us consider the following boundary value problem

$$y''(x) = f(x, y(x), y(x - \tau_0(x)), y'(x), y'(x - \tau_1(x)))+$$

$$+\int_a^b g(x, s, y(s), y(s - \tau_0(s)), y'(s), y'(s - \tau_1(s)))ds, \quad (1)$$

$$y^{(i)}(x) = \varphi^{(i)}(x),\ i = 0, 1,\ x \in [a^*; a],\ y(b) = \gamma, \quad (2)$$

where $a^* = \min\{\inf\limits_{x\in[a;b]} (x - \tau_0(x)),\ \inf\limits_{x\in[a;b]} (x - \tau_1(x))\},\ \gamma \in R.$

Let $f(x, u_0, u_1, v_0, v_1),\ g(x, s, u_0, u_1, v_0, v_1)$ be continuous functions in $G = [a, b] \times G_1^2 \times G_2^2$ and $Q = [a, b] \times G$, where $G_1 = \{u\ \in R :\ |u| < P_1\},\ G_2 = \{v\ \in R :\ |v| \le P_2\}$, $P_1,\ P_2$ are positive constants, $\varphi(t) \in C^1[a^*; a]$, delay $\tau_1(x)$ is a continuous on $[a, b]$ function such that the set $E = \{x_i \in [a, b] :\ x_i - \tau(x_i) = a,\ i = \overline{1, k}\}$ is finite.

We shall introduce the notations

$$P = \sup\Big\{|f(x, u, u_1, v, v_1)| + \Big|\int_a^b g(x, s, u, u_1, v, v_1)ds\Big| :$$

$$|u_i| < P_1, |v_i| < P_2,\ i = 0, 1,\ x, s \in [a, b]\Big\}, J = [a^*; a], I = [a, b],$$

$I_1 = [a, x_1], \ldots, I_k = [x_{k-1}, x_k],\ I_{k+1} = [x_k, b],\ B(J \cup I) = \{y(x) : y(x) \in (C(J\cup I)(C^1(J)\cup C^1(I)) \cap (\bigcup\limits_{j=1}^{k+1} C^2(I_j))), |y(x)| \le P_1, |y'(x)| \le P_2\}.$

A function $y = y(x)$ from the space $B(J \cup I)$ is called a solution of the problem (1)–(2) if it satisfies the equation (1) on $[a; b]$ (with the possible exception of a set of points E) and boundary conditions (2).

Theorem 1. *Let the following conditions hold:*

1) $\max\{\max\limits_{x\in J} |\varphi(x)|, \frac{(b-a)^2}{8} P + \max(|\varphi(a)|, |\gamma|)\} \le P_1,$

2) $\max\{\max\limits_{x\in J} |\varphi'(x)|, \frac{b-a}{2} P + |\frac{\gamma-\varphi(a)}{b-a}|\} \le P_2,$

3) the functions $f(x, u_0, u_1, v_0, v_1), g(x, s, u_0, u_1, v_0, v_1)$ satisfy in G and Q the Lipschitz condition for variables $u_i, v_i, i = \overline{0, 1}$ with constants $L_j, M_j,\ j = \overline{1, 4}$, *4)* $\frac{(b-a)^2}{8}\sum\limits_{j=1}^{2}(L_j+(b-a)M_j)+\frac{b-a}{2}\sum\limits_{j=3}^{4}(L_j+(b-a)M_j)<1.$

Then there exists a unique solution of the problem (1)-(2).

3 Cubic splines with defect two

Let us consider on a segment $[a;b]$ an irregular grid $\Delta=\{a=x_0<x_1<\ldots<x_n=b\}$, $E\subset\Delta$. We shall denote an interpolating cubic spline with defect two $S(y,x)$ on Δ, which belongs to the space $B(J\cup I)$. We can obtain a formula of $S(y,x)$ [4, 5]:

$$S(y,x)=M^+_{j-1}\frac{(x_j-x)^3}{6h_j}+M^-_j\frac{(x-x_{j-1})^3}{6h_j}+(y_{j-1}-\frac{M^+_{j-1}h_j^2}{6})\frac{x_j-x}{h_j}+$$

$$+(y_j-\frac{M^-_jh_j^2}{6})\frac{x-x_{j-1}}{h_j},\ x\in[x_{j-1};x_j],\ h_j=x_j-x_{j-1},\ j=\overline{1,n},\qquad(3)$$

where $M^+_j=S''(y,x_j+0),\ j=\overline{0,n-1},\ M^-_j=S''(y,x_j-0),\ j=\overline{1,n}$, satisfy the following system of equations

$$h_{j+1}y_{j-1}-(h_j+h_{j+1})y_j+h_jy_{j+1}=$$

$$=\frac{h_jh_{j+1}}{6}(h_jM^+_{j-1}+2h_jM^-_j+2h_{j+1}M^+_j+h_{j+1}M^-_{j+1}),j=\overline{1,n-1},\qquad(4)$$

$y_0=\varphi(a),y_n=\gamma.$

4 Computational scheme

A) Choose a cubic spline $S(y^{(0)},x)$ randomly so that the boundary conditions (2) are enforced.

B) Using the original equation (1) and the spline $S(y^{(k)},x)$, find $M^{+(k+1)}_j,j=\overline{0,n-1},M^{-(k+1)}_j,j=\overline{1,n}$.

C) Compute $\{y^{k+1}_j\},j=\overline{0,n}$, from the equations (4).

D) According to (3), build a cubic spline $S(y^{(k+1)},x)$ using the values of $\{y^{k+1}_j\},M^{+(k+1)}_j,M^{-(k+1)}_j$. This spline will be the next approximation.

Let us denote
$\lambda_1 = L_1+L_2+(b-a)(M_1+M_2)$, $\lambda_2 = L_3+L_4+(b-a)(M_3+M_4)$, $K = \frac{H}{h}$, $h = \min_i h_i$, $H = \max_i h_i$, $u = \frac{K^5}{8}(b-a)^2 + \frac{H^2}{8}$, $v = \frac{K^5}{2}(b-a) + \frac{2H}{3}$.

Theorem 2. *Assume that the conditions of Theorem 1 hold.*
If $u\lambda_1 + v\lambda_2 < 1$ is true, then there exists $H^* > 0$ such that for each $0 < H < H^*$ the sequence of splines $\{S(y^{(k)}, x)\}$, $k = 0, 1 \ldots$, converges uniformly on $[a; b]$.

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Integrability of some two-dimensional systems with homogeneous nonlinearities

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Abstract

In this paper we discuss the problem of integrability for planar polynomial systems. An approach to find reversible systems within polynomial families of Lotka-Volterra systems with homogeneous nonlinearities is proposed.

Keywords: integrability, reversibility, polynomial systems.

1 Introduction

The integrability problem for systems of differential equations is one of the main problems in the qualitative theory of differential systems. The integrability problem for systems with quadratic or cubic nonlinearities has been intensively studied. Recently several works have been also devoted to the investigation of systems with quartic and quintic nonlinearities, see e.g. [2, 3] and references given their.

In this paper we study the problem of existence for a polynomial system of the form

$$\begin{aligned} \dot{x} &= \mathcal{X}(x,y) = x - \sum_{j+k=2}^{n} a_{jk}\, x^j y^k, \\ \dot{y} &= \mathcal{Y}(x,y) = -y + \sum_{j+k=2}^{n} b_{jk}\, x^j y^k \end{aligned} \qquad (1)$$

(where a_{ij}, b_{ji} are complex parameters and $x(t)$ and $y(t)$ are complex unknown functions) of the first integral of the form

$$\Psi(x, y) = xy + \sum_{j+k\geq 3} \psi_{j,k}\, x^j y^k. \tag{2}$$

This problem of integrability can be considered as a generalization of the center problem for real systems (see e.g. [3] for more details).

2 Reversibility in systems with homogeneous nonlinearities

One of important mechanisms for integrability is the so-called *time-reversibility* (or just *reversibility*). By the definition it is said that the system (1) is (time-)reversible if there is an invertible transformation R, $(x_1, y_1) = R(x, y)$, such that the system is invariant under the transformation and the time inversion $t \to -T$. The simplest case of reversibility is when R is a linear transformation of the form

$$R : x_1 \to \gamma y, \qquad y_1 \to \gamma^{-1} x, \tag{3}$$

for some $\gamma \in \mathbb{C} \setminus \{0\}$. If a system (1) is reversible with respect to (3), then it admits a local analytic first integral of the form (2) (Theorem 3.5.5 of [3]).

Sometimes the reversibility is hidden and just through a change of variables and a scaling of time can be detected. The next theorem treats this situation when the system becomes reversible with respect to involution (3) after a change of coordinate and a time rescaling.

Consider a polynomial Lotka-Volterra system of the form

$$\dot{x} = x(1 + A(x, y)), \qquad \dot{y} = -y(1 + B(x, y)), \tag{4}$$

where A and B are homogeneous polynomials of degree d.

Theorem 1. *There exists a polynomial f of the form $f = 1+F(x, y)$, where F is a homogeneous polynomial of degree $d - 1$ such that the* *change of coordinates*

$$z = k_1 y/f(x,y)^{1/(d-1)} \text{ and } w = k_2 x/f(x,y)^{1/(d-1)}, \tag{5}$$

whose inverse change is given by

$$x = w/(k_2\tilde{f}(w,z)^{1/(d-1)}) \text{ and } y = z/(k_1\tilde{f}(w,z)^{1/(d-1)}), \tag{6}$$

where $\tilde{f}(w,z) = 1/f(x(w,z),y(w,z))$ *transforms (4) to a system of the form*

$$\frac{dz}{dT} = -z(1+h(z,w)), \qquad \frac{dw}{dT} = w(1-h(z,w)). \tag{7}$$

Moreover,

$$f = 1 + \frac{A+B}{2} \tag{8}$$

and

$$h(w,z) = \frac{1}{2}\left((\hat{B}-\hat{A}) + \frac{1}{d-1}\left(\left(x\frac{\partial(A+B)}{\partial x}\right)\Bigg|_{x=w/k_2,y=z/k_1}(\tilde{f}+\hat{A}) - \left(y\frac{\partial(A+B)}{\partial y}\right)\Bigg|_{x=w/k_2,y=z/k_1}(\tilde{f}+\hat{B})\right)\right), \tag{9}$$

where

$$\hat{A}(w,z) = A(w/k_2, z/k_1), \quad \hat{B}(w,z) = B(w/k_2, z/k_1). \tag{10}$$

As a direct corollary of this theorem we obtain the following criterion for integrability.

Theorem 2. *System (4) has a first integral of the form (2) if*

$$h(w,z) + h(z,w) \equiv 0, \tag{11}$$

where h is the function defined by (9).

Proofs of the theorems and their applications can be found in [1].

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Solution of the Chazy System

Valeri Gromak

Abstract

We construct the solution of the Chazy system, which specifies the conditions for the existence of the Painlevé property for a third-order nonlinear equation with six poles.

Keywords: Painlevé property, Painlevé equations, Chazy system.

Some classes of second-order equations were studied by P. Painlevé and B. Gambier, and 50 canonical equations, whose solutions have no movable critical singular points, were found. This property is now called the Painlevé property. Of the equations singled out by Painlevé, six equations are most important, which now bear his name and whose solutions are called Painlevé transcendents [1]. R.Fuchs suggested two approaches to obtaining the Painlevé equations. The first one deals with isomonodromic deformations of Fuchsian systems. The second, more geometrical, approach uses elliptic integrals. The Painlevé equations define nonlinear special functions.

For higher-order equations, the Painlevé classification problem has proved to be very hard, and so far the most complete results have been obtained only for higher-order polynomial equations.

The paper [2] is one of the first papers on the classification of higher-order equations with respect to the Painlevé property. It deals with the analysis of the Painlevé property of the Chazy equation

$$y''' = \sum_{k=1}^{6} \frac{A_k(y'-a_k')^3 + B_k(y'-a_k')^2 + C_k(y'-a_k')}{y-a_k} + Dy'' + Ey' +$$

$$+\sum_{k=1}^{6}\frac{(y'-a'_k)(y''-a''_k)}{y-a_k}+\prod_{k=1}^{6}(y-a_k)\sum_{k=1}^{6}\frac{F_k}{y-a_k}. \tag{1}$$

Here the poles $a_k = a_k(z)$ are finite and distinct and, in general, are functions of the independent variable z.

The paper [2] also presents the system of 31 algebraic and differential equations for 26 unknown functions $A_k = A_k(z)$, $B_k = B_k(z)$, $C_k = C_k(z)$, $D = D(z)$, $E = E(z)$, and $F_k = F_k(z)$ (Chazy system $(\mathcal{A})$–$(\mathcal{F})$), whose solution, as Chazy claims, determines necessary and sufficient conditions for the absence of movable critical points of solutions of (1). For example, Chazy system $(\mathcal{A})$–$(\mathcal{B})$ has the form

$$\sum_{j=1}^{6}A_j=0,\quad \sum_{j=1}^{6}a_jA_j=-6,\quad \sum_{j=1}^{6}a_j^2A_j=-2\sum_{j=1}^{6}a_j,$$

$$2A_k^2+\sum_{j=1}^{6}\frac{A_k-A_j}{a_k-a_j}=0,\ k=1,\dots,6\ (j\neq k), \tag{$\mathcal{A}$}$$

$$\sum_{j=1}^{6}(B_k-B_j)\Big(-\frac{A_k}{2}-\frac{1}{a_k-a_j}\Big)+A'_k-\frac{3}{2}A_k\sum_{i=1}^{6}a'_iA_i-$$

$$-\sum_{j=1}^{6}\frac{a'_k-a'_j}{a_k-a_j}(A_k-3A_j)=0, \tag{$\mathcal{B}$}$$

Here the poles a_k are parameters of the system. System $(\mathcal{A})$–$(\mathcal{F})$ was not studied in [2]. For solution of system $(\mathcal{A})$–$(\mathcal{F})$ first, note that the successive elimination of the variables A_k in $(\mathcal{A})$ permits uniquely expressing A_6, A_5, A_4, A_3, and A_2 via A_1.

We separately note the structure A_2:

$$A_2=\frac{n_0A_1^4+n_1A_1^3+n_2A_1^2+n_3A_1+n_4}{(a_2-a_4)(a_2-a_5)(a_2-a_6)Q_2},$$

where $Q_2 = d_0A_1^2+d_1A_1+d_2$, and the coefficients $n_0,\dots,n_4$ $d_0,\ d_1,\ d_2$ and $d_0,\ d_1,\ d_2$ are known functions of a_i.

Lemma. *System* $(\mathcal{A})$ *admits the symmetry* $(A_k, a_k) \leftrightarrow (A_j, a_j)$, $j, k = 1, \ldots, 6$.

Therefore, a permutation of arbitrary components (A_k, a_k) of the solution of system $(\mathcal{A})$ with arbitrary components (A_j, a_j) leads to a solution of system $(\mathcal{A})$.

Theorem 1 [3]. *System* $(\mathcal{A})$ *has the solution*

$$\begin{aligned}
A_j &= \frac{1}{a_5 - a_j} + \frac{1}{a_6 - a_j}, \quad j = 1, \ldots, 4,\\
A_5 &= \frac{1}{a_1 - a_5} + \frac{1}{a_2 - a_5} + \frac{1}{a_3 - a_5} + \frac{1}{a_4 - a_5} + \frac{2}{a_5 - a_6},\\
A_6 &= \frac{1}{a_1 - a_6} + \frac{1}{a_2 - a_6} + \frac{1}{a_3 - a_6} + \frac{1}{a_4 - a_6} + \frac{2}{a_6 - a_5}
\end{aligned}$$

under the following conditions on the poles:

$$\begin{aligned}
&-3s_3a_5 + 6s_4 + s_2a_5^2 + (-3s_3 + 4s_2a_5 - 3s_1a_5^2)a_6+\\
&\qquad +(s_2 - 3s_1a_5 + 6a_5^2)a_6^2 = 0, \qquad (2)
\end{aligned}$$

where $s_1, \ldots, s_4$ *are elementary symmetric polynomials in* $a_1, \ldots, a_4$.

In the general case, equation for A_1 has the form

$$U(p, A_1) = p_0A_1^5 + p_1A_1^4 + p_2A_1^3 + p_3A_1^2 + p_4A_1 + p_5 = 0, \qquad (3)$$

where $p_0, \ldots p_5$ are known functions of poles a_k.

Theorem 2. *Let* A_1 *be a solution of (3) for some fixed values of poles* a_k *such that* $Q_2 \neq 0$. *Then* A_1 *and* A_k $(k = 2, \ldots, 6)$, *evaluated on the basis of this value of* A_1, *define a solution of system* $(\mathcal{A})$.

Note that if the parameters a_k satisfy condition (2), then (3) can be factorized and represented in the form $(A_1Q_1 - P_1)\widetilde{U}(\tilde{p}, A_1)$, where the $\widetilde{U}(\tilde{p}, A_1)$ is polynomial in A_1. The vanishing of the factor $(A_1Q_1 - P_1)$ permits one to determine A_1 and successively the remaining A_k in a closed form, which coincides with the formulas in Theorem 1. It follows that Theorem 1 is a special case of Theorem 2. However, under condition (2), the A_k can be determined in the closed form

(2). Solution of system $(\mathcal{A})$–$(\mathcal{F})$ for known A_k can be reduced to the successive solution of three linear algebraic systems with additional constraints. In the general case, the solution of systems $(\mathcal{B})$, $(\mathcal{C})$, $(\mathcal{F})$ can be found by the Gauss method. For example, system $(\mathcal{B})$ for known A_k is the linear system $A_B B = R_B$ for $B = (B_1, \ldots, B_6)^T$, where the matrix A_B has the entries

$$\{A_B\}_{kj} = \frac{A_k}{2} + \frac{1}{a_k - a_j}, \quad \{A_B\}_{kk} = -\frac{5A_k}{2} - \sum_{i=1}^{6} \frac{1}{a_k - a_i}, \quad i, j \neq k.$$

In general, the rank of this matrix does not exceed 5. The vector R_B depends only on A_k, a_k and their derivatives. If the assumptions of Theorem 1 hold, then the matrix A_B can be represented in a closed form as well. To this end, one should substitute the values $A_1, \ldots, A_6$ (2) into the above-represented matrix A_B.

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The Averaging in the Multifrequency System of Differential Equations with Linearly Transformed Arguments and Integral Boundary Conditions

Inessa Krasnokutska

Abstract

In this work we researched the existence of solution and gave justifiction of averaging method on fast variables for multifrequency systems of differential equations with linearly transformed arguments and with integral boundary conditions. The coefficients in integral conditions depend on slow time, slow variables and on fast variables too.

Keywords: method of averaging, boundary value problems, linearly transformed arguments, multifrequency systems, resonances, slow and fast variables, integral conditions.

This paper presents the conditions of existence of solution of the system with slow and fast variables of the form

$$\frac{da}{d\tau} = X(\tau, a_\Lambda, \varphi_\Theta), \tag{1}$$

$$\frac{d\varphi}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + Y(\tau, a_\Lambda, \varphi_\Theta), \tag{2}$$

where $\tau \in [0, L]$, small parameter $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \ll 1$, $x \in D \subset \mathbb{R}^m$, $\varphi \in \mathbb{R}^m$; λ_i θ_j are numbers from semi-interval $(0, 1]$, $0 < \lambda_1 < \cdots < \lambda_{r_1} \leq 1$, $0 < \theta_1 < \cdots < \theta_{r_2} \leq 1$, $a_{\lambda_i}(\tau) = a(\lambda_i \tau)$, $\varphi_{\theta_j}(\tau) = \varphi(\theta_j \tau)$, $a_\Lambda = (a_{\lambda_1}, \ldots, a_{\lambda_{r_1}})$, $\varphi_\Theta = (\varphi_{\theta_1}, \ldots, \varphi_{\theta_{r_2}})$.

Let us set the system (1), (2) of boundary conditions

$$\int_0^L f(\tau, a_\Lambda, \varphi_\Theta) d\tau = d_1,$$

$$\int_0^L \left[\sum_{j=1}^{r_2} h_j(\tau, a_\Lambda, \varphi_\Theta) \varphi_{\theta_j} + g(\tau, a_\Lambda, \varphi_\Theta) \right] d\tau = d_2,$$

where $f, g, h_1, \ldots, h_{r_2}$ are 2π-periodic with fast variables vector-functions. d_1 and d_2 are n and m measurable vectors.

When vector-function h depends only on variables τ and a, in the work [1] systems without delay were investigated. In this work we researched the existence of solution and gave justifiction of averaging method on fast variables for multifrequency systems of differential equations with linearly transformed arguments and with integral boundary conditions, when all coefficients in integral conditions depend on slow time, slow variables and on fast variables too. This problem was solved in [2] for systems without delay.

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First integrals with polynomials not higher than second order of the mathematical model of the intrinsic transmission dynamics of tuberculosis

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Abstract

For the mathematical model of the intrinsic transmission dynamics of tuberculosis (TB) all first integrals with polynomials not higher than second order were found.

Keywords: tuberculosis, Lie algebra, first integral.

Consider three-dimensional autonomic real differential system which simulates the intrinsic transmission dynamics of tuberculosis [1], [2]

$$\begin{aligned} &\frac{dS}{dt} = \tau - \mu S - \beta ST, \quad \frac{dL}{dt} = -\delta L - \mu L + (1-p)\beta ST, \\ &\frac{dT}{dt} = \delta L - (\mu+\nu)T + p\beta ST. \end{aligned} \tag{1}$$

The parameters of the system (1) are described in Table 1 (see page 258).

According to [3] we obtain

Theorem 1. *The system (1) admits the noncommutative Lie algebra of operators of the form*

$$X_1 = S\frac{\partial}{\partial S} + L\frac{\partial}{\partial L} + T\frac{\partial}{\partial T} + D_1, \quad X_2 = (-\frac{\tau}{\beta} - \frac{\nu}{\beta}S + ST)\frac{\partial}{\partial S} +$$

$$+[\frac{\delta-\nu}{\beta}L + (p-1)ST]\frac{\partial}{\partial L} - (\frac{\delta}{\beta}L + pST)\frac{\partial}{\partial T} + D_2, \tag{2}$$

where

$$D_1 = -\beta\frac{\partial}{\partial\beta} + \tau\frac{\partial}{\partial\tau}, \quad D_2 = (\mu+\nu)\frac{\partial}{\partial\beta} - \frac{\tau}{\beta}(\mu+\nu)\frac{\partial}{\partial\tau}, \tag{3}$$

and the structural equation is $[X_1, X_2] = X_2$.

Table 1. Variables and parameters of the sistem (1)

Value	Description
$S(t)$	number of sensible persons in the moment t
$L(t)$	number of infected persons in the moment t
$T(t)$	number of infectious persons in the moment t
$\beta T(t)$	force of infection per capita in the moment t
τ	influx of young people
μ	average mortality from causes not related to TB
p	probability of rapid progression of the disease
δ	constant of speed of reactivation of TB infection
ν	additional mortality caused by active TB
β	transfer coefficient of TB infection

Note that the expressions

$$U_1 = \beta\tau, \quad U_2 = \mu, \quad U_3 = \nu, \quad U_4 = \delta, \quad U_5 = p, \tag{4}$$

are invariants of the system (1) with respect to the operators (2)–(3), i.e. $D_1(U_i) = D_2(U_i) = 0$ $(i = \overline{1,5})$.

Further we assume that U_i $(i = \overline{1,4})$ from (4) do not vanish. This guarantees us the existence of the quadratic part ST and of the free term τ in the system (1). The condition $\mu\nu\delta \neq 0$ arises from the medical sense of the parameters.

We determine the coordinates of the vector $(\tau, \beta, \mu, \delta, \nu, p)$ which contain the parameters of the system (1) when the invariants U_i $(i = \overline{1,4})$ (see (4)) are different from zero and first integral has the form

$$I_q(S, L, T, t) = P_q(S, L, T)\exp(\lambda t) \ (q \leq 2). \tag{5}$$

Assume that

$$P_q(S,L,T) = a + bS + cL + dT + eS^2 + fL^2 + gT^2 +$$
$$+2hSL + 2kST + 2lLT. \tag{6}$$

The coefficients of the polynomial (6) and the parameter λ are real unknown. From $\dfrac{dI_q}{dt} \equiv 0$ $(q \leq 2)$ using the relations (5)–(6) under system (1) we obtain the following system of polynomial equations:

$$\lambda a + \tau b = 0,\ 2\tau e + (\lambda - \mu)b = 0,\ 2\tau h - \mu c + \delta(d - c) + \lambda c = 0,$$
$$2\tau k + (\lambda - \mu - \nu)d = 0,\ (\lambda - 2\mu)e = 0,\ -2\mu f + 2\delta(l - f) + \lambda f = 0,$$
$$(\lambda - 2\mu - 2\nu)g = 0,\ 2\mu h + \delta(h - k) - \lambda h = 0,\ (2\mu + \nu)l - \delta(g - l) - \lambda l = 0,$$
$$\beta(-b + c - cp + dp) + 2(\lambda - 2\mu - \nu)k = 0,\ \beta(e - h + hp - kp) = 0,$$
$$\beta(k - l - gp + lp) = 0,\ \beta(f - h - fp + lp) = 0. \tag{7}$$

Consequently we arrive at the next result

Theorem 2. *Assume that the conditions* $U_1U_2U_3U_4 \neq 0$ *and* $0 \leq U_5 \leq 1$ *hold. Then the system (1) possessing the vector* $(\tau, \beta, \mu, \delta, \nu, p)$ *has* 5 *first integrals of the form (5)–(6) (see Table 2).*

Table 2. First integrals of the sistem (1)

$(\boldsymbol{\tau, \beta, \mu, \delta, \nu, p})$	**First integral**
$(\tau, \beta, \mu, p\nu, \nu, p)$	$I_1^{(1)} = (L + \frac{p-1}{p}T)\exp(t(\mu+\nu))$
$(\tau, \beta, \mu, \delta, \nu, 1)$	$I_1^{(2)} = L\exp(t(\delta+\mu))$
$(\tau, \beta, \mu, -\mu, \nu, 1)$	$I_2^{(1)} = a + L(c + fL)$
$(\tau, \beta, \mu, -p\mu, -\mu, p)$	$I_2^{(2)} = a + (L + \frac{p-1}{p}T)(c + f(L + \frac{p-1}{p}T))$
$(\tau, \frac{\mu(\nu^2-\mu^2)}{\nu\tau}, \mu, -\nu, \nu, 0)$	$I_2^{(3)} = ((\nu^2-\mu^2)((L+S)^2 + 2T(L+S))/(2\mu\tau) +$ $+(L + S + T) + T\nu/\mu - S\nu^2/\mu^2 -$ $-\tau/(2\mu) + \nu^2\tau/(2\mu^3))\exp(2t\mu)$

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Family of quadratic differential systems with invariant hyperbolas

Regilene D. S. Oliveira, Alex C. Rezende, Nicolae Vulpe

Abstract

In this article we consider the class $\mathbf{QS_f}$ of all quadratic systems possessing a finite number of singularities (finite and infinite). Using the algebraic invariant theory we provided necessary and sufficient conditions for a system in $\mathbf{QS_f}$ to have invariant hyperbolas in terms of its coefficients

Keywords: quadratic differential system, group action, affine invariant polynomial, algebraic invariant curve.

1 Introduction

Quadratic systems with an invariant algebraic curve have been studied by many authors (see for example [1, 2] and references from [1]). The main goal of this paper is to investigate nondegenerate quadratic systems having invariant hyperbolas and this study is done applying the invariant theory. More precisely in this paper we provided necessary and sufficient conditions for a quadratic system in $\mathbf{QS_f}$ to have invariant hyperbolas. We also determine the invariant criteria which provide the number and multiplicity of such hyperbolas.

2 Statement of the main result

Our main results are stated in the following theorem.

Main Theorem. (A) *The conditions* $\eta \geq 0$, $M \neq 0$ *and* $\gamma_1 = \gamma_2 = 0$ *are necessary for a quadratic system in the class* $\mathbf{QS_f}$ *to possess at least one invariant hyperbola.*

(B) *Assume that for a system in the class* $\mathbf{QS_f}$ *the condition* $\eta \geq 0$, $M \neq 0$ *is satisfied.*

- (B_1) *If* $\eta > 0$, *then the necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in* Diagram 1, *where we can also find the number and multiplicity of such hyperbolas.*

- ($B2$) *In the case* $\eta = 0$ *and* $M \neq 0$ *the corresponding necessary and sufficient conditions for this system to possess at least one invariant hyperbola are given in* Diagram 2, *where we can also find the number and multiplicity of such hyperbolas.*

(C) *The* Diagrams 1 *and* 2 *actually contain the global bifurcation diagram in the* 12*-dimensional space of parameters of the systems belonging to family* $\mathbf{QS_f}$ *, which possess at least one invariant hyperbola. The corresponding conditions are given in terms of invariant polynomials with respect to the group of affine transformations and time rescaling.*

Remark 1. In the case of the existence of two hyperbolas we denote them by $\mathcal{H}^p$ if their asymptotes are parallel and by $\mathcal{H}$ if there exists at least one pair of non-parallel asymptotes. We denote by $\mathcal{H}_k$ $(k = 2, 3)$ a hyperbola with multiplicity k; by $\mathcal{H}_2^p$ a double hyperbola, which after perturbation splits into two $\mathcal{H}^p$; and by $\mathcal{H}p_3$ a triple hyperbola which splits into two $\mathcal{H}^p$ and one $\mathcal{H}$.

Remark 2. The expressions for the invariant polynomials η, M, β_i $(i = 1, 2, ..., 13)$, γ_i $(i = 1, 2, ..., 16)$, δ_i $(i = 1, 2, ..., 6)$ and $\mathcal{R}_i$ $(i = 1, 2, ..., 11)$ are given in [3]

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- $\eta>0$
  - $\theta\neq 0$
    - $\beta_1\neq 0$
      - $\beta_2\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \mathcal{R}_1\neq 0$
      - $\beta_2=0$
        - $\beta_3\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_3=0,\ \mathcal{R}_1\neq 0$
        - $\beta_3=0$
          - $\beta_4\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \beta_5\mathcal{R}_2\neq 0$
          - $\beta_4=0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_3=0,\ \mathcal{R}_2\neq 0$
    - $\beta_1=0$
      - $\beta_6\neq 0$
        - $\beta_2\neq 0$ → $\exists \geq 1 \Leftrightarrow \gamma_4=0,\ \mathcal{R}_3\neq 0$ and either $\begin{cases}\delta_1\neq 0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \delta_1=0 \Rightarrow 2\mathcal{H}\end{cases}$
        - $\beta_2=0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_5=0, \mathcal{R}_4\neq 0$
      - $\beta_6=0$
        - $\beta_7\neq 0$ → $\exists \geq 1 \Leftrightarrow \gamma_5=0,\ \mathcal{R}_5\neq 0$ and either $\begin{cases}\beta_8^2+\delta_2^2\neq 0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \beta_8=\delta_2=0 \Rightarrow 2\mathcal{H}\end{cases}$
        - $\beta_7=0$
          - $\beta_9\neq 0$ → $\exists \geq 1 \Leftrightarrow \gamma_5=0, \mathcal{R}_5\neq 0$ and either $\begin{cases}\delta_3\neq 0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \delta_3=0,\ \beta_8\neq 0 \Rightarrow 2\mathcal{H}, \text{ or}\\ \delta_3=\beta_8=0 \Rightarrow 3\mathcal{H}\end{cases}$
          - $\beta_9=0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_6=0,\ \mathcal{R}_5\neq 0$
  - $\theta=0$
    - $N\neq 0$
      - $\beta_6\neq 0$
        - $\beta_{10}\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_7=0, \mathcal{R}_6\neq 0$
        - $\beta_{10}=0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \gamma_4=0,\ \beta_2\mathcal{R}_3\neq 0$
      - $\beta_6=0$
        - $\beta_2\neq 0$
          - $\beta_7\neq 0$ → $\exists \geq 1 \Leftrightarrow \gamma_8=0,\ \beta_{10}\mathcal{R}_7\neq 0$ and either $\begin{cases}\delta_4\neq 0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \delta_4=0 \Rightarrow 2\mathcal{H}\end{cases}$
          - $\beta_7=0$ → $\exists \geq 1 \Leftrightarrow \gamma_9=0,\ \mathcal{R}_8\neq 0$ and either $\begin{cases}\delta_5\neq 0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \delta_5=0 \Rightarrow 2\mathcal{H}\end{cases}$
        - $\beta_2=0$
          - $\beta_7\neq 0$
            - $\beta_{10}\neq 0$ → $\exists \geq 1 \Leftrightarrow \gamma_7\gamma_8=0, \mathcal{R}_5\neq 0$ and either $\begin{cases}\beta_8^2+\delta_2^2\neq 0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \beta_8=\delta_2=0 \Rightarrow 2\mathcal{H}\end{cases}$
            - $\beta_{10}=0$ → $\exists \geq 1 \Leftrightarrow$ $\mathcal{R}_3\neq 0$ and either $\begin{cases}\gamma_7\neq 0, \gamma_{10}>0 \Rightarrow 2\mathcal{H}^p, \text{ or}\\ \gamma_7\neq 0, \gamma_{10}=0 \Rightarrow 1\mathcal{H}_2^p, \text{ or}\\ \gamma_7=0, \gamma_{10}<0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \gamma_7=0, \gamma_{10}>0 \Rightarrow 1\mathcal{H}+2\mathcal{H}^p\end{cases}$
          - $\beta_7=0$ → $\exists\ 2\mathcal{H} \Leftrightarrow \gamma_7=0,\ \mathcal{R}_3\neq 0$
    - $N=0$
      - $\beta_2\neq 0$ → $\exists\ 1\mathcal{H} \Leftrightarrow \beta_1=\gamma_{11}=0, \mathcal{R}_9\neq 0$
      - $\beta_2=0$ → $\exists \geq 1 \Leftrightarrow \beta_1=0$ and either $\begin{cases}\gamma_{12}=0,\ \mathcal{R}_9\neq 0 \Rightarrow 1\mathcal{H}, \text{ or}\\ \gamma_{13}=0 \Rightarrow \infty\end{cases}$

Diagram 1. The case $\boldsymbol{\eta > 0}$

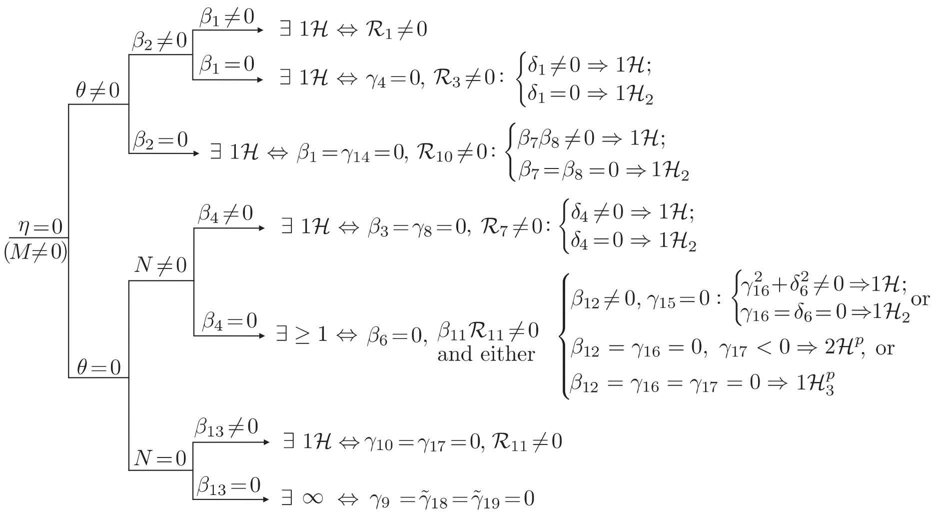

Diagram 2. The case $\boldsymbol{\eta=0}$

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The upper and lower bounds for number of conditions for center of the differential system $s(1, 2, ..., q)$

Mihail Popa, Victor Pricop

Abstract

We examine the system of differential equations that contains all nonlinearities of degree from 1 to q, denoted by $s(1, 2, ..., q)$. For this system the estimation of upper bound for the number of algebraically independent focal quantities, which take part in solving the center and focus problem for mentioned system, is known. It was made a comparison between this estimation and estimation of lower bound for number of essential conditions for center that was obtained by acad. C. S. Sibirschi for the system of differential equations $s(1, 2, ..., q)$.

Keywords: system of differential equations, essential conditions for center, the center and focus problem, Krull dimension, Sibirschi graded algebras.

Let us consider the system of differential equations

$$\frac{dx^j}{dt} = \sum_{i=1}^{q} P_i^j(x) \ (j = 1, 2), \tag{1}$$

where P_i^j are homogeneous polynomials of degree i in the coordinates of the vector $x = (x^1, x^2)$. The coefficients and variables of the system (1) take values from the field of the real numbers $\mathbb{R}$. We denote the system (1) by $s(1, 2, ..., q)$.

We examine the center and focus problem for this system. In the paper [1] it was demonstrated

Theorem 2.16. *The number of essential conditions for center for the system of differential equations* (1) *is not less than* $q^2 - q$, *for even* q *and* $q^2 - q - 1$, *for odd* q.

Let us consider the system of differential equations (1) when $q = 3$, i.e. $s(1, 2, 3)$. Then from Theorem 2.16 it results

Corollary 1. *If in the system of differential equations* (1) *we have* $q = 3$, *then the number of essential conditions for center is not less than* 5.

In the paper [2] it was examined the sequence of focal quantities for the system of differential equations $s(1, 2, 3)$ from (1)

$$L_1, L_2, ..., L_k, ..., \tag{2}$$

when the origin of coordinates is a singular point of the second type (center or focus).

For each focal quantity from this sequence it was put in correspondence a sequence of linear spaces of center-affine (unimodular) comitants of given type, i.e. for fixed k we have

$$L_k \to \left(2(k+1), \frac{1}{2}(5k^2 + 9k + 2) + i, 2(k-i), i\right), \ (i = \overline{0, k}),$$

where $2k+1$ is the degree of homogeneity of the comitant in coordinates of the vector x, $\frac{1}{2}(5k^2 + 9k + 2) + i$ is the degree of homogeneity of the comitant in coefficients of the polynomials $P_1^j(x)$ $(j = 1, 2)$, $2(k - i)$ is the degree of homogeneity of the comitant in coefficients of the polynomials $P_2^j(x)$ $(j = 1, 2)$, and i is the degree of homogeneity of the comitant in coefficients of the polynomials $P_3^j(x)$ $(j = 1, 2)$.

The characteristic of these spaces lies in the fact that for focal quantities L_k in each of these spaces there exists such comitant of the weight -1 [3] that the sum of their semiinvariants [2] on Sibirschi invariant variety for center accurately by a numerical constant is equal to L_k $(k = 1, 2, ...)$.

Also in the paper [2] it was shown that for any system of differential equations $s(1, m_1, ..., m_\ell)$, where m_i are some different integers, the maximal number of algebraically independent focal quantities for this system, when the origin of coordinates is a singular point of the second type (center or focus) does not exceed the Krull dimension of Sibirschi graded algebra $S_{1,m_1,...,m_\ell}$, and it is equal to

$$\varrho(S_{1,m_1,...,m_\ell}) = 2\left(\sum_{i=1}^{\ell} m_i + \ell\right) + 3. \tag{3}$$

Hence we have

Remark 1. *The number of algebraically independent focal quantities for the system of differential equations* $s(1,2,3)$, *when the origin of coordinates is a singular point of the second type* (*center or focus*) *does not exceed* 17.

Adapting the result from (3) for the system of differential equations $s(1,2,...,q)$ from (1) we have

$$\varrho(S_{1,2,...,q}) = 2\left(\sum_{i=2}^{q} i + q - 1\right) + 3 = 2[(2+3+...+q+q-1)+1] =$$

$$= 2\left[\frac{q(q+1)}{2} + q - 1\right] = q(q+1) + 2q - 2 + 1 = q^2 + 3q - 1.$$

We obtain that between relations from Theorem 2.16 and equality (3) the following equality takes place

$$\varrho(S_{1,2,...,q}) = q^2 + 3q - 1 = \underbrace{[q^2 - q] + 4q - 1}_{\text{for even q}} \text{ or } \underbrace{[q^2 - q - 1] + 4q}_{\text{for odd q}}.$$

From these equalities we conclude the following

Conclusion 1. *The estimation* (3) *for system of differential equations* $s(1,2,...,q)$ *from* (1) *is greater than the estimation from Theorem* 2.16 *with* $4q-1$, *for even* q *and with* $4q$, *for odd* q.

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Representations of the Lie tangential transformations

Sergiu Port

Abstract

In this work the images of the trigonometric functions at the Lie tangential transformations in the projective plan are represented.

Keywords: Lie tangential transformations, Lie operator.

Lie tangential transformations form a group described by the Lie operator. The subgroups of this group are known as the affine and the projective groups.

It is known that the number of invariants and respectively the rational basis of the invariants of the group are decreased at the extension of the group.

Tangential transformations of the point put in the correspondence a curve, analogous to polar correspondence, where in the projective plane, a straight line is corresponding to the point.

Lie described the tangential transformation in a plan, by the equation:

$$F(x, y, \overline{x}, \overline{y}) = 0, \tag{1}$$

which satisfies the necessary conditions of continuity.

The transformation may be illustrated graphically, as following:

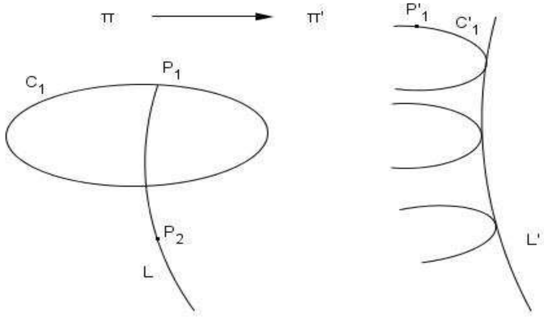

Fig. a

According to Fig. a the point P_1 corresponds to the curve C_1' and viceversa the point $P_1' \in C_1'$ corresponds the curve C_1 which passes through the point P_1. If the point P_1 will move into the plan π after a curve L, then at the tangential transformation the curve L will correspond $L' \in \pi'$. L' will be the wrapper of the curve family C_i'.

To determine the respective curves with the equation (1) the differential equation will be solved as:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} p = 0, \tag{2}$$

where $p = \frac{\partial y}{\partial x}$ is called a linear element and indicates the direction on the curve L. Actually, the given transformation puts in correspondence to each triplet x, y, p of the curve $L \in \pi$ the triplet $\overline{x}$, $\overline{y}$, $\overline{p}$ of the curve $L' \in \pi'$.

The Lie tangential transformation possesses invariant characteristic property: two tangent curves into π will transform in two tangent curves into π'.

This property allows a classification of curves into a plan. The suggested problem is the representation of the solutions of the differential equations (2).

In the projective plan, Felix Klein represented some of polynomial curves, which he had classified. As far as the complex roots of characteristic equation of the linear differential system involves expression of solutions by trigonometric functions. Therefore the graphical representation is worth to investigate.

We examine images to the Lie tangential transformation of the functions $y = sin(x)$ and $y = tg(x)$. The graphical images of the functions $y = sin(x)$ and $y = tg(x)$ are represented respectively in Fig. b and Fig. c.

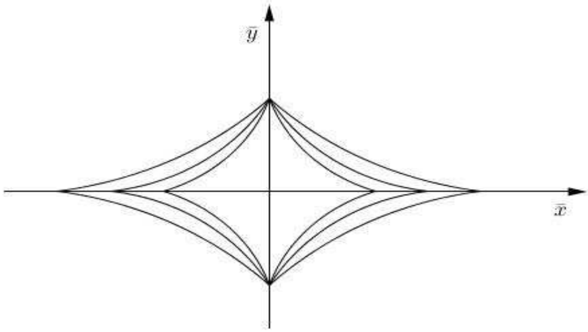

Fig. b

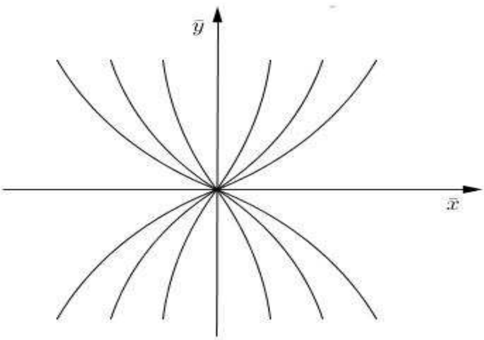

Fig. c

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Isochronous Hamiltonians with quadratic nonlinearities

A.E. Roudenok

Abstract

In this paper we study nontrivial and nonglobal polynomial Hamiltonian isochronous center associated to Hamiltonians of the form $H(x,y) = F(x) + 2G(x)y + Q(x)y^2$.

Keywords: Hamiltonian, Jacobian pair, trivial (nontrivial) isochronous center, global (nonglobal) isochronous center.

Consider real polynomial Hamiltonian system

$$\dot{x} = -H_y(x,y),\ \dot{y} = H_x(x,y). \tag{1}$$

We will assume that $H(0,0) = 0$ and that system (1) has a nondegenerate center at the origin.

There is a simple method to generate polynomial Hamiltonian isochronous centers. Take two polynomials $u(x,y)$ and $v(x,y)$ in two variables with $u(0,0) = v(0,0) = 0$ such that the determinant of the Jacobian of the mapping

$$(x,y) \to (u(x,y), v(x,y)) \tag{2}$$

is constant. A couple of such polynomials is called a Jacobian pair. It is readily seen that the Hamiltonian system associated to

$$H(x,y) = \frac{1}{2}\left(u^2(x,y) + v^2(x,y)\right) \tag{3}$$

is linearizable by means of the canonical change of coordinates $u = u(x,y)$, $v = v(x,y)$ and hence that the origin is an isochronous center.

We call [1] trivial isochronous centers the centers constructed with this method.

Authors of the paper [1] first give examples of nontrivial and non-global polynomial Hamiltonian isochronous centers.

We study nontrivial and nonglobal polynomial Hamiltonian isochronous centers in the case when the polynomial Hamiltonian (3) has the form

$$H(x, y) = F(x) + 2G(x)y + Q(x)y^2, \tag{4}$$

where

$$u(x, y) = x + \varphi(x),\ v(x, y) = y + g(x, y) \tag{5}$$

and $\varphi(x) \neq 0$.

It is easily proved that in this case

$$v(x, y) = \frac{y + g(x)}{1 + \varphi'(x)}, \tag{6}$$

where $g(x)$ is some differentiable function.

Substituting functions (5), (6) in (3) and equating coefficients under equal powers of a variable y in got Hamiltonian and in Hamiltonian (4) we get equalities

$$F(x) = (x + \varphi(x))^2 + \frac{g^2(x)}{(1 + \varphi'(x))^2},$$

$$G(x) = \frac{g(x)}{(1 + \varphi'(x))^2},\ Q(x) = \frac{1}{(1 + \varphi'(x))^2}. \tag{7}$$

From (7) we have

$$\varphi(x) = -x + \int\limits_0^x \frac{1}{\sqrt{Q(x)}} dx,\ g(x) = \frac{G(x)}{Q(x)}. \tag{8}$$

Substituting these expressions in the first equality of (7) we have

$$\frac{F(x)Q(x) - G^2(x)}{Q(x)} = \left(\int\limits_0^x \frac{1}{\sqrt{Q(x)}} dx \right)^2. \tag{9}$$

Denote

$$F(x)Q(x) - G^2(x) = h(x). \tag{10}$$

In this case we can write (9) in the form

$$\int\limits_0^x \frac{1}{\sqrt{Q(x)}}dx = \frac{\sqrt{h(x)}}{\sqrt{Q(x)}}.$$

Differentiation of the last equality means that

$$Q(x)h'(x) - h(x)Q'(x) = 2\sqrt{h(x)}Q(x). \tag{11}$$

Since $F(x),\ G(x),\ Q(x)$ are polynomials, the function $h(x)$ from (10) is polynomial.

It follows from (11) that $h(x)$ is quadrate of some polynomial.

Denote $h(x) = f^2(x)$. Then equality (11) is rewritten in the form

$$\frac{Q(x)}{Q'(x)} = \frac{f(x)}{2(f'(x) - 1)}. \tag{12}$$

From (10) we have

$$F(x) = \frac{f^2(x) + G^2(x)}{Q(x)}. \tag{13}$$

So we obtain the following result.

Theorem 1. *If the Hamiltonian system associated to polynomial Hamiltonian of the form (3), (4) is linearizable by means of the change of coordinates (5), (6), then polynomials $F(x),\ G(x),\ Q(x)$ are connected with relations (12), (13), where $f(x)$ is polynomial and $f(0) = 0,\ f'(0) = 1,\ Q(0) = 1$.*

One can prove the following assertions.

Theorem 2. *Polynomials $Q(x),\ f(x)/x$ have not real roots.*

Theorem 3. *The degree of the polynomial $f(x)$ is $1/2$ of degree of the polynomial $Q(x)$.*

Corollary. The degree of polynomial $Q(x)$ of real isochronous Hamiltonian (4) can be equal only to $4n + 6,\ n \in \mathbb{N}\bigcup\{0\}$.

Theorem 4. *Hamiltonians*

$$H_1 = x^2(1 - y - x^2y - g(x))^2 + (y + x^2y + g(x))^2,$$

$$H_2 = x^2(1 + y + x^2y + g(x))^2 + (y + x^2y + g(x))^2,$$

where $g(x)$ is a polynomial, a least degree of which is nonless as 2, are isochronous nonglobal ones. These Hamiltonians are the only ones among the polynomial isochronous Hamiltonians (4), where the degree of polynomial $Q(x)$ is equal to 6.

Proof. It was proved above that for a polynomial $Q(x)$ of degree 6 there exists the only pair $(Q(x), f(x)) = \big((1 + x^2)^3, x(1 + x^2)\big)$. Pick out a polynomial $G(x)$ such function (13) is a polynomial. We have $F(x) = (x^2 + 2x^4 + x^6 + G^2(x))/(1 + x^2)^3$. The function $F(x)$ is a polynomial if and only if the polynomials $P(x) = x^2+2x^4+x^6+G^2(x)$, $P'(x)$, $P''(x)$ are such that $P(i) = P'(i) = P''(i) = 0$. It is easy to prove that then $G(x) = (1 + x^2)(-x^2 + (1 + x^2)g(x))$ and $F(x) = x^2 - 2x^2g(x) + (1 + x^2)g^2(x)$ or $G(x) = (1 + x^2)(x^2 + (1 + x^2)g(x))$, $F(x) = x^2+2x^2g(x)+(1+x^2)g^2(x)$. Hamiltonians H_1, H_2 are nonglobal ones so far as their level curve $H_1 = 1$, $H_2 = 1$ have the points of intersection with an equator of a Poincaré sphere.

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About one irreducible component of the center variety of cubic system with two invariant straight lines and one invariant conic

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Abstract

The center conditions for the cubic system in the case of equations (24) in [1, p.188] are presented.

Keywords: center-focus problem, center variety, cubic differential system, invariant algebraic curve.

1 Introduction

In [1] necessary and sufficient conditions for the cubic system with two invariant straight lines and one invariant conic are obtained. One of the most difficult cases is described by the system of equations (24) [1, p.188]. The coefficients of this system are expressed through coefficients of invariant conic. In this paper we give one irreducible component of the center variety for the cubic system in the case of equations (24).

2 One center case for the cubic system

We will consider system

$$\begin{aligned} \dot{x} &= y + Ax^2 + Cxy + Fy^2 + Kx^3 + Mx^2y + Pxy^2 + Ry^3, \\ \dot{y} &= -x - Gx^2 - Dxy - By^2 - Sx^3 - Qx^2y - Nxy^2 - Ly^3, \end{aligned} \tag{1}$$

where A, B, C, D, F, G, K, L, M, N, P, Q, R, $S \in \mathbb{C}$.

Theorem 1. *Let V be the center variety of the system (1). Then $\mathbb{V}(J) \subset V$, where*

$$J = \langle -35L^9 - L^8B(75A - 131F) - 2B^9F^5(A + F)(16F^3+ \\ +2(B^2 + 12F^2)A + 9A^2F + 3B^2F) + B^3L^6(375A^3 - 702F^3+ \\ +61B^2A - 1286F^2A + 415A^2F + 189B^2F) - 5B^2L^7(7B^2- \\ -35A^2 - 22F^2 - 156AF) + B^6F^2L^3(66A^2B^2 - 5F^2(671A^2- \\ -35B^2) - 866F^4 - 3002F^3A - 30(39A^2 - 8B^2)AF) + B^7F^3L^2\times \\ \times(7F^2(5B^2 + 18F^2) + A^2(38B^2 + 517F^2) + 193A^3F + 73(B^2+ \\ +6F^2)AF) - B^5L^4F(2A^2(5B^2 - 2703F^2) + 2B^2F^2 - 861F^4- \\ -2010A^3F - A(4075F^3 - 76B^2F)) - B^4L^5(286B^2F^2 + A^2(30B^2+ \\ +3322F^2) - 295F^4 + 1450A^3F + 2A(567F^3 + 80B^2F)) + 4B^3L^2C\times \\ \times(BF - L)^5 - 2B^8F^4(A^2(6B^2 - 115F^2) + 11B^2F^2 - 56F^4- \\ -30A^3F + 2A(8B^2F - 71F^3))L,\ 7L^8 + L^7B(11A - 37F) + B^3L^5\times \\ \times(25A^3 + 3(B^2 + 64F^2)A + 55A^2F + B^2F) - 2B^8F^4(A + F)\times \\ \times(16F^3 + 2(B^2 + 12F^2)A + 9A^2F + 3B^2F) - B^2L^6(B^2 - 25A^2- \\ -82F^2 + 18AF) - B^6F^2L^2(18A^2B^2 + (85A^2 + 27B^2)F^2 + 86F^4+ \\ +180F^3A + 2(A^2 + 22B^2)AF) + B^4L^4(10B^2F^2 - 2A^2(B^2 + F^2)- \\ -99F^4 + 20A^3F + 6(B^2 - 39F^2)AF) - B^5L^3F(2A^2(5B^2 + 62F^2)- \\ -69F^4 + 76A^3F + A(-65F^3 + 14B^2F)) + 4L^2BD(BF - L)^5- \\ -B^7F^3(A^2(14B^2 - 197F^2) + 25B^2F^2 - 96F^4 - 51A^3F- \\ -A(244F^3 - 37B^2F))L,\ -15L^7 - L^6B(35A - 27F) + B^7F^3\times \\ \times(A + F)(16F^3 + 2(B^2 + 12F^2)A + 9A^2F + 3B^2F) + B^3L^4\times \\ \times(175A^3 - 160F^3 + 29B^2A - 48F^2A + 355A^2F + 45B^2F)- \\ -B^2L^5(11B^2 - 75A^2 - 124F^2 - 296AF) - B^4L^3(14A^2B^2 + 12(95A^2+ \\ +B^2)F^2 + 165F^4 + 878F^3A + 410A^3F + 32B^2AF) - B^5L^2F(2A^2\times \\ \times(13B^2 - 510F^2) + 72B^2F^2 - 309F^4 - 328A^3F - A(1003F^3- \\ -82B^2F)) - B^6F^2(10A^2B^2 + (343A^2 + B^2)F^2 + 136F^4 + 378F^3A+ \\ +2(51A^2 + 8B^2)AF)L + 4B^3GL(BF - L)^4,\ -13L^8 - 5L^7B(5A- \\ -7F) + B^8F^4(A + F)(9A^2F + 2(B^2 + 12F^2)A + 16F^3 + 3B^2F)+ \\ +B^3L^5(125A^3 - 130F^3 + 23B^2A - 173F^2A + 191A^2F + 41B^2F)- \\ -B^2L^6(-65A^2 + 9B^2 - 61F^2 - 238AF) - B^4L^4(10A^2B^2+ \\ +31(23A^2 + B^2)F^2 + 33F^4 + 385F^3A + 300A^3F + 34B^2AF)- \\ -B^6F^2L^2(7B^2F^2 + A^2(4B^2 + 41F^2) - 17F^4 + 29A^3F+ \\ +11A(B - F)F(B + F)) - B^5L^3F(2A^2(9B^2 - 290F^2)+$$

$$+32B^2F^2-103F^4-225A^3F-A(436F^3-47B^2F))+B^7F^3\times$$
$$\times(A^2(6B^2-115F^2)+11B^2F^2-56F^4-30A^3F-2A(71F^3-$$
$$-8B^2F))L+4B^4KL(BF-L)^4,-11L^7-7L^6B(A-9F)+$$
$$+B^4L^3(2A^2(B^2+6F^2)-4B^2F^2-145F^4+18F^3A+30A^3F)-$$
$$-B^3L^4(25A^3-208F^3+3B^2A+16F^2A+5A^2F+3B^2F)+$$
$$+B^7F^3(A+F)(16F^3+2(B^2+12F^2)A+9A^2F+3B^2F)+$$
$$+B^2L^5(B^2-25A^2-168F^2-12AF)+B^5L^2F(6A^2B^2+$$
$$+4(25A^2+2B^2)F^2+97F^4+111F^3A+2(8A^2+7B^2)AF)+$$
$$+B^6F^2(A^2(6B^2-115F^2)+11B^2F^2-60F^4-30A^3F-2A\times$$
$$\times(67F^3-8B^2F))L-4B^2M(BF-L)^5,-13L^8-L^7B(29A-$$
$$-48F)+3B^8F^4(A+F)(16F^3+2(B^2+12F^2)A+9A^2F+$$
$$+3B^2F)+B^3L^5(25A^3-172F^3+3B^2A-500F^2A-70A^2F+$$
$$+6B^2F)-B^5F^2L^3(268F^3-74A^3+14B^2A+477F^2A+64A^2F+$$
$$+20B^2F)-B^2L^6(B^2-25A^2+15F^2-195AF)+B^6F^2L^2\times$$
$$\times(12A^2B^2+F^2(415A^2+13B^2)+267F^4+623F^3A+26(3A^2+$$
$$+B^2)AF)-B^4L^4(5B^2F^2+A^2(2B^2+27F^2)-289F^4+105A^3F-$$
$$-A(534F^3-9B^2F))+B^7F^3(2A^2(8B^2-189F^2)+30B^2F^2-$$
$$-184F^4-99A^3F-A(466F^3-43B^2F))L-4B^2(BF-L)^5LN,$$
$$-75L^8-L^7B(155A-174F)+3B^8F^4(A+F)(16F^3+2(B^2+$$
$$+12F^2)A+9A^2F+3B^2F)+B^3L^5(775A^3-748F^3+125B^2A-$$
$$-660F^2A+1360A^2F+256B^2F)-B^2L^6(-375A^2+63B^2-457F^2-$$
$$-1403AF)-B^4L^4(141B^2F^2+A^2(62B^2+4611F^2)+441F^4+$$
$$+1805A^3F+137(B^2+22F^2)AF)-B^6F^2L^2(29B^2F^2+A^2(40B^2+$$
$$+727F^2)+181F^4+270A^3F+A(621F^3+70B^2F))-B^5L^3F(4A^2\times$$
$$\times(29B^2-954F^2)+260B^2F^2-918F^4-1354A^3F-A(3313F^3-$$
$$-370B^2F))+B^7F^3(4A^2(5B^2-78F^2)+4(9B^2-38F^2)F^2-$$
$$-81A^3F-A(386F^3-53B^2F))L-4B^4(BF-L)^4LP,-38L^7-$$
$$-L^6B(90A-67F)+3B^7F^3(A+F)(16F^3+2(B^2+12F^2)A+$$
$$+9A^2F+3B^2F)+B^3L^4(450A^3-414F^3+70B^2A-113F^2A+$$
$$+921A^2F+103B^2F)-B^4L^3(36A^2B^2+8(372A^2+B^2)F^2+436F^4+$$
$$+2307F^3A+71(15A^2+B^2)AF)-B^2L^5(26B^2-(2A+7F)\times$$
$$\times(95A+46F))-B^5L^2F(A^2(66B^2-2708F^2)+204B^2F^2-827F^4-$$
$$-867A^3F-A(2675F^3-221B^2F))+B^6F^2(6B^2F^2-24A^2B^2-$$
$$-942A^2F^2-376F^4-1042F^3A-279A^3F-33B^2AF)L+4B^4\times$$
$$\times(BF-L)^4Q,\ L(BF-L)-B^2R,\ 6L^7+L^6B(14A-11F)+$$

$$+B^7F^3(A+F)(16F^3+2(B^2+12F^2)A+9A^2F+3B^2F)-B^3L^4\times$$
$$\times(50A^3-50F^3+6B^2A-45F^2A+89A^2F+7B^2F)+B^2L^5(2B^2-$$
$$-26A^2-38F^2-105AF)+B^4L^3(4A^2(B^2+51F^2)-4B^2F^2+8F^4+$$
$$+85A^3F+3(B^2+37F^2)AF)+B^5L^2F(10A^2(B^2-4F^2)+16B^2F^2+$$
$$+9F^4-23A^3F-A(3F^3-25B^2F))+B^6F^2(A^2(8B^2-82F^2)+$$
$$+14B^2F^2-40F^4-21A^3F-3A(34F^3-7B^2F))L+4B^2(BF-$$
$$-L)^4LS,\ 5L^6BF-L^7+B^7F^2(A+F)^216F^3+2(B^2+12F^2)A+$$
$$+9A^2F+3B^2F)+B^3L^4(4(B^2-20F^2)A-34F^3-36A^2F+7B^2F)-$$
$$-B^4L^3(5A^2(5A^2+B^2)+7(19A^2+2B^2)F^2-19F^4+24F^3A+$$
$$+2(55A^2+8B^2)AF)+B^5L^2(2A^3(B^2+123F^2)+2B^2F^3+49F^5+$$
$$+4F^2(B^2+60F^2)A+55A^4F+A^2(383F^3+5B^2F))-B^2L^5(B^2-$$
$$-2(5A^2+F^2+11AF))+B^6F(A+F)(A^2(4B^2-139F^2)+11B^2F^2-$$
$$-56F^4-39A^3F-A(158F^3-13B^2F))L,\ 1-B(BF-L)Lt\rangle\bigcap\mathbb{C}[p],$$
$$p=(A,B,C,D,F,G,K,L,M,N,P,Q,R,S).$$

The variety $\mathbb{V}(J)$ is an irreducible component of V. In this case the system (1) has invariant straight lines and one invariant conic

$$1+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2=0,$$

where $a_{02}\neq 0$.

3 Conclusion

The obtained result says us about extreme difficulty of the center-focus problem for the cubic system (1).

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Cozma's centers with invariant straight lines and conics

Anton P. Sadovskii, Tatsiana V. Shcheglova

Abstract

We consider the center-focus problem for cubic systems with two invariant straight lines and one invariant conic. In [1] it is proved that a weak focus is a center for such systems if and only if the first four Liapunov quantities vanish. Seven classes of cubic systems with the center at the origin with two invariant straight lines and one invariant conic which is linear by variable y are presented.

Keywords: center-focus problem, center variety, cubic differential system, integrating factor, invariant algebraic curve.

1 Introduction

In 2013 in the monograph of D. Cozma [1] the solution of the center-focus problem for systems of the differential equations with homogeneous nonlinearities of the second and the third degree in the case when this system has either three invariant straight lines or a bundle of two invariant straight lines and one invariant conic was presented.

Especially difficult there was a problem of definition of the center necessary and sufficient conditions for cubic systems with a bundle of two invariant straight lines and one invariant conic. The uncommon ingenuity and erudition was necessary to find the solution of this problem. In many cases the author needed to represent new parametrization which, eventually, allowed to solve the center-focus problem.

In this paper we will specify systems with the center at the origin for some Cozma's cases. These will be systems with invariant conics which are linear relatively variable y. We will widely use the theory of polynomial ideals and varieties which allows to present the received results in simple way.

Before passing to the main result we will mark out the paper of H. Żołądek [2,3], where reversible systems with the center at the origin are presented, which appear in this paper and the paper [4], where we present the solution of the center-focus problem for the cubic system with nine parameters with two parallel straight lines.

2 Cubic systems with the center at the origin

We will consider the system

$$\begin{aligned} \dot{x} &= y + Ax^2 + Cxy + Fy^2 + Kx^3 + Mx^2y + Pxy^2 + Ry^3, \\ \dot{y} &= -x - Gx^2 - Dxy - By^2 - Sx^3 - Qx^2y - Nxy^2 - Ly^3, \end{aligned} \tag{1}$$

where A, B, C, D, F, G, K, L, M, N, P, Q, R, $S \in \mathbb{C}$.

Let us denote vector $p = (A, B, C, D, F, G, K, L, M, N, P, Q, R, S)$ and the following ideals

$$\begin{gathered} J_1 = \langle (B-2C)(B+C)(7B+4C) - 2F^2(31B+32C) - 2A(13B+ \\ +16C)F + 20(B+C)DF, (B-2C)^2(B+C) + 4F^2(B+13C)- \\ -18A(B-2C)F - 20F^2G, (B-2C)^2(B+C) + 2F^2(7B+16C)- \\ -8A(B-2C)F + 20FK, BF - L, (B-2C)(B+C)(3B+2C)- \\ -4F^2(7B+8C) - 4A(B+4C)F - 20(B+C)M, (B-2C)(B+C)\times \\ \times(4B+C) - 2F^2(12B+13C) - 6A(2B+3C)F + 10(B+C)N, \\ BF + 2P, (B-2C)^2(B+C)(3B+2C) + 2F^2(B^2+52C^2+52BC)- \\ -2A(B-2C)(17B+18C)F - 40(B+C)FQ, R, (B-2C)^3(B+C)^2+ \\ +8C^3(A-2F)F - 16F^3B(6A+7F) - 2B^2CF(3A+19F)+ \\ +2B^3F(A+23F) - 4F^2C(34F^2+25BC+42AF) - 360F^2(B+C)S, \\ 36A^2F^2 + 4F^4 - (B-2C)^2(B+C)^2 - 4F^2(B+C)(7B+11C)+ \\ +A(24F^3+16(B-2C)(B+C)F), 1-(B+C)Ft\rangle \bigcap \mathbb{C}[p], \end{gathered}$$

$$\begin{gathered} J_2 = \langle F(A+F) - M - N, C(A+F) - FG - K, FG - Q, \\ 2F^2(A+F) - (2A - D + 4F)N, F^2G - CN, (D-2F)(DF- \\ -F^2 - N)N + 2F(F^2-N)S, B, L, P, R, 1 - F(F^2-N)t\rangle \bigcap \mathbb{C}[p], \end{gathered}$$

$$J_3 = \langle B(B-2C)^2 - 24F^2B + 4(B-2C)DF, B(B-2C)- \\ -4F(3A-D+3F), 3B+2G, BF-2K, BF-L, B(B-2C)-4M, \\ B(B-2C)-2F(A+F)+2N, BF+2P, B(5A-D+F)-2Q, \\ B^2+2(A+F)(2A+F)-2S, R, 1-(B-2C)Ft\rangle \bigcap \mathbb{C}[p],$$

$$J_4 = \langle 4B^2+9F(A+F), 3F^2-BC, 2B^2+3(D-2F)F, 2B+G, \\ 8B^3+9F(2BF-3K), BF-L, 5B^2+9(2F^2+M), B^2-3(4F^2- \\ -N),\ R,\ BF+3P, 2B^3+9F(4BF+Q), 4B^4+9F^2(4B^2-3S), \\ 1-BFt\rangle \bigcap \mathbb{C}[p],$$

$$J_5 = \langle 41B^2-3F^2+40BC+(24C-13G)G, 7B^2-5F^2-3G^2- \\ -8AF,\ 47B^3+B(20DF-41F^2-11G^2)+20B^2G+4G(3DF- \\ -GF^2-G^2),\ A(B-2C)-F(2C-G)+K, BF-L, 2B(4B-3C)+ \\ +7F(5A+2F)-9M,\ 4B(B-3C)-(5A-11F)F+9N, \\ F(2B+G)+P,\ 2AC+F(C+G)-Q, R, 2B(2B+3C)- \\ -9A(A-D)-F(41A+11F)-9S,\ 11B^2-F^2+ \\ +G^2+8BG,\ 1-F(5B+3G)t\rangle \bigcap \mathbb{C}[p],$$

$$J_6 = \langle D-3(A+F), 2B^2A-(F^2-BC)F, 2B+G, \\ 2(AC+(B+C)F)-3K, BF-L, AF+M, 2F^2-B(2B+C)-N, \\ AB+P, AC-(2B-C)F-Q, R, F^2+2B(2B+C)+3A(A+ \\ +2F)-S, B^2(2B+C)^2+F^2(3F^2-4BC), 1-Bt\rangle \bigcap \mathbb{C}[p],$$

$$J_7 = \langle C(A+F)+4B(2A+F), 2D-3(3A-F), \\ 2B+G, 6AB+K, BF-L, 3B^2-M, 3F(A+F)+4N, \\ 3BF+P, 3B(7A-F)+2Q, R, 7(3A-F)(A+F)-4S, \\ 7F(A+F)^2+4B^2(9A+F), 1-(A+F)t\rangle \bigcap \mathbb{C}[p].$$

Theorem 1. *Let V be the center variety of the system (1). Then*

$$\bigcup_{k=1}^{7} \mathbb{V}(J_k) \subset V.$$

3 Conclusion

In this paper we give seven irreducible components $\mathbb{V}(J_k)$ of the center variety of system (1). All these systems have invariant straight lines

$$1 + s_i x - s_3 y = 0,\ i = \overline{1,2},$$

where s_i, $i = \overline{1,3}$, are arbitrary complex parameters, and one invariant conic

$$1 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy = 0.$$

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Existence and Uniqueness of Positive Solutions for a Boundary Value Problem with a Laplacian Operator

Erdogan Sen, Kamil Orucoglu

Abstract

In this work we study the existence and uniqueness of positive and nondecreasing solutions for a fractional two-point boundary value problem with p-Laplacian operator.

Keywords: Positive solution, fixed point theorem, p-Laplacian operator, fractional q-difference equation.

In this work, we study a q-difference boundary-value problem with p-Laplacian operator

$$D_q^\gamma(\phi_p(D_q^\delta y(t))) + f(t, y(t)) = 0, \quad 0 < t < 1, \ n-1 < \delta < n, \tag{1}$$

$$\begin{cases} (D_q^{(i)} y)(0) = 0, \quad i = 0, 1, ..., n-2 \\ (D_q^{n-2} y)(1) = (D_q y)(1), \quad D_{0+}^\gamma y(t)|_{t=0} = 0, \quad 0 < \gamma < 1, \end{cases} \tag{2}$$

where $0 < q < 1$, D_{0+}^α is the Riemann-Liouville fractional derivative, $\phi_p(s) = |s|^{p-2} s$, $p > 1$. We prove the existence and uniqueness of a positive and nondecreasing solution for the boundary value problem (1)-(2) by using a fixed point theorem in partially ordered sets.

The basic space used in this paper is $E = C[0, 1]$. Then E is a real Banach space with the norm $\|u\| = \max_{0 \le t \le 1} |u(t)|$. Note that this space can be equipped with a partial order given by

$$x, y \in C[0, 1], \quad x \le y \Leftrightarrow x(t) \le y(t), \quad \forall t \in [0, 1].$$

Main result of our paper is the following:

Theorem 1. *The boundary-value problem (1)-(2) has a unique positive and increasing solution* $u(t)$ *if the following two conditions are satisfied:*

i.) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and nondecreasing with respect to the second variable*;

ii.) *There exists* $0 < \lambda + 1 < M$ *such that for* $u, v \in [0,+\infty)$ *with* $u \geq v$ *and* $t \in [0,1]$

$$\phi_p(\ln(v+2)) \leq f(t,v) \leq f(t,u) \leq \phi_p(\ln(u+2)(u-v+1)^{\lambda}).$$

Moreover

iii.) *If* $f(t,0) > 0$ *for all* $t \in [0,1]$, *then the solution* $u(t)$ *of boundary value problem (1)-(2) is strictly increasing.*

Example 1. *We can show that the fractional boundary-value problem*

$$D^{6}_{1/3}u(t) + (\frac{8}{3}t^{1/7} + \pi)\ln(2 + u(t)) = 0, \quad 0 < t < 1, \tag{3}$$

$$\left(D^{(i)}_{1/3}u\right)(0) = 0, ...i = \overline{0,4}, \left(D^{4}_{1/3}u\right)(1) = \left(D_{1/3}u\right)(1) \tag{4}$$

has a unique and strictly increasing solution. In this problem, $p = 2$, $q = 1/3$, $f(t,u(t)) = (\frac{8}{3}t^{1/7} + \pi)\ln(2 + u(t))$ and let $\delta = 11/2$ and $\gamma = 1/2$ for $(t,u) \in [0,1] \times [0,\infty)$. Moreover $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and nondecreasing with respect to the second variable u since $f_u = (\frac{8}{3}t^{1/7} + \pi)\frac{1}{u+2} > 0$. Also $f(t,u) - f(t,v) = (\frac{8}{3}t^{1/7} + \pi)\ln\left(1 + \frac{u-v}{2+v}\right) \leq \frac{8+3\pi}{3}\ln(1 + u - v)$ and we can choose $\lambda = \frac{8+3\pi}{3} \approx 5.8082593$. Thus the Theorem 1 says that the boundary-value problem (3)-(4) has a unique strictly increasing solution $u(t)$.

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Algebraic limit cycles in polynomial differential systems with a weak focus

Alexandru Şubă, Dumitru Cozma

Abstract

In this paper we show that the algebraic limit cycles of a polynomial differential system of degree n, $n \geq 3$ with a weak focus of order $[(n-1)/2]$ lie on at most $(n^2-3)/2$ irreducible algebraic invariant curves if n is odd and on $(n^2-2)/2$ ones if n is even. In particular, the limit cycles of a cubic system with a weak focus of order one lie on at most three algebraic invariant curves.

Keywords: polynomial differential system, invariant algebraic curve, algebraic limit cycle.

We consider the real polynomial system of differential equations

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad GCD(P,Q) = 1 \tag{1}$$

and the vector field $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to system (1). Denote $n = \max\{\deg(P), \deg(Q)\}$ and suppose that $n \geq 3$. If $n = 3$, then the system (1) is called cubic.

A singular point (x_0, y_0) of a system (1) is a *weak focus* if the eigenvalues of the linearization at (x_0, y_0) are pure imaginary. Without loss of generality we may suppose that the singular point (x_0, y_0) is placed at the origin. In this case via rotation of axes and time rescaling, the system (1) becomes

$$\begin{aligned} \dot{x} &= y + P_2(x,y) + \cdots + P_n(x,y), \\ \dot{y} &= -x + Q_2(x,y) + \cdots + Q_n(x,y), \end{aligned} \tag{2}$$

where P_j, Q_j are homogeneous polynomials in x, y of degree j. It is known there is a function $F(x, y)$ defined in a neighborhood of $(0, 0)$ such that its rate of change along trajectories of (2) is of the form

$$\mathbb{X}(F) = \sum_{j=1}^{\infty} L_j \cdot (x^2 + y^2)^{j+1},$$

where L_j, $j = \overline{1, \infty}$ are polynomials in the coefficients of (2). The order of the weak focus is r if $L_1 = \cdots = L_{r-1} = 0$ but $L_r \neq 0$.

An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ (a function $f = \exp(g/h)$, $g, h \in \mathbb{C}[x, y]$) is called *invariant algebraic curve* (*exponential factor*) of the vector field $\mathbb{X}$ if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $\deg(K) \leq n - 1$ such that the identity $\mathbb{X}(f) \equiv f(x, y)K_f(x, y)$, $(x, y) \in \mathbb{R}^2$ holds.

A *limit cycle* of a polynomial vector field $\mathbb{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathbb{X}$. An *algebraic limit cycle of degree* d of $\mathbb{X}$ is an oval of a real irreducible invariant algebraic curve $f = 0$ of degree d which is a limit cycle of $\mathbb{X}$.

In 1900 D. Hilbert published a list of problems for being solved during the 20^{th} century. From this list two problems remain open. One is the 16th Hilbert problem on the limit cycles of the polynomial differential equations. This problem remains to be one of the most difficult task and J. Llibre [3, 4] proposed to study it firstly in the cases of algebraic limit cycles.

In this paper we will show that the presence of a singular point of the weak focus type in a polynomial differential system imposes restrictions on the number of degrees of the algebraic limit cycles. For this we bring some results from [1, 2, 5].

Theorem 1. *Let the system* (1) *to have: 1) a singular point* (x_0, y_0) *with pure imaginary eigenvalues; 2)* $n(n+1)/2 - [(n+1)/2]$ *irreducible invariant algebraic curves that do not contain* (x_0, y_0) *and 3) the first* $[(n-1)/2]$ *Lyapunov quantities at* (x_0, y_0) *vanish, then* (x_0, y_0) *is of a center type.*

The following two Theorems are the first corollary of Theorem 1:

Theorem 2. *If the system* (1) *has a weak focus of order* $[(n-1)/2]$, *then it can have algebraic limit cycles of at most* $(n^2-3)/2$ *different degrees if* n *is odd, and* $(n^2-2)/2$ *ones if* n *is even. In particular, a cubic system with a weak focus can have algebraic limit cycles of at most three different degrees.*

Theorem 3. *Any cubic system* $[(1),\ n=3]$ *with a weak focus* (x_0, y_0) *of order at least one and three invariant straight lines* l_1, l_2, l_3, $(x_0, y_0) \notin l_1 \cup l_2 \cup l_3$ *does not have algebraic limit cycles.*

In [4] two examples of cubic system with two algebraic cycles of order four there are given:

$$\dot{x} = 2y(10+xy), \quad \dot{y} = 20x + y - 20x^3 - 2x^2y + 4y^3; \tag{3}$$

$$\begin{aligned} \dot{x} &= y(a^2 - r^2 - 3ax + 3x^2 - ay + 2xy + y^2), \\ \dot{y} &= -a^2x + 3ax^2 - 2x^3 - r^2y + axy - x^2y + y^3, \end{aligned} \tag{4}$$

with $0 < r < a/2$. The cycles of (3) have degree four and are contained on the invariant curve $2x^4 - 4x^2 + 4y^2 + 1 = 0$. The cycles of (4) are $x^2 + y^2 = r^2$ and $(x-a)^2 + y^2 = r^2$. The systems (3) and (4) do not have invariant straight lines and singular points of weak focus type.

Theorem 4. *Any cubic system* $[(1),\ n=3]$ *with a weak focus of order at least four and two invariant straight lines does not have algebraic limit cycles.*

The cubic system

$$\begin{aligned} \dot{x} &= (2y + 4x^2 + 2xy + 6x^3 + 25x^2y - 6xy^2 - 23y^3)/2, \\ \dot{y} &= -x(2 + 2x - 4y + x^2 - 12xy - 47y^2)/2 \end{aligned}$$

has a weak focus of degree one at $(0,0)$, two complex invariant straight lines $x \pm iy = 0$ and an algebraic limit cycle of degree two $3x^2 + 3y^2 - 12y + 2 = 0$.

N.A. Lukashevich (1965) proved that a quadratic system ($n = 2$) with a center has no limit cycles. N.P. Erughin (1970) raised the problem of coexisting centers and limit cycles in polynomial differential systems. M.V. Dolov (1972) gave an example of cubic system with a center and a non-algebraic limit cycle.

The following cubic system

$$\begin{aligned}\dot{x} &= y - xy - 2y^2 + 2x^3 + 3x^2y - 6xy^2 - 3y^3,\\ \dot{y} &= -x + x^2 + 2xy + 6x^2y + 6xy^2 - 2y^3,\end{aligned}$$

has a center at $(0,0)$, two complex invariant straight lines $x \pm iy = 0$, and the algebraic limit cycles of degree two: $1 - 2x + x^2 - 4y + y^2 = 0$ and $1 + x + x^2 + 2y + y^2 = 0$.

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Cubic differential systems with a straight line of maximal multiplicity

Alexandru Şubă, Olga Vacaraş

Abstract

In this paper we show that in the class of cubic systems of differential equations the maximal multiplicity of an affine invariant straight line and the maximal multiplicity of the line at infinity are equal to 7.

Keywords: cubic differential system, invariant straight line, multiplicity of invariant algebraic curve.

1 Definitions

We consider the real polynomial system of differential equations

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \qquad GCD(P,Q) = 1 \tag{1}$$

and the vector field $\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ associated to the system (1). Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 3$, then the system (1) is called cubic.

Definition 1. *An algebraic curve $f(x,y) = 0$, $f \in \mathbb{C}[x,y]$ (a function $f = \exp(g/h)$, $g, h \in \mathbb{C}[x,y]$) is called invariant algebraic curve (exponential factor) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$, $\deg(K) \leq n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y)$, $(x,y) \in \mathbb{R}^2$ holds.*

In the work [1] there are introduced the following definitions of the multiplicty of an algebraic curve:

Definition 2. *An invariant algebraic curve f of degree d for the vector field $\mathbb{X}$ has algebraic multiplicity k when k is the greatest positive integer such that the k-th power of f divides $\mathrm{E}_d(\mathbb{X})$, where*

$$E_d(\mathbb{X}) = det \begin{pmatrix} \upsilon_1 & \upsilon_2 & ... & \upsilon_l \\ \mathbb{X}(\upsilon_1) & \mathbb{X}(\upsilon_2) & ... & \mathbb{X}(\upsilon_l) \\ ... & ... & ... & ... \\ \mathbb{X}^{l-1}(\upsilon_1) & \mathbb{X}^{l-1}(\upsilon_2) & ... & \mathbb{X}^{l-1}(\upsilon_l) \end{pmatrix},$$

and $\upsilon_1, \upsilon_2, ..., \upsilon_l$ *is a basis of* $\mathbb{C}_d[x, y]$. If $d = 1$, then $\upsilon_1 = 1$, $\upsilon_2 = x$, $\upsilon_3 = y$ and $E_1(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$.

Definition 3. *Let $f = 0$ be an invariant algebraic curve of degree d of (1). We say that $F = f_0 + f_1\epsilon + \cdots + f_{k-1}\epsilon^{k-1} \in \mathbb{C}[x, y, \epsilon]/(\epsilon^k)$ defines a nondegenerate generalized invariant algebraic curve of order k based on $f = 0$ if $f_0 = f, ..., f_{k-1}$ are polynomials in $\mathbb{C}[x, y]$ of degree at most d, f_1 is not a multiple of f, and F satisfies the equation $\mathbb{X}(F) = FL_F$, for some polynomial $L_F = L_0 + L_1\epsilon + \cdots + L_{k-1}\epsilon^{k-1} \in \mathbb{C}[x, y, \epsilon]/(\epsilon^k)$, which must necessarily be of degree at most $n - 1$ in x and y.*

Definition 4. *Let $f = 0$ be an invariant algebraic curve of degree d of system (1). We say that $f = 0$ is of infinitesimal multiplicity m with respect to $\mathbb{X}$ if m is the maximal order of all nondegenerate generalized invariant algebraic curves of $\mathbb{X}$ based on f.*

Definition 5. *We shall say that the invariant algebraic curve $f = 0$ has integrable multiplicity m with respect to $\mathbb{X}$ if m is the largest integer for which the following is true: there are $m - 1$ exponential factors $exp(g_j/f^j), j = 1, ..., m - 1$, with $deg\ g_j \leq j\ deg\ f$, such that each g_j is not a multiple of f.*

Definition 6. *An invariant algebraic curve $f = 0$ of degree d of the vector field $\mathbb{X}$ has geometric multiplicity m if m is the largest integer for which there exists a sequence of vector fields $(\mathbb{X}_i)_{i>0}$ of bounded degree, converging to $h\mathbb{X}$, for some polynomial h, not divisible by f, such that each $\mathbb{X}_r$ has m distinct invariant algebraic curves, $f_{r,1} = 0, ..., f_{r,m} = 0$, of degree at most n, which converge to $f = 0$ as r goes to infinity.*

2 Maximal multiplicity of an affine invariant straight line of cubic systems

Theorem 1. *For cubic systems the algebraic (infinitesimal, integrable, geometric) multiplicity of an affine invariant straight line is at most* *seven. Any cubic system having an affine invariant straight line of the algebraic (infinitesimal, integrable, geometric) multiplicity seven via affine transformations and time rescaling can be brought to the form*

$$\dot{x} = x^3, \quad \dot{y} = 1 + 3x^2 y. \tag{2}$$

Proof. Algebraic multiplicity. For algebraic multiplicity the statement of Theorem 1 was established by direct calculation. The system (2) only has the invariant straight line $x = 0$ and for this line $E_1(\mathbb{X}) = 6x^7 y$, i.e. $k = 7$.

Infinitesimal multiplicity. For the system (2) we have $F = x + (1 + x)\epsilon(1 + \epsilon) + (1 + x + y)\epsilon^3(1 + \epsilon + \epsilon^2 + \epsilon^3)$ and $L_F = x^2 - x\epsilon + \epsilon^2 + 2xy\epsilon^3 - y\epsilon^4 - 2y^2\epsilon^6$.

Integrable multiplicity. For (2) and the invariant straight line $f \equiv x = 0$ we have: $g_1 = g_2 = 1$, $g_3 = 1 + 3x^2 y$, $g_4 = 1 + 4x^2 y$, $g_5 = 1 + 5x^2 y$, $g_6 = 1 + 6x^2 y + 3x^4 y^2$.

Geometric multiplicity. The following cubic system

$$\begin{aligned} \dot{x} = {} & x(x - 3\epsilon)(x - 3\epsilon + 6\epsilon^3), \\ \dot{y} = {} & 1 + 3x^2 y - 3\epsilon^2 - 6\epsilon^4 + 8\epsilon^6 + 9\epsilon^2(1 - 2\epsilon^2)y \\ & -12\epsilon(1 - \epsilon^2)xy + 24\epsilon^4(1 - \epsilon^2)y^2 - 12\epsilon^3 xy^2 + 16\epsilon^6 y^3 \end{aligned} \tag{3}$$

has seven distinct invariant straight lines: $l_1 = x$, $l_2 = x - 3\epsilon$, $l_3 = x - 3\epsilon + 6\epsilon^3$, $l_4 = x - \epsilon - 2\epsilon^3 - 4\epsilon^3 y$, $l_5 = x - \epsilon + 4\epsilon^3 - 4\epsilon^3 y$, $l_6 = x - 4\epsilon + 4\epsilon^3 - 4\epsilon^3 y$, $l_7 = x - 2\epsilon + 2\epsilon^3 - 2\epsilon^3 y$.

If $\epsilon \to 0$, then the system (3) tends to the system (2) and the invariant straight lines l_i, $i = 1, ..., 7$ of the system (3) converge to the invariant straight line $l :\ x = 0$ of the system (2). $\square$

3 Maximal multiplicity of the line at infinity

Theorem 2. *For cubic systems the algebraic multiplicity of the line at infinity is at most seven and any cubic system having the line at infinity of the algebraic multiplicity seven can be written as one of the following two forms:*

$$\dot{x} = 1, \ \dot{y} = x^3 + ax, \ a \in \mathbb{R}. \tag{4}$$

$$\dot{x} = -x, \;\; \dot{y} = x^3 + 2y. \tag{5}$$

The statement of Theorem 2 was established by direct calculation.

In the class of cubic systems, the geometric multiplicity of the line at infinity for the systems (4) and (5) is also 7. Indeed, the following cubic system

$$\begin{array}{l} \dot{x} = 1 - 3x\epsilon + ax\epsilon + x^3\epsilon - 4\epsilon^2 + 2x^2\epsilon^2 + 14x\epsilon^3 - 4ax\epsilon^3 - \\ 4x^3\epsilon^3 - 14x^2\epsilon^4 + 4x^3\epsilon^5, \\ \dot{y} = ax + x^3 - 3y\epsilon - 4ax\epsilon^2 - 4x^3\epsilon^2 + 4xy\epsilon^2 + 14y\epsilon^3 - \\ 2y^2\epsilon^3 - 28xy\epsilon^4 + 12x^2y\epsilon^5 + 14y^2\epsilon^5 - 12xy^2\epsilon^6 + 4y^3\epsilon^7 \end{array} \tag{6}$$

has six distinct invariant straight lines: $l_1 \cdot l_2 \cdot l_3 = 1 - 3x\epsilon + ax\epsilon + x^3\epsilon - 4\epsilon^2 + 2x^2\epsilon^2 + 14x\epsilon^3 - 4ax\epsilon^3 - 4x^3\epsilon^3 - 14x^2\epsilon^4 + 4x^3\epsilon^5, l_4 = 1 - \epsilon x + \epsilon^2 y$, $l_5 = 1 - 2\epsilon x + 2\epsilon^2 y$, $l_6 = -1 + 4\epsilon^2 - 2\epsilon^3 x + 2\epsilon^4 y$. If $\epsilon \to 0$, then (6) tends to (4) and l_i, $i = 1, ..., 7$ converge to the line at infinity.

The cubic system

$$\begin{array}{l} \dot{x} = x(-1 + 3x\epsilon)(1 - 3x\epsilon + 6x\epsilon^3), \\ \dot{y} = x^3 + 2y - 6xy\epsilon - 3x^3\epsilon^2 + 6xy\epsilon^3 - 12y^2\epsilon^3 - 6x^3\epsilon^4 + \\ 24xy^2\epsilon^4 + 8x^3\epsilon^6 - 24xy^2\epsilon^6 + 16y^3\epsilon^6 \end{array} \tag{7}$$

has seven distinct invariant straight lines: $l_1 = x$, $l_2 = -1 + 3x\epsilon, l_3 = 1 - 3x\epsilon + 6x\epsilon^3$, $l_4 = 1 - 4x\epsilon + 4x\epsilon^3 - 4y\epsilon^3$, $l_5 = 1 - x\epsilon + 4x\epsilon^3 - 4y\epsilon^3$, $l_6 = 1 - 2x\epsilon + 2x\epsilon^3 - 2y\epsilon^3$, $l_7 = -1 + x\epsilon + 2x\epsilon^3 + 4y\epsilon^3$. If $\epsilon \to 0$, then (7) tends to (5) and l_i, $i = 2, ..., 7$ converge to the line at infinity.

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The Painlevé equations in Hamiltonian form

Henryk Żołądek

The Painlevé equations $P_I - P_{VI}$ appeared in the beginning of the previous century in works of B. Gambier and P. Painlevé as a result of study of a class of second order differential equations with rational right-hand sides and satisfying certain rigidity property (the Painlevé property). These equations turned out not integrable in terms of known functions. Nowadays the solutions to these equations are called the Painlevé transcendents. The Painlevé equations have numerous applications in mathematics and mathematical physics.

Due to works of Japanese mathematicians (see [Oka]) the equations $P_I - P_{VI}$ can be written in the Hamiltonian form

$$\dot{x} = \frac{\partial h}{\partial y}, \quad \dot{y} = -\frac{\partial h}{\partial x},$$

where $h = h(x, y, t)$ is some (time dependent) Hamilton function; therefore they have 3/2 degrees of freedom. After renaming the 'time' t by a new 'coordinate' q, introducing a new 'momentum' p and extending the Hamilton function,

$$H(x, y, q, p) = h(x, y, q) + p,$$

one obtains an autonomous Hamiltonian system with two degrees of freedom.

We develop a rather new approach to the Hamiltonian property of the Painlevé equations. Firstly, one notes that the Painlevé equations take the generalized Liénard form

$$\ddot{x} = A(x,t)\dot{x}^2 + B(x,t)\dot{x} + C\,(x,t)\,,$$

with rational coefficients (with possible poles at $t=0$, $t=1$, $t=\infty$, $x=0$, $x=1$, $x=\infty$ and $x=t$). We postulate a momentum coordinate of the form $y=\dot{x}/D(x,y)$. It leads to the condition $D(x,t)=\exp\left(\int 2A\mathrm{d}x+B\mathrm{d}t\right)$.Thus D is a well defined function iff $2A_t'=B_x'$; it turns out that the latter condition holds in the case of Painlevé equations. The corresponding Hamilton function equals

$$h(x,y,t)=D(x,t)\frac{y^2}{2}+h_0(x,t),\quad h_0=-\int^x\frac{C}{D}\mathrm{d}x.$$

We have the following extended Hamilton functions H_J, each associated with the corresponding Painlevé equation P_J :

$$\begin{aligned}
H_I &= \frac{1}{2}y^2-2x^3-qx+p,\\
H_{II} &= \frac{1}{2}y^2-\frac{1}{2}x^4-\frac{1}{2}qx^2-\alpha x+p,\\
H_{III} &= \frac{x^2}{q}\cdot\frac{y^2}{2}-\alpha x+\frac{\beta}{x}-\frac{\gamma}{2}x^2q+\frac{\delta}{2}\frac{q}{x^2}+p,\\
H_{IV} &= x\cdot\frac{y^2}{2}-\frac{x^3}{2}-2qx^2-2(q^2-\alpha)x+\frac{\beta}{x}+p,\\
H_V &= \frac{x(x-1)^2}{q}\cdot\frac{y^2}{2}-\alpha\frac{x}{q}+\frac{\beta}{qx}+\frac{\gamma}{x-1}+\delta\frac{qx}{(x-1)^2}+p,\\
H_{VI} &= \frac{x(x-1)(x-q)}{q(q-1)}\cdot\frac{y^2}{2}\\
&\quad -\frac{1}{q(q-1)}\left\{\alpha x-\beta\frac{q}{x}-\gamma\frac{q-1}{x-1}-\delta\frac{q(q-1)}{x-q}\right\}+p.
\end{aligned}$$

The Okamoto Hamiltonians [Oka], denoted by $\tilde{H}_J$, take rather different forms, but they are related with the above Hamiltonians by means of symplectic transformations (with respect to the symplectic form $\mathrm{d}x\wedge\mathrm{d}y+\mathrm{d}q\wedge\mathrm{d}p$). For example, we have

$$\tilde{H}_{II}=y^2/2-\left(x^2+q/2\right)y-(\alpha+1/2)\,x+p$$

and the corresponding symplectic map equals

$$(x,y,q,p)\longmapsto\left(x,y-x^2-q/2,q,p-q^2/8-x/2\right).$$

We realize the Bäcklund transformations of the Painlevé equations as symplectic transformations. Such a transformation replaces a given Hamiltonian H_J to the same type Hamiltonian but with changed parameters. The induced groups of parameter changes are isomorphic to affine Weyl groups $W_a(R)$ associated with some root systems R. We have:

$$\begin{aligned} &W_a(A_1) \quad \text{for} \quad P_{II}, \\ &W_a(B_2) \quad \text{for} \quad P_{III}, \\ &W_a(A_2) \quad \text{for} \quad P_{IV}, \\ &W_a(A_3) \quad \text{for} \quad P_V, \\ &W_a(D_4) \quad \text{for} \quad P_{VI}. \end{aligned}$$

Also the cases with so-called classical solutions are interpreted as the partial integrability of the corresponding Hamiltonian system. This means existence of invariant surfaces

$$\Sigma_h = \{f = 0, \quad H = h\},$$

where $f(x, y, q, p)$ is a rational function on the phase space. These invariant surfaces are of quite special form, with $f = y - E(x, q)$ which, together with the relations $y = \dot{x}/D(x,t)$ and $q = t$, lead to Riccati equations of the form

$$\dot{x} = a(t)x^2 + b(t)x + c(t).$$

The later equations, for different Painlevé equations, are related with the classical second order equations. More precisely, we have:

Airy equation for P_{II},

Bessel equation for P_{III},

Hermite-Weber equation for P_{IV},

confluent hypergeometric equation for P_V,

Gausshypergeometric equation for P_{VI}.

Another aim of the work is to study the question of integrability of the 4−dimensional Hamiltonian systems in the Arnold–Liouville sense. It means that there should exist a first integral $F(x, y, q, p)$ for the vector field X_H, independent of H. If F is an algebraic function (respectively elementary function), then we say that the vector field X_H is algebraically integrable in the Liouville–Arnold sense (respectively elementarily integrable in the Liouville–Arnold sense). This problem was firstly considered by Ts. Stoyanova [Sto]. We prove the following results.

Theorem 1 *The Hamiltonian system associated with any of $P_I - P_{VI}$ excluding the cases: (a) $\alpha = \gamma = 0$ in P_{III}, (b) $\beta = \delta = 0$ in P_{III}, (c) $\gamma = \delta = 0$ in P_V, does not admit any first integral which is an algebraic function of x, y, q, p and is independent of H.*

Theorem 2 *Any of the equations $P_I - P_{VI}$, excluding the cases (a), (b) and (c) above, does not admit a first integral which is an elementary function of x, dx/dt and t.*

The above cases (a), (b) and (c) of Theorem 1 are well known (see [GLS], for example); the corresponding first integrals are given explicitly. We must underline that Theorem 1 is not new, only its proof is new. It was proved by V. Gromak; the cases of equations P_I and P_{II} are described in [GLS]. [1]

Stoyanova [Sto] applied a version of the Ziglin method, developed by J.-P. Ramis with J. Morales-Ruiz [M-R]. It uses the monodromy group (or the differential Galois group) of the normal variation equation for a particular algebraic solution of the corresponding Hamiltonian system. In the case of complete integrability with meromorphic first integrals the identity component of this differential Galois group should be abelian. In the case of the equation P_{VI} suitable algebraic solutions

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The Gromak's approach uses so-called second Malmquist theorem which states that if a solution $x = \varphi(t)$ to some of the P_J's satisfies an algebraic relation between t, φ and $\dot{\varphi}$, then this relation is of special type: $\dot{\varphi}^m + P_1(t, \varphi)\dot{\varphi}^{m-1} + \ldots + P_m(t, \varphi) \equiv 0$ with $P_j \in \mathbb{C}(t)[\varphi]$. Therefore we have a monic polynomial in $\dot{\varphi}$ with polynomial in φ coefficients.

Our approach is direct, without reference to other theorems.

exist for special values of the parameters. By direct computation of the monodromy group Stoyanova shows that the identity component of the differential Galois group of the normal variation equation is not abelian. Similar arguments, with use of some Stokes operators (rather than monodromy operators), are applied for the equation P_V. But we underline that only for special (but not discrete) values the parameters $\alpha, \beta, \gamma, \delta$ in P_V and P_{VI} invariant algebraic curve is given explicitly and the method works.

Our method of proof of the non-integrability is different. Its idea is described in [Zol], where the algebraic non-integrability of the 2 degrees of freedom Hamiltonian systems associated with P_I and P_{II} is proved, and in the preprint [ZoFi] (with Galina Filipuk). By a suitable normalization of the variables we arrive at a perturbation of a completely integrable system with two algebraic first integrals. Then we consider the equation in variations with respect to a parameter (denoted by ε) around a particular solution which is a rather general elliptic curve. Then analysis of few initial terms in powers of ε of a possible first integral of the perturbed system leads to some properties of elliptic integral which cannot be true.

In the proof of Theorem 2 we use the Liouville–Ritt [Rit] theory of elementary functions.

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Section 3

Mathematical Modelling and Numerical Analyses

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The New Method for Calculation of the Diffraction Integrals

Alla Albu, Vladimir Zubov

Abstract

An algorithm for calculating integrals of rapidly oscillating functions given on a smooth two-dimensional surface is proposed. The surface is approximated by a collection of flat triangles with the values of the integrand known at their vertices. These values are used as reference ones to extend the function to other points of a triangle. The integral of the extended function over the surface of a triangle is calculated exactly. The desired value of the full diffraction integral is determined as the sum of the integrals calculated over the surfaces of all triangles.

Keywords: diffraction integral, algorithm for computing two-dimensional integrals of rapidly oscillating functions.

1 Introduction

The Fresnel–Kirchhoff diffraction theory with related Kirchhoff methods and physical optics techniques are widely used to solve numerous problems associated with propagation, diffraction, and scattering of waves of various natures, which are problems lacking in strict analytical solutions. An attractive feature of the theory is that the solution can be immediately written in the form of a diffraction integral and the approach itself is rather simple and visual.

Much interest has recently been expressed in the creation of application packages intended to help engineers in the design of optical systems. The widely known Samsung Electronics Company develops, improves, and uses its Software Laser Optics Design and Analysis Tool (SLO-DAT) intended for the design of laser optical systems (LOS). A combined method is used in SLO-DAT to determine the field parameters at any surface point behind the LOS. Specifically, the original laser beam is represented as a set of rays, and the field distribution of the transformed beam over the surface of the LOS exit pupil is determined. Next, the diffraction distortions of the beam introduced by the LOS are taken into account with the use of the finite size of the LOS exit pupil; i.e., the complex field amplitude distribution of the LOS-transformed beam is determined by computing the diffraction integral over the surface of the exit pupil.

The computation of the diffraction integral is not a new problem. It is well known that its analytical evaluation is possible only in rare cases. Accordingly, an important problem is to design efficient numerical algorithms for diffraction integral computation. This is an integral of a rapidly oscillating function. Some methods for its computation can be found in textbooks and special scientific literature. Integrals of rapidly oscillating functions can also be computed using built-in procedures in Maple, Mathcad, Mathematica, etc. However, there is always a risk of a result being incorrect.

When a method for diffraction integral computation is chosen, it is desirable to use the specific features of the problem under consideration, specifically, take into account the distribution of the complex field amplitude on the input surface (integration surface).

We present an algorithm for diffraction integral computation that takes into account the specific features of SLO-DAT. Note that the algorithm is rather universal, so it can be efficient and useful as applied to other problems.

2 Mathematical statement of the problem

Suppose that we know the field distribution of the transformed laser beam over the surface of the exit pupil of a laser optical system. The problem is to calculate the field of the transformed laser beam at points of the observation surface by applying the diffraction integral:

$$U(X^{'},Y^{'},Z^{'})=\iint\limits_{S}\frac{(cos\beta+cos\gamma)U(X,Y,Z)e^{ik\cdot r(X^{'},Y^{'},Z^{'},X,Y,Z)}}{2\lambda i\cdot r(X^{'},Y^{'},Z^{'},X,Y,Z)}dS \quad (1)$$

Here, S is the integration domain, $U(X,Y,Z)$ is the complex field amplitude given at the point with coordinates (X,Y,Z) in the integration domain, $U(X^{'},Y^{'},Z^{'})$ is the desired complex field amplitude at the point with coordinates $(X^{'},Y^{'},Z^{'})$ on the observation surface, β is the angle between the normal to the wavefront and the normal to the integration surface at the point (X,Y,Z), γ is the angle between the normal to the integration surface and the vector with coordinates $(X^{'}-X,Y^{'}-Y,Z^{'}-Z)$, λ is the radiation wavelength, k is the wave number ($k=2\pi/\lambda$), and $r(X^{'},Y^{'},Z^{'},X,Y,Z)$ is the distance from the point (X,Y,Z) to the point $(X^{'},Y^{'},Z^{'})$.

The integral has the following features:

(a) The integrand performs fast oscillations in the integration domain; i.e., (1) is an integral of a rapidly oscillating function.

(b) Integral (1) is a two-dimensional integral of a rapidly oscillating function.

The computation of complex integral (1) is reduced to the computation of two real surface integrals, of which one is the integral of a real function $\omega(X,Y,Z)$ times $cos(k\psi(X,Y,Z))$, while the other is the integral of $\omega(X,Y,Z)$ times $sin(k\psi(X,Y,Z))$. Even if the arguments of $\psi(X,Y,Z)$ vary rather moderately, the functions $cos(k\psi(X,Y,Z))$ and $sin(k\psi(X,Y,Z))$ vary very rapidly for large k.

3 Method for computing diffraction integrals

In the algorithm, the integration surface S is approximated by a piecewise smooth surface that is the union of flat triangles. Each vertex of a triangle is a point on at which the complex field amplitude is given. The values of the phase and the complex amplitude at the points of the triangle are approximated by a bilinear function, using the known values of the field at the vertices of the triangle. Next, the elementary diffraction integral over a triangle is calculated exactly (analytically). The diffraction integral over S is calculated as the sum of elementary diffraction integrals over all triangles.

The following algorithm is used to approximate the integration surface by flat triangles. The points with input data lying on the integration surface (on the exit pupil surface of the optical system) are projected onto the plane OXY (orthogonal to the axis of the optical system). For this purpose, the point with coordinates (X, Y, Z) is put in correspondence with the point $(X, Y, 0)$. The projected points are the input data for partitioning the projection of the integration surface into triangles. The vertices of the desired triangles must be projected points. The desired set of triangles is constructed by applying Delaunay triangulation. Next, the integration surface (the surface of the exit pupil) is approximated by triangles. More specifically, each vertex with coordinates $(X, Y, 0)$ in a triangle produced by the triangulation is assigned new coordinates (X, Y, Z) such that this point lies on the surface of the exit pupil.

When calculating the diffraction integrals using the computer, often erroneous results occur. This happens due to the use of formulas that contain a division of two expressions, each of which is close to zero. The computer evaluates the indeterminate form of type 0/0 incorrectly. The significance of the proposed algorithm for calculation of the diffraction integrals is in the developed regular formulas that do not contain indeterminacies.

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Decisional analysis of supply planning in monopolistic economic systems

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Abstract

In this paper, a decisional model for monopolistic business activity is analyzed in order to establish the optimal input of factors of production and output of products and services. The decisions are based on resources supply, demand of monopolistic goods and on their prices. The investigated models are non-differentiable, and their solving can be effectively accomplished using numerical algorithms built on method of generalized gradient.

Keywords: monopoly, supply and demand, profit, the optimal decision, generalized gradient method, non-differentiable models.

An economic system of production is identified as a monopoly if it is a unique bidder of certain goods in a given market. Further, we will consider a decision situation that arises from the main goal: to maximize profit for a monopolistic firm.

Description of the mathematical model. Goal function:

$$R^0(x,y) = \sum_{j=1}^{n} \left[c_j(\cdot) \min\{y_j;\ Y_j\} - p_j \max\{0;\ y_j - Y_j\} \right] - \tag{1}$$

$$- \sum_{i=1}^{m} q_i x_i \to \max_{(x,y)}$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} y_j = x_i, i = \overline{1,m}. \tag{2}$$

Vector of goods:

$$y = (y_1, ..., y_j, ..., y_n) \in D_y = \left\{ y \in E^n : \underline{y}_j \leq y_j \leq \bar{y}_j,\ j = \overline{1,\, n} \right\}, \quad (3)$$

Vector of resources:

$$x = (x_1, ..., x_i, ..., x_m) \in D_x = \left\{ x \in E^m : \underline{x}_i \leq x_i \leq \bar{x}_i,\ i = \overline{1,\, m} \right\}. \quad (4)$$

Notations: the sets D_x and D_y are assumed known; $c_j(\cdot)$ – selling price of a unit of good j – depends on the amount of offer y_j and on the quantity of demand Y_j of this good; p_j – losses (in monetary units), when a unit of product j is not sold; q_i-price of the resource i; a_{ij} – technological coefficient. Further, it will be assumed that the selling price of product j, depending on quantity y_j, is expressed as

$$c_j\left(\cdot\right) = \left\{ \begin{array}{ll} c_j(y_j), & \text{if} \quad y_j \leq Y_j \\ c_j(Y_j), & \text{for} \quad y_j > Y_j \end{array} \right. ,$$

$$c_j\left(Y_j\right) = \left\{ \begin{array}{l} \bar{c}_j, \quad \text{if} \quad Y_j \leq \underline{y}_j \\ c_j\left(\tilde{y}_j\right), \quad \text{if} \ \underline{y}_j \leq Y_j \leq \bar{y}_j,\ Y_j \leq y_j; \ \text{where} \ \tilde{y}_j = Y_j, \end{array} \right. \quad (5)$$

$$c_j(y_j) = \bar{c}_j - (\bar{c}_j - \underline{c}_j) * \frac{y_j - \underline{y}_j}{\bar{y}_j - \underline{y}_j} \quad (6)$$

and decreases linearly for $y_j \in \left[\ \underline{y}_j;\ \bar{y}_j \right]$.

Taking into consideration the nature of the relations (2) we obtain $R^0\left(x, y\right) = R(y) = \sum_{j=1}^{n} R_j(x_j)$, where

$$R_j(y_j) =$$

$$\sum_{j=1}^{n} \left[c_j(\cdot) \min\left\{y_j;\ Y_j\right\} - p_j * \max\left\{0;\ y_j - \ Y_j\right\} - (\sum_{i=1}^{m} a_{ij} q_i)\, y_j \right]. \quad (7)$$

Analyzing the properties of the function $R_j(y_j)$, it is observed that for $y_j < Y_j$:

$$R_j(y_j) = \left(\frac{\bar{c}_j \bar{y}_j - \underline{c}_j \underline{y}_j}{\bar{y}_j - \underline{y}_j} - \sum_{i=1}^{m} a_{ij} q_i \right) \cdot y_j - \frac{\bar{c}_j - \underline{c}_j}{\bar{y}_j - \underline{y}_j} \cdot y_j^2, \quad (8)$$

and for $y_j \geq Y_j$,

$$R_j(y_j) = (c_j(Y_j) + p_j) \cdot Y_j - (p_j + \sum_{i=1}^{m} a_{ij} q_i) y_j. \tag{9}$$

Structure of the model

$$R(y) \rightarrow \max \tag{10}$$

in conditions (3–7) allows to apply, in the process of solving, the method of generalized gradient [3]. Relations (8), (9) admit to "adjust" the values $\underline{c}_j$ and $\bar{c}_j$ in order to assure the positive output of the economic system for the category of products of prime necessity, but not necessarily to satisfy the demand Y_j completely. In other words, for the interval $\left[\underline{y}_j; \bar{y}_j\right]$ the values $\bar{c}_j > \underline{c}_j$ can be indicated, that satisfy the following inequality $\frac{\bar{c}_j \bar{y}_j - \underline{c}_j \underline{y}_j}{\bar{y}_j - \underline{y}_j} - \sum_{i=1}^{m} a_{ij} q_i > 0$.

The method of solving. Let us assume that we know the estimations for values Y_j, $j = \overline{1, n}$.

1. We build the functions $\varphi_i^1(y) = \sum_{j=1}^{n} a_{ij} y_j - \bar{x}_i; \varphi_i^2(y) = \underline{x}_i - \sum_{j=1}^{n} a_{ij} y_j$, $i = \overline{1, m}$ and $\varphi(y) = \max\limits_{1 \leq i \leq m} \left\{\varphi_i^1(y), \varphi_i^2(y)\right\}$ – the maximum value of these $2m$ functions;

2. The generalized gradients of functions $R(y)$ and $\varphi(y)$ are represented by $g_R(y)$ and $g_\varphi(y)$ respectively;

3. An arbitrary element $y^0 \in D_y$ is taken, then, calculate the elements $y^1, y^2, ..., y^k, y^{k+1} ... \in D_y$. Suppose that the element y^k is already obtained. To find the element y^{k+1} the following algorithm is used:

$$y^{k+1} = \prod_{D_y} (y^k - h_k \cdot g^k), \tag{11}$$

where the step size:

$$h_k \geq 0, \;\; h_k \rightarrow 0, \;\; \sum_{k=0}^{\infty} h_k = \infty, \tag{12}$$

and the vector g^k, which determines the direction of displacement, is built according to the following scheme [2]:

$$g^k = \begin{cases} -g_R(y^k), & \text{if} \quad \varphi(y^k) \le 0 \\ g_\varphi(y^k), & \text{for} \quad \varphi(y^k) > 0. \end{cases} \tag{13}$$

Remark 1. *In practical applications, a priori is indicated a number $\bar{K}$, and the calculations are stopped as soon as the $k = \bar{K}$. The obtained solution at iteration $\bar{K}$, if it is necessary, can be taken as a starting point y^0 for a new release of the algorithm. This process can be repeated until the decision $y^{\bar{K}}$ is taken as an optimal solution.*

Remark 2. *More frequently, of course, the situations are met, where the demand volume Y_j is not known, but the values $\underline{Y}_j$ and $\bar{Y}_j$, $\underline{Y}_j < \bar{Y}_j$, and $Y_j \in \left[\underline{Y}_j; \bar{Y}_j\right]$ can be estimated. These situations could lead to decision models under risky conditions or to the complex models with uncertainty factors, otherwise [1].*

Remark 3. *Method (11-13) is general, in the sense of ability to solve some classes of non-differentiable problems. Also, this method can facilitate the obtaining of acceptable solutions (decisional alternatives) in real time, which is very important for businesses.*

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Bayesian Experimental Design for Network Loss Tomography

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Abstract

We consider a sequential experimental design problem for network loss tomography with a random choice of paths. We apply a Bayesian approach and Kullback-Leibler divergence to maximise the informational gain obtained at each step of the network probing experiment and show that the choice of paths is, in fact, deterministic. We discuss practical aspects of this result.

Keywords: path probing, statistical network loss tomography, Bayesian experimental design, Kullback–Leibler divergence

1 Introduction

Let $\mathcal{N} = (V, L)$ denote a network with the set of nodes V and the set of links L. Each link $l \in L$ is characterised by the *loss rate* $1 - \theta_l$ which is the probability that a message/packet/probe will fail to successfully pass this link if sent along. Let $\mathcal{P}$ be a given set of probing paths in $\mathcal{N}$. A probing path consists of one or more pairwise adjacent links in $\mathcal{N}$. We assume that the monitoring system can inject probes on all paths in $\mathcal{P}$ and observe whether they successfully reach their destinations or not. Link failures are independent and if a probe is sent along path $y \in \mathcal{P}$, the distribution of the outcome x (success or failure in transmitting the probe along the entire length of y) is

$$\mathbb{P}_y(x) = [\Psi_y(\boldsymbol{\theta})]^x \, [1 - \Psi_y(\boldsymbol{\theta})]^{(1-x)}, \; x = 0, 1,$$

where $\Psi_y(\boldsymbol{\theta}) := \prod_{l \in y} \theta_l$ is the probability of successful transmission along y. Thus, θ_l is the reliability of link l, whereas Ψ_y is the reliability of path y.

The goal in loss tomography is to infer the success rates (complements of which are the link loss rates) of all or some links in $\mathcal{N}$ by observing results of end-to-end probes on a given set of probing paths $\mathcal{P}$. The goal of loss tomography experimental design is to identify such a set of probing paths optimally (with respect to some optimality criterium).

2 Statistical Design in Network Tomography

Statistical tomography models each link state as a random variable with unknown probability distributions, and apply various estimation techniques to estimate these distributions or their characteristics from path measurements [1]. Consequently, experimental designs derived with such a purpose use a metric which aims to maximise precision of this estimation. We shall take a different approach in devising experimental designs which aim to maximally increase the knowledge gain obtained in an experiment.

2.1 Experimental Model

Various experimental loss network tomography models are possible, but in this work we are only concentrating on the following experimental scenario:

Experiment: Let $\phi_y^{(i)}$ be a probability of choosing to probe path y at step i. Use N probes by determining sequentially at each step the optimal vector $\boldsymbol{\phi}^{(i)}$ for randomly choosing a path to probe after observing the result of the previous probing.

2.2 Experimental Design

Let the model vector parameter be $\boldsymbol{\theta}$, the model is described by a probability distribution $f(\mathbf{z} \,|\, \boldsymbol{\theta}, \mathbf{d})$ of the outcome $\mathbf{z}$ of the studied process under experimental conditions described by the **design parameter $\mathbf{d}$**. Our knowledge/uncertainty about $\boldsymbol{\theta}$ is described by a **prior distribution** $\pi(\boldsymbol{\theta})$. Using a typical Bayesian approach, after observing $\mathbf{z}$ this knowledge updates to $\pi(\theta|\mathbf{z})$. We shall measure the usefulness of the experiment using the Kullback–Leibler divergence:

$$U_{\mathrm{KL}}(\mathbf{d}) := \mathbb{E}_{\mathbf{z},\boldsymbol{\theta}}\left[\log\frac{\pi(\boldsymbol{\theta}\,|\,\mathbf{z},\mathbf{d})}{\pi(\boldsymbol{\theta})}\right] \equiv \mathbb{E}_{\mathbf{z}}\left[D_{KL}\{\pi(\boldsymbol{\theta}\,|\,\mathbf{z},\mathbf{d}) \parallel \pi(\boldsymbol{\theta})\}\right], \quad (1)$$

where $D_{KL}\{h(t) \parallel g(t)\} := \int\limits_{\mathbb{R}} h(t)\log\frac{h(t)}{g(t)}\,\mathrm{d}t$. This quantity measures the informational gain obtained after updating the prior information to posterior.

3 Main Result

Lemma 1 (First-order conditions) The partial derivative of the expected utility $U_{\mathrm{KL}}(\mathbf{d})$ with respect to its continuous component d_i can be calculated as follows:

$$\frac{\partial U_{\mathrm{KL}}(\mathbf{d})}{\partial d_i} = \int\limits_{\Theta}\int\limits_{\mathcal{Z}} \log\frac{\pi(\boldsymbol{\theta}\,|\,\mathbf{z},\mathbf{d})}{\pi(\boldsymbol{\theta})} f'_{d_i}(\mathbf{z}\,|\,\boldsymbol{\theta},\mathbf{d})\pi(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\mathbf{z}, \quad (2)$$

provided the functions $f(\cdot\,|\,\cdot,\cdot)$, $\pi(\cdot)$ are such that differentiation of $U_{\mathrm{KL}}(\mathbf{d})$ and the corresponding integration are interchangeable.

Theorem 1
In the above random sequential design the following ***deterministic*** *choice of the path maximises the expected Kullback-Leibler divergence between the current and updated knowledge on* $\boldsymbol{\theta}$ *:*

$$y^i = \arg\max_y\left(\sum_{x=0,1}\int\limits_{\Theta}\log\frac{\pi^i(\boldsymbol{\theta}\,|\,y,x)}{\pi^{i-1}(\boldsymbol{\theta}|y^{i-1},x^{i-1})}\mathbb{P}_y(x;\boldsymbol{\theta})\pi^{i-1}(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}\right).$$

Proof
Let $\mathbf{z} = (y,x)$ and the design vector $\mathbf{d}^i$ be the vector of path choice probabilities $\boldsymbol{\phi}^i$. The distribution of observables $\mathbf{z}$ is as follows:

$f(\mathbf{z}\,|\,\boldsymbol{\theta},\boldsymbol{\phi}^i) = \sum_{y\in\mathcal{P}} \phi^i_y \mathbb{P}_y(x;\theta)$, where $\mathbb{P}_y(x) = (\prod_{l\in y}\theta_l)^x(1-\prod_{l\in y}\theta_l)^{(1-x)}$.
We next use Lemma 1 to find the following representation:

$$\frac{\partial U_{\mathrm{KL}}(\boldsymbol{\phi}^i)}{\partial\phi^i_w} = \int\limits_{\Theta}\int\limits_{\mathcal{Z}} \log\frac{\pi^i(\boldsymbol{\theta}\,|\,y,x,\boldsymbol{\phi}^i)}{\pi^{i-1}(\boldsymbol{\theta}|y^{i-1},x^{i-1})} f'_{\phi^i_w}(\mathbf{z}\,|\,\boldsymbol{\theta},\boldsymbol{\phi})\pi^{i-1}(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}\mathrm{d}\mathbf{z}$$
$$= \sum_{x=0,1}\int\limits_{\Theta}\log\frac{\pi^i(\boldsymbol{\theta}\,|\,y=w,x)}{\pi^{i-1}(\boldsymbol{\theta}|y^{i-1},x^{i-1})}\mathbb{P}_w(x;\theta)\pi^{i-1}(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}.$$

Thus, $\frac{\partial U_{\mathrm{KL}}(\boldsymbol{\phi}^i)}{\partial\phi^i_w}$ does not depend on $\boldsymbol{\phi}^i$. Together with the feasibility conditions $\sum_{y\in\mathcal{P}}\phi^i_y = 1$ and $\phi^i_y \geq 0$, this implies that the maximal component of the gradient vector of $U_{\mathrm{KL}}(\boldsymbol{\phi}^i)$ indicates which path to probe. This completes the proof. $\square$

The talk will discuss practical aspects of computation of y^i.

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Auxiliary busy periods for $M_2|G_2|1$ system with PH distribution and strategy "reset-to-zero"

Diana Bejenari, Attahiru S. Alfa,
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Abstract

An $M_2|G_2|1$ system with two priority classes is considered. There is orientation time when service is switching from one class of priority to another. Analytical results are obtained for the auxiliary busy period under the "reset-to-zero" strategy when the switching times have PH distribution.

Keywords: $M_2|G_2|1$ system, busy period, PH distribution.

1 Introduction

In this paper an $M_2|G_2|1$ system with two priority classes and orientation time while switching from serving one class of priority to another is studied. The requests of class 1 are endowed with absolute (preemptive) priority. If they arrived in the system while serving a request of class 2 or orientation to class 2, the service of this class 2 message and/or the orientation towards the service of messages of class 2 will be interrupted. The switching time from class 1 to class 2 is C_{12} and from class 2 to class 1 is C_{21}. The switching times are assumed to have PH distribution. For the case of "reset-to-zero" strategy we obtain analytical results for auxiliary busy periods. Given that the distribution functions of service and orientation time are of general order, this type of system can be used for modeling and analyzing a wide spectrum of real problems. The research methods are based on Laplace-Stieltjes transforms, generating functions and PH distribution.

2 The busy period for strategy “reset-to-zero”

The distribution of the busy period and auxiliary periods for strategy “reset-to-zero” in the free state and absolute priority schemes are given in [1]. All the distributions are presented in terms of Laplace-Stieltjes transform and are obtained by the method of catastrophes. In conditions that $C_{01} = C_{21}$ and $C_{02} = C_{12}$ the distribution of busy period for strategy ”reset to zero” of the $M_2|G_2|1$ system is given by [1]:

$$\pi(s) = \frac{a_1}{a_1 + a_2}\pi_{21}(s) + \frac{a_2}{a_1 + a_2}\pi_{22}(s),$$

$$\pi_{21}(s) = \pi_1(s+a_2)+\{\pi_1(s+a_2(1-\overline{\pi}_2(s)))-\pi_1(s+a_2)\}\nu_2(s+a_2(1-\overline{\pi}_2(s))),$$

$$\pi_{22}(s) = \nu_2(s + a_2(1 - \overline{\pi}_2(s)))\overline{\pi}_2(s),$$

$$\pi_1(s) = c_{21}(s + a_1(1 - \overline{\pi}_1(s)))\overline{\pi}_1(s), \tag{1}$$

$$\overline{\pi}_2(s) = h_2(s + a_2(1 - \overline{\pi}_2(s))), \tag{2}$$

$$\overline{\pi}_1(s) = \beta_1(s + a_1(1 - \overline{\pi}_1(s))), \tag{3}$$

where $\nu_2(s)$ and $h_2(s)$ are given in [1].

3 Matrix forms for auxiliary busy periods

The matrix forms and matrix algorithms for auxiliary busy periods $\overline{\pi}_1(s)$ and $\pi_1(s)$ are obtained in this section. Suppose that $B_1(x)$ is a PH distribution [2] with representation (α_1^t, T_1), where

$$T_1 = \begin{pmatrix} -a_1 & a_1 & \dots & 0 & 0 \\ 0 & -a_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -a_1 & a_1 \\ 0 & 0 & \dots & 0 & -a_1 \end{pmatrix},$$

and

$$T_1^0 = \begin{pmatrix} 0 & 0 & \dots & a_1 \end{pmatrix}^t.$$

The Laplace-Stieltjes transform $\beta_1(s)$ of probability distribution $B_1(x)$ of the time until absorbtion in the state 0 is $\beta_1(s) = \alpha_1^t(sI - T_1)^{-1}T_1^0$. Then the matrix form of equation (3) is

$$\overline{a}_1(s) = \alpha_1^t g_1(s) y_1(s), \tag{4}$$

where $\overline{a}_1(s) = 1 - \overline{\pi}_1(s)$ and $y_1(s) = (g_1(s)I + A_1)^{-1}e$ can be obtained by solving the simultaneous linear equations

$$(g_1(s)I + A_1)y_1(s) = e,$$

where $A_1 = -T_1$. The matrix algorithm for equation (3) is $\overline{a}_1(s) = 1 - (a_1\omega_1)^n$, where $\omega_1 = 1/(g_1(s) + a_1)$, $\alpha_1^t e = 1$ and we start with $\overline{a}_1(s) = 1$ and $\alpha_1^t = (10\ldots0)$, the remaining values are input values $(a_1, \widetilde{a}_1 = a_1$ and $s)$.

Suppose that $C_{21}(x)$ is a PH distribution with representation (α_{21}^t, T_{21}), where

$$T_{21} = \begin{pmatrix} -\delta_{21} & \delta_{21} & \ldots & 0 & 0 \\ 0 & -\delta_{21} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -\delta_{21} & \delta_{21} \\ 0 & 0 & \ldots & 0 & -\delta_{21} \end{pmatrix},$$

and

$$T_{21}^0 = \begin{pmatrix} 0 & 0 & \ldots & \delta_{21} \end{pmatrix}^t.$$

The Laplace-Stieltjes transform $c_{21}(s)$ of probability distribution $C_{21}(x)$ of the time until absorbtion in the state 0 is

$$c_{21}(s) = \alpha_{21}^t(sI - T_{21})^{-1}T_{21}^0.$$

Then the matrix form of equation (2) $(a_1 = \widetilde{a}_1)$ is

$$b_1(s) = \alpha_{21}^t g_1(s)(1 - \overline{a}_1(s))\widetilde{y}_{21}(s), \tag{5}$$

where $\widetilde{y}_{21}(s) = (g_1(s)(1-\overline{a}_1(s))I + D_{21})^{-1}e$ can be obtained by solving the simultaneous linear equations $(g_1(s)(1-\overline{a}_1(s))I + D_{21})\widetilde{y}_{21}(s) = e$, where $D_{21} = -T_{21}$.

The matrix algorithm for equation (2) is $b_1(s) = 1-(\delta_{21}\gamma_1)^n$, where $\gamma_1 = 1/(g_1(s)(1-\overline{a}_1(s)) + \delta_{21})$, $\alpha_{21}^t e = 1$ and we start with $\overline{a}_1(s) = 1$ and $\alpha_{21}^t = (10\ldots0)$, the remaining values are input values ($\delta_{21}, \widetilde{a}_1 = a_1$ and s).

4 Conclusion

The presented results will be used for obtaining the matrix form of busy period of the mentioned system.

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Derivation of Boolean functions by the blocks method

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Abstract

A method for calculating using blocks of the partial derivatives of the boolean function which are represented in algebraic form – a polynomial form, disjunctive normal form and conjunctive normal form is proposed. The method can be used successfully for functions which depend on a large number of variables (tens and hundreds). This method can be applied to solve many problems from different areas such as: elaborating of encryption system with variable keys, some problems from discrete mathematics (systems isomorphism problem, problem of determining of chromatic number in graphs ect.).

Keywords: boolean function, derivative of function, subsets of columns, block of partition.

1 Introduction

In [2] the keys of encryption system are built on the basis of subsets column of partial derivatives of Boolean functions using subsets of column of functions from the key. With increasing of number of functions variables the encryption speed decreases. The situation can be changed if the subsets of column of partial derivatives are calculated without calculating the subsets of column of functions. This we can do by using blocks which correspond to conjunctions (disjunctions) of the algebraic form of functions [2].

2 The subsets of column of derivatives of boolean functions

Suppose we have a Boolean function

$$F(x_1, ..., x_\tau, x_{\tau+1}, ..., x_n) = U_1 \oplus ... \oplus U_r, \tag{1}$$

where conjunctions U_i are of the form

$$U_i = x_{i_1}^{\sigma_{i_1}} \wedge ... \wedge x_{i_s}^{\sigma_{i_s}} \wedge x_{j_1}^{\sigma_{j_1}} \wedge ... \wedge x_{j_k}^{\sigma_{j_k}} \text{ and } \forall a, b: \ \sigma_{i_a}, \sigma_{j_b} \in \{0, 1\}.$$

We construct the sets

$$\tilde{X}_1 = \{x_1, ..., x_\tau\} = \{1, ...\tau\}, \tilde{X}_2 = \{x_{\tau+1}, ..., x_n\} = \{\tau + 1, ..., n\}$$

and find the subset of column [1] of the partial derivative

$$S^{z_j}_{\left(\frac{\partial^t U_i}{\partial x_{1_1} ... \partial x_{1_m} \partial x_{2_1} ... \partial x_{2_b}}\right)^1},$$

where $1_1, ..., 1_m \in \{1, ..., \tau\}$, $2_1, ..., 2_b \in \{\tau + 1, ..., n\}$, $t = m + b$. Denote $\{x_{i_1}, ..., x_{i_s}\} = \{i_1, ..., i_s\}$ and $\{x_{j_1}, ..., x_{j_u}\} = \{j_1, ..., j_k\}$ and construct the sets:

$$\{c_1, ..., c_q\} = \{i_1, ..., i_s\} \backslash \{1_1, ..., 1_m\},$$

$$\{t_1, ..., t_p\} = \{j_1, ..., j_k\} \backslash \{2_1, ..., 2_b\}.$$

The following **Theorem** is true:

$$S^{z_j}_{\left(\frac{\partial^t U_i}{\partial x_{1_1} ... \partial x_{1_m} \partial x_{2_1} ... \partial x_{2_b}}\right)^1} = \begin{cases} Y, \, if \begin{cases} 1) \ \{1_1, ..., 1_m\} = \{i_1, ..., i_s\} \\ 2) \ \{2_1, ..., 2_b\} \ \subseteq \{j_1, ..., j_k\} \\ 3) \forall \ a \in \{t_1, ..., t_p\} : \tilde{x}_a = \tilde{\sigma}_a \end{cases} \\ \bigcap_{u=1}^{q} \bar{m}_{c_u}^{\sigma_{c_u}}, if \begin{cases} 1) \ \{1_1, ..., 1_m\} \subset \{i_1, ...1_s\} \\ 2) \ \{2_1, ..., 2_b\} \subseteq \{j_1, ..., j_k\} \\ 3) \forall a \in \{t_1, ..., t_p\} : \tilde{x}_a = \tilde{\sigma}_a \end{cases} \\ \emptyset, \ in \, other \, cases \end{cases}$$

where $Y = \{0, 1, 2, ..., 2^{\tau} - 1\}$.
The subsets of column of derivatives of function are obtained by the symmetrical difference of subsets of column of all conjunctions from (1).

3 Some applications

- Suppose the secret key represent a set of Boolean functions $F = \{F_1, ..., F_k\}$[2] and for information encryption the subsets of column of partial derivatives of functions from this set are used. The method of derivation by blocks allows a fast calculation of derivatives of functions that depend on tens and hundreds of variables. This makes possible to quickly change the encryption key during information encrypting. The change of the number of variables in the functions from F practically does not influence the speed of information encryption. The increasing of the number of variables in the functions from F implies a higher cryptographic resistance of system without having a great impact of encryption speed. Encryption system from [2] is improved based on derivation by blocks of the functions from key. In this system the keys are variable and change from one to another message and during information encryption. Generally the same symbol from message is encrypted differently. This makes it practically impossible to restore the message if we know part of it.
- The method of derivation by blocks allows to solve the problem of determining the upper and lower boundaries for the chromatic number of a graph [3].

4 Conclusion

- The proposed method allows increasing encryption system reliability by increasing of the number of variables in functions from key without essential changing of encryption speed and volume of message.

- It is welcome to generalise this method for functions which are represented in other analytic forms.

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Covering undirected graphs by convex sets

Radu Buzatu

Abstract

This paper is focused on some aspects of undirected graphs covering, in particular convex sets problem (CCS) and partitioning undirected graph into convex sets problem (PCS). We prove theorems regarding existence of graphs with fixed number of convex sets which serve as solutions to CCS and PCS problems.

Keywords: convexity, d-convexity, graphs, convex covers, vertex covers.

1 Introduction

Let $G(X, G)$ be a simple connected graph. We use the notion of *d-convexity* that is defined in [1]. The concepts of *convex p-cover* and *convex p-partition* are defined in [2,3] as follows: a graph G has convex p-cover if $X(G)$ can be covered by p convex sets, that is, there exists $\mathcal{X} = \{X_1, X_2, \ldots, X_p\}$, $p \in N$, such that $X(G) = \bigcup_{1 \leq i \leq p} X_i$; set X_i is convex and $X_i \nsubseteq \bigcup_{\substack{1 \leq j \leq p \\ i \neq j}} X_j$, for $1 \leq i \leq p$. If all sets of $\mathcal{X}$ are disjoint, then $\mathcal{X}$ is a convex p-partition of $X(G)$. Additionally, there are defined numbers $\varphi_c(G)$ and $\theta_c(G)$. The *convex cover number* $\varphi_c(G)$ of a graph G is the least integer $p \geq 2$ for which G has convex p-cover. The *convex partition number* $\theta_c(G)$ of a graph G is the least integer $p \geq 2$ for which G has a convex p-partition.

We define new concepts: *non-trivial convex p-cover* and *non-trivial convex p-partition.* A graph G has a non-trivial convex p-cover if G has convex p-cover and all sets of $\mathcal{X}$ are non-trivial, $|X_i| \geq 3$, for $1 \leq i \leq p$.

If a graph G has a non-trivial convex p-cover and all sets of $\mathcal{X}$ are disjoint, then $\mathcal{X}$ is a non-trivial convex p-partition of $X(G)$.

The *non-trivial convex cover number* $\varphi_{cn}(G)$ of a graph G is the least integer $p \geq 2$ for which G has a non-trivial convex p-cover. The *non-trivial convex partition number* $\theta_{cn}(G)$ of a graph G is the least integer $p \geq 2$ for which G has a non-trivial convex p-partition.

2 Covering and partitioning of graphs

In this section there are presented several proved theorems regarding existence of graphs with fixed number of convex sets which serve as solutions to CCS and PCS problems and some properties of graph covers and graph partitions.

Let $P_\varphi(X(G))$ and $P_\theta(X(G))$ be minimum families of sets, which serve as solutions to CCS and PCS problems.

Theorems which describe properties are presented below.

Theorem 1. *If $\varphi_c(G) \geq 3$, then for any $P_\varphi(X(G))$, $\forall A, B \in P_\varphi(X(G))$, where $A \neq B$, $\exists C \in (P_\varphi(X(G))\backslash A)\backslash B$, such that $\exists a \in A$, $\exists b \in B$, $\exists c \in C$, where $c \in \langle a, b \rangle$.*

Corollary 1. *If $\theta_c(G) \geq 3$, then for any $P_\theta(X(G))$, $\forall A, B \in P_\theta(X(G))$, where $A \neq B$, $\exists C \in (P_\theta(X(G))\backslash A)\backslash B$, such that $\exists a \in A$, $\exists b \in B$, $\exists c \in C$, where $c \in \langle a, b \rangle$.*

Theorem 2. *If $\varphi_c(G) \geq 3$, then for any $P_\varphi(X(G))$, $\forall A \in P_\varphi(X(G))$, $\exists B \in P_\varphi(X(G))\backslash A$, $\exists C \in P_\varphi((X(G))\backslash A)\backslash B$, such that $\exists a \in A$, $\exists b \in B$, $\exists c \in C$, where $a \in \langle b, c \rangle$.*

Corollary 2. *If $\theta_c(G) \geq 3$, then for any $P_\theta(X(G))$, $\forall A \in P_\theta(X(G))$, $\exists B \in P_\theta(X(G))\backslash A$, $\exists C \in P_\theta((X(G))\backslash A)\backslash B$, such that $\exists a \in A$, $\exists b \in B$, $\exists c \in C$, where $a \in \langle b, c \rangle$.*

Next few theorems regard existence of graphs with fixed number of convex sets.

Theorem 3. *$\forall p, n \in N$, where $2 \leq p \leq n$, an undirected connected graph G exists, such that:*

a) $|X(G)| = n$;

b) $\varphi_c(G) = p$;

c) $\theta_c(G) = p$.

Theorem 4. *$\forall q, n \in N$, where $2 \leq q \leq \lfloor \frac{n}{3} \rfloor$, an undirected connected graph G exists, such that:*

a) $|X(G)| = n$;

b) $\theta_{cn}(G) = q$.

Theorem 5. *$\forall q, n \in N$, where $2 \leq q = n - 2$, there is no undirected connected graph G, such that:*

a) $|X(G)| = n$;

b) $\varphi_{cn}(G) = q$.

Theorem 6. *$\forall q, n \in N$, where $2 \leq q \leq n - 3$, an undirected connected graph G exists, such that:*

a) $|X(G)| = n$;

b) $\varphi_{cn}(G) = q$.

Theorem 7. *Let G be an undirected connected graph, conditions a)-b) are met, $G \notin F$, where F is a family of graphs (see Fig.1), then $\varphi_{cn}(G) = 2$.*

a) $|X(G)| \geq 5$;

b) $\varphi_c(G) = 2$.

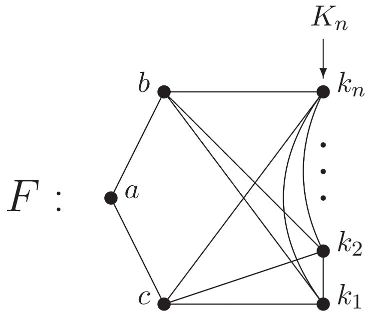

Figure 1.

3 Conclusion

These findings can help in characterization of graphs for which $\varphi_c(G)$, $\varphi_{cn}(G)$, $\theta_c(G)$, $\theta_{cn}(G)$ are predefined. Furthermore, results describing relation between solutions of CCS and PCS problems can be obtained.

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A Method for Solving Stochastic Discrete Control Problems on Networks with Varying Time of States' Transitions of the System

Maria Capcelea

Abstract

The stochastic version of discrete optimal control problem on networks with an average cost optimization criterion and with varying time of state transitions is studied.

Keywords: stochastic discrete control problem, stationary strategies.

1 Problem Formulation

In this paper we consider the stationary stochastic discrete optimal control problem on networks with an average cost optimization criterion, when the time of systems' transitions from one state to another may vary in the control process. The problem will be reduced to the case with unit time of states' transitions of the system.

Let a discrete dynamical system L with finite set of states X be given. At every discrete moment of time $t = t_0, t_1, t_2, ...$ the state of L is $x(t) \in X$ and at the starting moment of time $t_0 = 0$ the state of the dynamical system is $x_0 = x(0)$. Assume that the dynamics of the system is described by a directed graph of state's transitions $G = (X, E)$. An arbitrary vertex x of G corresponds to a state $x \in X$ and an arbitrary directed edge $e = (x, y) \in E$ expresses the possibility of the system L to pass from the state $x(t)$ to the state $x(t + \tau_e)$, where τ_e is the time of the system's transition from the state $x = x(t)$ to the state $y = x(t+\tau_e)$ through the edge $e = (x, y)$. So, on the edge set E it is defined the transition time function $\tau : E \to \mathbb{N}$ and, also, the cost function $c : E \to \mathbb{R}$, which associates to each edge the cost c_e of the system's transition from the state x to the state y.

We assume that the set X may admit states in which system L makes transitions to a next state in the random way, according to given distribution function of probabilities on the set of possible transitions from these states. So, the set X is divided into two subsets X_C and X_N ($X = X_C \bigcup X_N$, $X_C \bigcap X_N = \emptyset$), where X_C represents the set of controllable states $x \in X$, in which the transitions of the system to a next state y can be controlled by the decision maker, and X_N represents the set of uncontrollable states $x \in X$, in which the decision maker is not able to control the transition, because the system passes to a next state y randomly. The probability distribution function $p : E_N \to [0, 1]$ on the set $E_N = \{e = (x, y) \in E \mid x \in X_N\}$ is defined in such a way that $\sum_{y \in X^+(x)} p_{x,y} = 1$, $X^+(x) = \{y \in X | e = (x, y) \in E\}$. Here $p_{x,y}$ expresses the probability of system's transition from the state x to the state y.

A directed edge $e = (x, y)$ in G corresponds to a stationary control of the system in the state $x \in X$. A sequence of directed edges $\tilde{E} = \{e_0, e_1, ..., e_t, ...\}$, where $e_j = \left(x(t_j), x\left(t_j + \tau_{e_j}\right)\right)$, $j = 0, 1, 2, ...$, determines in G a control of the system with fixed starting state $x(0)$. An arbitrary control in G generates a trajectory $x(t_0), x(t_1), x(t_2), ...$ for which mean integral-time cost by a trajectory can be defined by the formula $f\left(\tilde{E}\right) = \lim\limits_{t\to\infty} \left(\frac{1}{\sigma} \sum_{j=0}^{t-1} c_{e_j}\right)$, where $\sigma = \sum_{j=0}^{t-1} \tau_{e_j}$. The control problem on network (G, X_C, X_N, c, p, x_0) with an average cost optimization criterion consists in finding the stationary strategy s^* that provides the minimal mean integral-time cost by a trajectory.

We define a stationary strategy for the control problem as a map $s : x \to y \in X^+(x)$ for $x \in X_C$. For the arbitrary stationary strategy s the graph $G_s = (X, E_s \bigcup E_N)$, where $E_s = \{e = (x, y) \in E \mid x \in X_C, y = s(x)\}$, corresponds to a Markov process with the probability matrix $P^s = \left(p^s_{x,y}\right)$, where

$$p^s_{x,y} = \begin{cases} p_{x,y}, & if\ x \in\ X_N\ and\ y \in X, \\ 1, & if\ x \in\ X_C\ and\ y = s(x), \\ 0, & if\ x \in\ X_C\ and\ y \neq\ s(x). \end{cases}$$

2 Reduction to the problem with unit time of states' transitions

Our problem can be reduced to the case with unit time of states' transitions on an auxiliary graph $G' = (X',\, E')$. Graph G' is obtained from G, where each directed edge $e = (x,\, y) \in E$ with corresponding transition time τ_e is changed by a sequence of directed edges $e'_1 = (x,\, x^e_1)$, $e'_2 = (x^e_1,\, x^e_2)$, ..., $e'_{\tau_e} = \left(x^e_{\tau_e - 1},\, y\right)$. So, the set of vertices X' of the graph G' consists of the set of states X and the set of intermediate states $XI = \{x^e_i |\ e \in E,\ i = 1, 2, ..., \tau_e - 1\}$, i.e., $X' = X \bigcup XI$. Also, we consider the sets X'_C and X'_N, so that $X' = X'_C \bigcup X'_N$, $X'_C = X_C$ and $X'_N = X' \backslash X_C$. The set of edges E' is defined as

$$E' = \bigcup_{e \in E} \mathcal{E}^e, \quad \mathcal{E}^e = \left\{ (x,\, x^e_1),\ \ (x^e_1,\, x^e_2),\, ...,\ \left(x^e_{\tau_e - 1},\, y\right) \middle|\ (x,\, y) \in E \right\}$$

and the cost function $c' : E' \to \mathbb{R}$ by $c'_{x, x^e_1} = c_{x,y}$ if $e = (x,\, y) \in E$, $c'_{x^e_1, x^e_2} = c_{x^e_2, x^e_3} = ... = c_{x^e_{\tau_e - 1}, y} = 0$. The probability function $p' : E'_N \to [0, 1]$ is defined as follows:

$$p'_{x', y'} = \begin{cases} p_{x,y}, & if\ \ x' = x,\ \ x' \in X_N \subset X'_N\ \ and\ \ y' = x^e_1, \\ 1, & if\ \ x' \in X'_N \backslash X_N. \end{cases}$$

Between the set of stationary strategies $s : x \to y \in X^+(x)$ for $x \in X$ and $s' : x' \to y' \in X'^+(x')$ for $x' \in X'$, there exists a bijective mapping such that the corresponding average costs on G and on G' are the same. So, if s'^* is the optimal stationary strategy of the problem with unit transitions on G' , then the optimal stationary strategy s^* on G is determined by fixing $s^*(x) = y$ if $s'^*(x) = x_1^e$, where $e = (x, y)$. The linear programming algorithms for solving the control problem with a unit time of states' transitions have been developed in [1].

We consider that the network (G, X_C, X_N, c, p, x_0) is perfect, i.e., the graphs G and G_s are strongly connected.

Theorem 1. *Let $\alpha^*_{x',y'}$ $(x' \in X'_C,\, y' \in X')$, $q^*_{x'}$ $(x' \in X')$ be a basic optimal solution of the following linear programming problem:*

Minimize $\bar{\psi}(\alpha, q) = \sum_{x' \in X'_C} \sum_{y' \in X'^+(x')} c'_{x',y'} \alpha_{x',y'} + \sum_{z' \in X'_N} \mu_{z'} q_{z'}$, *subject to*

$$\begin{cases} \sum_{x' \in (X'_C)^-(y')} \alpha_{x',y'} + \sum_{z' \in X'_N} p_{z',y'} q_{z'} = q_{y'}, \quad \forall y' \in X', \\ \sum_{x' \in X'_C} q_{x'} + \sum_{z' \in X'_N} q_{z'} = 1, \\ \sum_{y' \in X'^+(x')} \alpha_{x',y'} = q_{x'}, \quad \forall x' \in X'_C, \\ \alpha_{x',y'} \geq 0,\ \forall x' \in X'_C,\ y' \in X'; \quad q_{x'} \geq 0,\ \forall x' \in X', \end{cases}$$

where $\mu_{z'} = \sum_{y' \in X'(z')} p'_{z',y'} c'_{z',y'}$, $\forall z' \in X'_N$. Then the optimal stationary strategy s'^* can be found on the base of the formula $(s'_{x',y'})^* = \begin{cases} 1, & if\ \alpha^*_{x',y'} > 0 \\ 0, & if\ \alpha^*_{x',y'} = 0 \end{cases}$, *where* $x' \in X'_C,\ y' \in X'^+(x')$.

The stochastic control problem on the network with an arbitrary structure can be reduced to an auxiliary problem on perfect network.

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Euler characteristic of abstract cubes complex

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Abstract

The complex of abstract cubes $\mathfrak{I}^n = \{\mathcal{I}^0, \mathcal{I}^1, \dots, \mathcal{I}^n\}$, defined by m-dimensional cubes family $\mathcal{I}^n = \{I_1^m, I_2^m, \dots, I_{\beta_1}^m\}$, $0 \leq m \leq n$ is studied. Formula for the Euler characteristics for the $\mathfrak{I}^n$ complex is deduced. The formula is expressed by cardinals of the groups of the homologies and by a number of m-dimensional abstract cubes from $\mathfrak{I}^n$.

Keywords: : Abstract cubes, Euler characteristic, group of homologies, group rank, Betti number.

1 Introduction

Euler characteristic is frequently used in algebraic topology and combinatorics of the polyhedrons, which represents a topological invariant – number, which describes form and structure of the space. Initially, Euler characteristic was introduced in the study of polyhedrons, particularly, it was used to study the Platon bodies. In modern mathematics this number appears in study of homologies of spaces in correlation with a list of another invariants.

Formula which connects a number of *3*-dimensional facets of the polyhedrons for the first time was introduced by Leonard Euler in 1752, despite the fact that some its references can be found in manuscripts of Rene Descartes. This formula was lately generalized by H. Poincare for n-dimensional polyhedrons.

Next we will introduce some relations for Euler characteristic, also known as Euler-Poincare characteristic for complex of abstract cubes $\mathfrak{I}^n$. Euler characteristic of the complex $\mathfrak{I}^n$ will be denoted as $\chi(\mathfrak{I}^n)$.

2 Groups in the abstract cubes complex

In paper [1] groups of $\Box-$homologies of the complex $\mathfrak{I}^n$ were defined. Ranks of these groups represent topological invariants of the complex. For the group $\Box^m(\mathfrak{I}^n, Z)$, $0 \leq m \leq n$ the complete system of invariants is defined by rank of this group and torsion coefficients of the group. These invariants play an important role in construction of Euler formula of the complex.

We will mention some properties of the abstract cubes complex $\mathfrak{I}^n = \{\mathcal{I}^0, \mathcal{I}^1, \ldots, \mathcal{I}^n\}$, which are studied in papers [2].

A) Set $\mathcal{L}^m$ of all cubic $m-$dimensional chains from $\mathfrak{I}^n$ form a commutative group in relation to addition operation.

B) Group $\mathcal{L}^m$ has a finite system of generators. Therefore, in $\mathcal{L}^m$ there exist a finite number of $m-$dimensional chains $L^m_{i_1}, L^m_{i_2}, \ldots, L^m_{i_t}$, so that each chain $L^m \in \mathcal{L}^m$ can be expressed as a linear combination $L^m = p_1 L^m_{i_1} + p_2 L^m_{i_2} + \ldots + p_t L^m_{i_t}$, where $p_1, p_2, \ldots, p_t$ are integer numbers.

C) Homologies groups $\Box^0(\mathfrak{I}^n, Z), \Box^1(\mathfrak{I}^n, Z), \ldots, \Box^n(\mathfrak{I}^n, Z)$ of the complex $\mathfrak{I}^n$ has a finite system of generators.

Definition 1. Rank of the group of homologies $\Box^m(\mathfrak{I}^n, Z)$ of the $n-$dimensional complex of abstract cubes $\mathfrak{I}^n$ is called Betti number with dimension m of this complex, and can be denoted as $\rho^m card \Box^m(\mathfrak{I}^n, Z), 0 \leq m \leq n$.

Let the $\mathcal{Z}^m$ and $\mathcal{Z}^m_0$ be a group of $m-$dimensional cycles and group of $m-$dimensional cycles $\Box-$homologous to 0 of the complex I^n.

Theorem 1. For the quotient factor $\mathcal{Z}^m/\mathcal{Z}^m_0$ with dimension m of the cubic complex $\mathfrak{I}^m$, $\rho(\mathcal{Z}^m) = \rho(\mathcal{Z}^m_0) + \rho(\mathcal{Z}^m/\mathcal{Z}^m_0)$ is true.

Theorem 2. Group of $(m-1)-$dimensional cycles and $\Box-$homologuous to zero $\mathcal{Z}^{m-1}_0$ of $n-$dimensional complex of abstract cubes $\mathfrak{I}^n$ is isomorph to the quotient factor $\mathcal{Z}^m/\mathcal{Z}^m_0$.

3 Euler characteristic

We will denote by α_m the number of $n-$dimensional abstract cubes of the complex $\mathfrak{I}^n$.

$$\alpha_m = card\mathcal{I}^m, 0 \leq m \leq n.$$

Theorem 3. If $\mathfrak{I}_1^n, \mathfrak{I}_2^n, \ldots, \mathfrak{I}_p^n$ are connex components of the complex $\mathfrak{I}^n$ and $\mathfrak{I}^m = \mathfrak{I}_1^n \cup \mathfrak{I}_2^n \cup \ldots \cup \mathfrak{I}_p^n$, then the following is true:

$$\square^0(\mathfrak{I}^n, Z) = \underbrace{Z + Z + \ldots + Z}_{N\ times}.$$

Theorem 4. For the complex of abstract cubes $\mathfrak{I}^n$ the following equality is true:

$$\sum_{m=1}^{n} (-1)^m \alpha_m = \sum_{m=1}^{n} (-1)^m \rho_m.$$

Proof: For families $\mathcal{I}^m$ and $\mathcal{L}^m$ the following is true: $\rho(\mathcal{L}^m) = card\mathcal{I}^m = \alpha_m$. Based on Theorem 1, for the groups $\mathcal{L}^m$, $\mathcal{Z}^m$ and $\mathcal{L}^m/\mathcal{Z}^m$ we have:

$$\rho(\mathcal{L}^m) = \rho(\mathcal{Z}^m) + \rho(\mathcal{L}^m/\mathcal{Z}^m). \tag{1}$$

Based on Theorem 2, groups $\mathcal{L}^m/\mathcal{Z}^m$ and $\mathcal{Z}_0^{m-1}$ are isomorph. Therefore, relation (1) can be written as $\rho(\mathcal{L}^m) = \rho(\mathcal{Z}^m) + \rho(\mathcal{Z}_0^{m-1})$.

If we accept that $\rho(\mathcal{Z}_0^{-1}) = 0$, then (2) is true for every m, $0 \leq m \leq n$. Considering definition of the rank of the group and applying Theorem 1 for groups $\mathcal{Z}^m$, $\mathcal{Z}_0^m$ and for quotient factor $\mathcal{Z}^m/\mathcal{Z}_0^m$ we have:

$$\rho(\mathcal{Z}^m) = \rho(\mathcal{Z}_0^m) + \rho(\mathcal{Z}^m/\mathcal{Z}_0^m) = \rho(\mathcal{Z}_0^m) + \rho^m, 0 \leq m \leq n. \tag{2}$$

After substitution of relation (3) in (2) we have:

$$\rho(\mathcal{Z}^m) = \rho(\mathcal{Z}_0^m) + \rho^m + \rho(\mathcal{Z}_0^{m-1}), 0 \leq m \leq n. \tag{3}$$

Considering that $\mathcal{Z}_0^n = \{0\}$, from the mentioned results we obtain $\rho(\mathcal{Z}_0^{-1}) = \rho(\mathcal{Z}_0^n) = 0$.

Because $\rho(L^m) = cardI^m = \alpha_m$, we have that:

$$\alpha^m = \rho^m + \rho(\mathcal{Z}_0^m) + \rho(\mathcal{Z}_0^{m-1}), 0 \leq m \leq n. \tag{4}$$

Multiplying both parts of the relation (5) with $(-1)^m$ and sum by index m, $0 \leq m \leq n$. The next equation is true:

$$\sum_{m=1}^{n} (-1)^m \rho(\mathcal{Z}_0^{m-1}) + \sum_{m=1}^{n} (-1)^m \rho(\mathcal{Z}_0^m) = 0.$$

In condition that: $\mathcal{Z}_0^m = \{0\}$, $\rho(\mathcal{Z}_0^{-1}) = \rho(\mathcal{Z}_0^n) = 0$. Finally, from relation (5) we have that:

$$\sum_{m=1}^{n} (-1)^m \alpha_m = \sum_{m=1}^{n} (-1)^m \rho_m. \blacksquare$$

4 Conclusion

Formula for calculation of Euler characteristic is deduced for a complex of abstract cubes I^n, using invariants of groups of homologies I^n. Using Theorem 4 we ascertain an important relation of the Euler characteristic of the complex used for solving problems with applicative aspect.

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Gromov hyperbolicity and cop and robber game

J. Chalopin, V. Chepoi, P. Papasoglu, T. Pecatte

Abstract

In these notes, I will briefly describe the results which I will present in the talk entitled *Cop and robber game and hyperbolicity*, which are based on paper [3]. Generally speaking, our main result is that Gromov hyperbolicity can be characterized by the cop and robber game with speeds: in a δ-hyperbolic graph a slower cop can always catch a faster robber and, conversely, if in a graph G a slower cop can always catch a faster robber, then the hyperbolicity of G is quadratic in the speed of the cop. As a byproduct of our results, we obtain a constant factor approximation of hyperbolicity of an n-vertex graph in $O(n^2)$ time.

Keywords: geometry, computer science, Gromov hyperbolicity, linear isoperimetric inequality, cop and robber game.

1 Cop and robber game

The cop and robber game originated in the 1980's with the work of Nowakowski, Winkler, Quilliot, and Aigner, Fromme, and since then has been intensively investigated by many authors under numerous versions and generalizations. Cop and robber is a pursuit-evasion game played on finite undirected graphs $G = (V, E)$. Player cop $\mathcal{C}$ attempts to capture the robber $\mathcal{R}$. At the beginning of the game, $\mathcal{C}$ chooses a vertex of G, then $\mathcal{R}$ chooses another vertex. Thereafter, the two sides move alternatively, starting with $\mathcal{C}$, where a move is to slide along an edge of G or to stay at the same vertex. The objective of $\mathcal{C}$ is to capture $\mathcal{R}$ and the objective of $\mathcal{R}$ is to continue evading $\mathcal{C}$. A *cop-win graph* is a graph in which $\mathcal{C}$ captures $\mathcal{R}$ after a finite number of moves. We investigate a natural extension of the cop and robber game in which the cop $\mathcal{C}$ and the robber $\mathcal{R}$ move at speeds $s' \geq 1$ and $s \geq 1$, respectively. The unique difference of this "(s, s')-cop and robber game" and the classical cop and robber game is that at each step, $\mathcal{C}$ can move along a path of length at most s' and $\mathcal{R}$ can move along a path of length at most s not traversing the position occupied by the cop.

2 Gromov hyperbolicity

δ-Hyperbolic metric spaces have been defined by M. Gromov in 1987 via a simple 4-point condition [1, 4]: for any four points u, v, w, x, the two larger of the distance sums $d(u,v)+d(w,x), d(u,w)+d(v,x), d(u,x)+d(v,w)$ differ by at most 2δ. They play an important role in geometric group theory, geometry of negatively curved spaces, and have recently become of interest in several domains of computer science. In case of geodesic metric spaces and graphs, δ-hyperbolicity can be defined in several equivalent ways. We will use the definition of hyperbolicity via the linear isoperimetric inequality, which I recall now.

A *loop* c is a sequence of vertices $(v_0, v_1, v_2, \ldots, v_{n-2}, v_{n-1}, v_0)$ of a graph such that for each $0 \leq i \leq n-1$, either $v_i = v_{i+1}$, or $v_i v_{i+1} \in E$; n is the *length* $\ell(c)$ of c. A *non-expansive map* Φ from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a function $\Phi\colon V \to V'$ such that for all $v, w \in V$, if $vw \in E$, then $\Phi(v) = \Phi(w)$ or $\Phi(v)\Phi(w) \in E'$. For an integer $N > 0$ and a loop c in G, an *N-filling* (D, Φ) of c consists of a 2-connected planar graph D and a non-expansive map Φ from D to G such that the following conditions hold:

1. Φ is a bijection between c and the external face ∂D of D,

2. every internal face of D has at most $2N$ edges.

The *N-area* $\mathrm{Area}_N(c)$ of c is the minimum number of faces in an N-filling of c. A graph G satisfies a *linear isoperimetric inequality* if there exists an $N > 0$ and a K such that any loop c of G has an N-filling such that $\mathrm{Area}_N(c) \leq K \cdot \ell(c))$. The following result of Gromov [4] proven in [1] is the basic ingredient of our proof:

Theorem 1 (Gromov). *If a graph G is δ-hyperbolic, then any loop of G admits a 16δ-filling of linear area. Conversely, if a graph G satisfies the linear isoperimetric inequality $Area_N(c) \leq K \cdot \ell(c)$ for some integers N and K, then G is δ-hyperbolic, where $\delta \leq 108K^2N^3 + 9KN^2$.*

3 Results

Generally speaking, our main result is that Gromov hyperbolicity can be characterized by the cop and robber game with speeds: in a δ-hyperbolic graph a slower cop can always catch a faster robber and, conversely, if in a graph G a slower cop can always catch a faster robber, then the hyperbolicity of G is quadratic in the speed of the cop. We continue with the formal formulations.

A (non-necessarily finite) graph $G = (V, E)$ is called (s, s')-*dismantlable* if the vertex set of G admits a well-order $\preceq$ such that for each vertex v of G there exists another vertex u with $u \preceq v$ such that $B_s(v, G - \{u\}) \cap X_v \subseteq B_{s'}(u, G)$, where $X_v := \{w \in V : w \preceq v\}$. From the definition immediately follows that if G is (s, s')-dismantlable, then G is also (s, s'')-dismantlable for any $s'' > s'$ (with the same dismantling order). In case of finite graphs, the following result holds (if $s = s' = 1$, this is the classical characterization of cop-win graphs):

Theorem 2. *[2] A finite graph G is (s, s')-cop win if and only if G is (s, s')-dismantlable.*

We will also consider a stronger version of (s, s')-dismantlability: a graph G is $(s, s')^*$-*dismantlable* if the vertex set of G admits a well-order $\preceq$ such that for each vertex v of G there exists another vertex u with $u \preceq v$ such that $B_s(v, G) \cap X_v \subseteq B_{s'}(u, G)$. In [2] it was shown that δ-hyperbolic graphs are $(s, s')^*$-dismantlable for some values s, s' depending of δ. For sake of completeness, we recall here these results.

Proposition 1. *[2] For a δ-hyperbolic graph G and any integer $r \geq \delta$, any breadth-first search order $\preceq$ is a $(2r, r + 2\delta)^*$-dismantling order of G.*

We continue with the main results of this section:

Theorem 3. *[3] If a graph G is $(s,s')^*$-dismantlable with $0 < s' < s$, then G is δ-hyperbolic with $\delta = 16(s+s')\left\lceil \frac{s+s'}{s-s'} \right\rceil + \frac{1}{2} \leq 32\frac{s(s+s')}{s-s'} + \frac{1}{2}$.*

Idea of the proof: At the first step, we will establish that for any cycle c of G, $\text{Area}_{s+s'}(c) \leq \left\lceil \frac{\ell(c)}{2(s-s')} \right\rceil$ (for this we use Lemma 1). At the second step, we will present a modified and refined proof of Theorem 2.9 of [1], which will allow us to deduce that if $\text{Area}_{s+s'}(c) \leq \frac{\ell(c)}{2(s-s')} + 1$, then G is $O(\frac{s^2}{s-s'})$-hyperbolic.

Lemma 1. *If a graph G is $(s,s')^*$-dismantlable with $s' < s$ and $c = (v_0, v_1, \ldots, v_{n-1}, v_0)$ is a loop of G of length $n > 2(s+s')$, then c contains two vertices $x = v_p, y = v_q$ with $q - p = 2s \mod n$ such that $d(x,y) \leq 2s'$.*

Proposition 2. *If a graph G is $(s,s')^*$-dismantlable with $s' < s$ and c is a loop of G, then $\text{Area}_{s+s'}(c) \leq \left\lceil \frac{\ell(c)}{2(s-s')} \right\rceil$.*

Proposition 3. *For a graph G and constants $K \in \mathbb{Q}$ and $N \in \mathbb{N}$ such that $2KN$ is a positive integer, if for every cycle c of G, $Area_N(c) \leq \lceil Kl(c) \rceil$, then the geodesic triangles of G are $16KN^2$-slim and G is $(32KN^2 + \frac{1}{2})$-hyperbolic.*

The assertion of Theorem 3 follows from Propositions 2 and 3 by setting $N := s + s'$ and $K := \frac{1}{2N} \cdot \left\lceil \frac{N}{(s-s')} \right\rceil \geq \frac{1}{2(s-s')}$.

Here are the main consequences of Theorem 3:

Corollary 1. *[3] If a graph G is (s,s')-dismantlable with $s' < s$ (in particular, G is a finite (s,s')-cop-win graph), then G is δ-hyperbolic with $\delta = 64s^2$.*

Corollary 2. *[3] If a graph G is $(s,s')^*$-dismantlable with $s - s' \geq ks$ for some constant $k > 0$, then G is $\frac{64s}{k}$-hyperbolic. Conversely, if G is δ-hyperbolic, then G is $(2r, r+2\delta)^*$-dismantlable for any $r > 0$.*

4 Algorithmic consequences

The *hyperbolicity* δ^* of a metric space (X, d) is the least value of δ for which (X, d) is δ-hyperbolic. By a remark of Gromov [4], if the four-point condition in the definition of hyperbolicity holds for a fixed base-point u and any triplet x, y, v of X, then the metric space (X, d) is 2δ-hyperbolic. This provides a factor 2 approximation of hyperbolicity of a metric space on n points running in cubic $O(n^3)$ time. Using fast algorithms for computing (max,min)-matrix products, it was noticed in [5] that this 2-approximation of hyperbolicity can be implemented in $O(n^{2.69})$ time. In this section, we will describe a fast $O(n^2)$ time algorithm for constant-factor approximation of hyperbolicity δ^* of a graph $G = (V, E)$ with n vertices and m edges, assuming that its distance-matrix has already been computed. Our main algorithmic result is the following

Proposition 4. *There exists a constant-factor approximation algorithm to approximate the hyperbolicity δ^* of a graph G with n vertices running in $O(n^2)$ time if G is given by its distance-matrix. The algorithm returns a* 1569*-approximation of δ^*.*

The hyperbolicity δ^* of a graph G is an integer or a half-integer belonging to the list $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots n-1, \frac{2n-1}{2}, n\}$. Without loss of generality, we will assume that $\delta^* \geq \frac{1}{2}$. Due to the space constraints, here we will present an auxiliary algorithm (Algorithm 1) that for a parameter α either ensures that G is $(784\alpha + \frac{1}{2})$-hyperbolic or that G is not $\frac{\alpha}{2}$-hyperbolic. Algorithm 1 is based on Theorem 3 and Corollary 1. Algorithm 1 can be easily transformed into a $O(n^2)\log\delta^*$ time algorithm for approximating δ^*.

Algorithm 1: Approximated-Hyperbolicity($G = (V, E), \alpha$)

Construct a BFS-order $\preceq$ starting from an arbitrary vertex v_0;
For each $v \in V$, let $f_\alpha(v)$ be the vertex at distance $\min\{2\alpha, d(v, v_0)\}$ from v on the path of the BFS-tree from v to v_0 ;
for each $v \in V$ **do**
  if $B_{4\alpha}(v) \cap X_v \not\subseteq B_{3\alpha}(f_\alpha(v))$ **then return** No
return Yes

First, suppose that Algorithm 1 returns Yes. This means that the BFS-order $\preceq$ is a $(4\alpha, 3\alpha)^*$-dismantling order of the vertices of G. Consequently, from Theorem 3, G is $(784\alpha + \frac{1}{2})$-hyperbolic. Now, suppose that the algorithm returns No. This means that there exists a vertex v such that $B_{4\alpha}(v) \cap X_v \not\subseteq B_{3\alpha}(f_\alpha(v))$. From Corollary 1 with $r = 2\alpha$, this implies that G is not $\frac{\alpha}{2}$-hyperbolic and thus $\delta^* > \frac{\alpha}{2}$.

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Dynamic Games in Informational Extended Strategies

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Abstract

In this article we'll have a short view over the dynamic form of the informational extended games. For constructing the dynamic form of the double-sided informational extended games we'll use the dynamic Bayesian games. A parallel backward induction algorithm, for solving informational extended games with informational flow is oriented in two directions, based on Zermelo's algorithm will be given.

Keywords: informational extended games, dynamic form, Bayesian games, backward induction, parallel algorithm.

1 Introduction

Information becomes central as soon as players move in sequence. We can't analyze a dynamic game not taking into consideration what do players know, how much do they know and when do they know that. 4 informational types of games are known:

- **Perfect information:** a game in which a player knows exactly in what node of the game tree is he now, and what is the way that brought him here.
- **Imperfect information:** a game in which some players do not know what actions were chosen by other participants or the nature, earlier in the game.

- **Complete information:** the game in which the knowledge about all participants of the game and about game in general is known to all players.
- **Incomplete information:** the game in which some players have private information about the game, before it begins.

This informational aspects are used to define and construct the dynamic form of the informational extended games.
Informational extended games generated by a one-way directional informational flow, notation $j \overset{\text{inf}}{\rightarrow} i$, are games where player i, and only him, knows exactly what value of the strategy will be chosen by the player j. Besides informational extended games with informational flow oriented in one direction we also know informational extended games with informational flow oriented in two directions, so called double-sided informational extended games. This type of games are denoted by $i \overset{\text{inf}}{\leftrightarrows} j$ and its meaning is that this game is a $j \overset{\text{inf}}{\rightarrow} i$ game for player i and in the same time is a $i \overset{\text{inf}}{\rightarrow} j$ game for player j. For a detailed description of this type of games see [1].

2 The dynamic form of the informational extended games

Finding the dynamic form of one-way informational extended games is quite an easy thing if we have the normal form of the game, an algorithm for this can be found here [2], but how to act in case we have a double-sided informational extended game? At this point a helpful tool will be the Bayesian Games. Double-sided informational extended game $i \overset{\text{inf}}{\leftrightarrows} j$ can be seen as a sequence of games where both, player i and j know the moves of each other but none of them knows the type of the other players. From this point of view the double-sided informational extended games are games of incomplete information. Taking this in consideration, we can represent the double-sided informational extended games in the dynamic form as a Bayesian Game that results in joining the both game trees of $i \stackrel{\inf}{\rightarrow} j$ and $j \stackrel{\inf}{\rightarrow} i$ games and the first move will be done by nature. Nature will assign a random variable to each player which can take values of types for each player and associating probabilities or a probability density function with those types. If the probabilities will be equal to $\frac{1}{2}$, then the game will be of type $i \stackrel{\inf}{\leftrightarrows} j$.

3 A parallel backward-induction algorithm for solving the dynamic informational extended games

Zermelo's backward induction algorithm is used as base, with some modification, to solve the dynamic double-sided informational extended games. We'll divide the initial problem in some subproblems and run the backward induction algorithm for every of these problems, a good idea, in this order of ideas will be to use a parallel machine to solve the double-sided informational extended games. In this order of ideas we'll give some specifications on a parallelized backward induction algorithm. The above algorithm uses the MPI library standard.
Parallel Algorithm for solving double-sided informational extended games in the dynamic form:

1. Use the MPI functions $MPI\_comm\_group, MPI\_Group\_incl$ and MPI_Comm_create to create the new communicator and new group of processes;

2. Use the function $MPI\_Scatter$ to send the data of the sequence games to different processes in the communicator, that processes should determine the set of penultimate nodes $P_t = \{m_i \in M : s(m_i) \in T\}$ and for every penultimate node processes should construct the set of successor nodes $S(m_i) = \{m \in M : s(m) : M \to \{m \cup \emptyset\}\} \subset T$;

3. For every set of successor nodes, determined earlier, use the reduction function MPI_Reduce with reduce operation MPI_MAXLOC to determine the maximum payoff of the player who is to choose also the action that leads to it;

4. The root node will perform the replacements: $T \leftarrow (T \setminus S(m)) \cup \{m\}$, $X \leftarrow X \setminus \{m\}$, $u(m) \leftarrow u(\nu(a|m))$;

5. If the initial nodes of every sequential game were reached, the root process will calculate the payoff of every player as an average value of payoffs obtained in all subgames $u_i = \sum p_i \cdot u_i(m_i)$, else will go to step 2;

Theorem 1. *The backward induction algorithm finds, for every fixed P_t, a Nash equilibrium in the informational extended games with finite number of strategies.*

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The application of modern information technologies in the port activity

Alina Costea

Abstract

Quality of Service (QoS) and Class of Service (CoS) technologies play nowadays an important role in the analysis of network traffic which is highly variable and can be characterized in terms of bandwidth, delay, loss and availability.

Keywords: Quality of service, Class of Service.

1 Introduction

Today the majority of traffic is based on the IP protocol. On the one hand, this is useful because it provides a single traffic protocol and it simplifies the maintenance of hardware and software products. However, IP-based technologies also have many shortcomings. According to the IP protocol, packages are delivered over the network without having a well-determined way. This leads to the fact that the quality of service in such networks cannot be predicted. QoS and CoS technologies serve to guarantee that various applications can be properly maintained in IP networks.

2 The application of QoS (quality of service) standards

Quality of Service is a concept that generally refers to the ability of the network to provide the best service for the traffic of the selected network by various technologies. Quality of service is the ability to provide different priorities to different applications, users or data flows, or to guarantee a certain level of performance to a data flow. QoS (Quality of Service) refers to a wide range of network technologies and techniques. The purpose of QoS is to provide guarantees on the ability of a network to achieve predictable results. Elements of network performance within the scope of QoS often include the availability of the bandwidth (throughput), latency (delay), and error rate. QoS involves prioritization of the network traffic. QoS may be oriented towards a network interface, a particular server or a performance router, or according to specific applications. A monitoring system of the port network should typically be implemented as part of the QoS in order to ensure that the network is performing as desired. QoS is especially important for the new generation of Internet applications such as VoIP, video-on-demand and other consumer services.

3 The application of CoS (class of service) standards

CoS is a way of managing the network traffic by grouping together similar types of traffic (e.g., e-mail, streaming video, voice, large file transfer) and by treating each type as a class with its own level of service priority. Unlike QoS traffic management, CoS does not guarantee a level of service in terms of bandwidth and delivery time. On the other hand, CoS technology is simpler to manage and more scalable like a network that increases in structure and traffic. Class of service is a concept of network input flow divided into different classes. This concept provides class-dependent service for each package of flow, depending on each priority class that belongs to it. CoS provides the continuous setting of priorities for the retransmission structure and ATM traffic over IP networks. In the structure of CoS movement, priorities are set by the Differentiated Services code at the beginning of an IP package. We need to understand how the queuing system works under different configurations and rules. Some features of interest to a standby system operator are the following:

- Queue Length: queue length is the number of elements, in this case the packages that are queuing in a certain location or place waiting to be processed. This is often an indication of how qualitative a queuing system is. The longer the queue length is, the worse the quality of service is from the point of view of the user, although this is not always correct.

- Probability of requirement loss (Loss Probability): If the waiting place where the elements have to wait is limited, which often occurs in the real systems, then the elements that will arrive after the waiting place is occupied by other elements, will be considered lost and they can return in a subsequent moment of time. In packages data systems, the loss of a package might be very unacceptable and customers are worried about the probability of the waiting period. The higher this value is, the worse the quality of service of the system from the customer's perspective is.

- Waiting Time: Waiting time is the time between the arrival of an item in the system and the onset of its servicing. This is the most used quality of the system, by customers. Of course the higher this feature is, the worse the quality of service of the system is, from the customer's point of view.

- System Time: It is the waiting time plus the time needed in order to be served. This is perceived in the same way as the waiting time of the beginning of the service, except when dealing with a preventive system where some elements may be sometimes interruptedly served.

- Work Load: The work load is the time required to process the queuing elements and is equal to the sum of the remaining time for serving the element in service and the service time of all items waiting in a conservation working system. In a conservation working system, the service which is not complete is repeated and no work is removed. A queuing system becomes available and the server becomes inactive at the moment, when the work load is reduced to zero.

- Busy Period: Busy period is the time that begins with the change of the server to a new queue after the previous served queue is available, and ends when the respective queue becomes empty. This measure shows more interest for the ISP that wants to keep its resources fully used. So, the higher this value is, the more satisfied an ISP is. However, if the source used to provide services is human like in a bank, a grocery store, and so on, then there is a time limit when the service provider wants to keep a server busy before the server becomes ineffective.

4 Conclusion

An important role in analysing and optimizing information processes in the port activity is played by the queuing theory, in particular by the priority queuing systems theory. As recently demonstrated, the serving with priority appears as optimum service in the class of all service legalities. Moreover, the diversification of traffic information in priority classes becomes inevitable even in the activity of information flow in the seaport.

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AIRSEC
Integration and Prototypeimplementation Supporting Airport Security – Modelling, Simulation and Optimization of Datafusion Processes in Multilayered Networks

Matthias Dehmer, Dmitrii Lozovanu, Stefan Pickl

Abstract

This paper introduces the joint project AIRSEC which was initiated in 2013 between Moldova, Academy of Science, and Germany, UBw Munich. Many processes from various technical, social and economic areas such as information technology, transportation systems, multi-agent resource management, power distribution etc. can be modelled as multilayered decision processes which lead to multiobjective discrete control and optimal flow problems on dynamic networks. For such kind of problems especially in the context of aviation management, efficient discrete algorithms as well as their robust implementation will be needed in the future. In the project AIRSEC a novel combination of optimization methods and basic game-theoretic concepts will be exploited. This contribution will give an introduction into this joint research project and present first results.

Keywords: Computational Networks, Decision Process, Positional Games.

1 Introduction

Decision support on networks structures becomes more and more important: Therefore, the discrete algorithms of the project AIRSEC will be embedded in a comfortable software environment which will be developed as a web-based solution to determine optimal solutions on such general networks. The gained solutions will be implemented and tested within the CENETIX (Center for network innovation and experimentation at the Naval Postgraduate School in Monterey (USA)) environment. CENETIX do research for the analysis and optimization of IT-based decision procedures. For such an application special visualization techniques on such networks are crucial. The aim of the project is now a novel combination of both expertise – especially in the optimization and visualization of certain multilayered decision problems. New solution procedures will be developed and embedded in real-world scenarios which are gained by the related project RIKOV. In the project RIKOV vignettes are central; they can be considered as bases of such a multilayered decision problem.

2 The Project RiKoV – Vignette Analysis

RiKoV stands for strategic planning and intelligent scenario development for the security of public transport. Efficient measures are developed and embedded in an intelligent decision support tool which is based on an automated generated scenario process. RIKOV is supported by the BMBF (Bundesministerium für Bildung und Forschung) within the special program "Sicherheitsforschungsprogramm der Bundesrepublik Deutschland". RIKOV consists of several institutes and departments from Universities (Karlsruhe Institut für Technologie KIT, Fachhochschule Köln) as well as partners from industry (Airbus Defence and Security). The aim of RiKoV is to develop a comprehensive holistic approach of risk management and strategic planning in critical infrastructures. The approach is based on an intelligent scenario planning tool. The heart of this approach is the generation of suitable vignettes as bases of a multilayered decision process (see Fig. 1).

The vignettes model measurements, priorities and also the recovery time. Our approach is that these processes can be modelled via a specific multilayered decision framework [1-8]. As its bases we take a multilayered network structure. Measurements, capabilities and recovery times are now simulated via special visualization techniques on

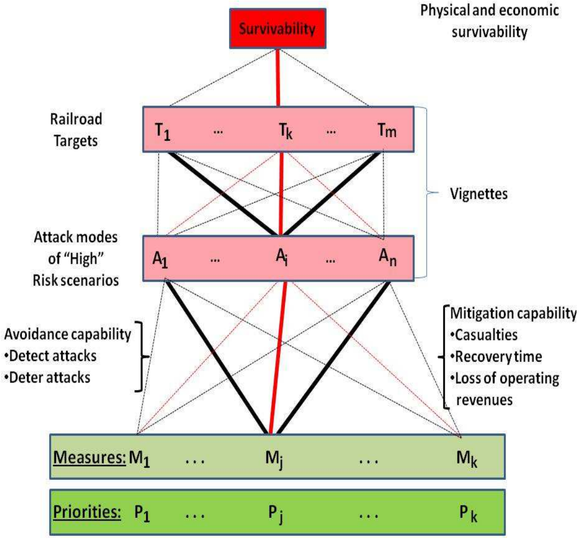

Figure 1. Structure of Vignette Analisys

these networks. For this purpose we will introduce special graph measures. The success of this approach depends on the research outcome and feasibility of the preceding steps. If successful, this task will lead to novel techniques that might be very useful to localize special properties within the process. Depending on the used models including graph matching and other analysis methods, these properties can mark special locations with critical potential or regions with high probabilities for quick recovery existence.

3 Multilayered Decision Framework – Visualization

This special structure should be used now as a general multilayered decision framework for the AIRSEC project. The construction of critical scenarios according to the vignettes takes into account past and possible future scenarios in the context of aviation management. Those scenarios will be simulated and visualized using agent-based modelling dynamical systems data farming.

Appropriate security measures will be identified and evaluated to eliminate unacceptable risks and attenuate unwanted consequences, taking into consideration social values, legal restrictions, and an analysis of cost-effectiveness. For this, the social acceptances of security measures as well as the challenge of limited financial budgets are fused to the IT-based risk management (and stochastic vulnerability analysis) in order to afford the best constellation of measures to the identified risks. The correlation between measures and risks will be presented in a multicriteria risk matrix (vulnerability, threat, consequences) which will be correlated with the graph measures. This new approach will be embedded in a comfortable decision support tool.

4 New Class of Stochastic Poistional Games

This approach leads us also to a new class of so-called stochastic games. In previous works an essentially new class of stochastic positional games was formulated and studied applying the game-theoretical concept to Markov decision problems with average and expected total discounted costs optimization criteria. To formulate this class of games it is assumed that Markov decision processes may be controlled by several actors (players). The set of states of the system in such processes is divided into several disjoint subsets which represent the corresponding position sets of the players. Each player has to determine which action should be taken in each state of his position set in order to minimize (or maximize) his own average cost per transition or the expected total discounted cost. For the stochastic discrete optimal control problems with infinite time horizon this approach is developed in a similar way and a new class of stochastic positional games on networks is obtained. Nash equlibria can be derived and an elaboration of the algorithms for determining the optimal stationary strategies of the players is possible. In [9] Nash equilibria conditions for the stochastic positional games with average and discounted payoff functions of the players are formulated and proved and algorithms for determining the optimal strategies for different classes of games are developed. These results are specified for antagonistic stochastic positional games and algorithms for determining the optimal strategies of the players are gained.

We show that the obtained results generalize the well known results for deterministic positional games and new conditions for determining the solutions of the problems can be derived. Moreover, we show that the considered class of stochastic positional games can be used for studying cyclic games and Shapley stochastic games. New polynomial time algorithms for deterministic antagonistic positional games are described. The algorithms for determining the optimal strategies of the players in deterministic cases are developed for a more general class of positional games on networks.

Both approaches will be combined in the AIRSEC project.

5 Conclusion

The presented approach helps especially decision makers of aviation transportation providers to decide and manage the implementation of different security measures. These measures are visualized on a graph theoretic structure and embedded in a decision support tool. The holistic risk management supports the development to a customized optimal security, including prevention and civil protection in times of tension between the technical and organizational maximum attainable on the one hand, and the striving of economically meaningful concepts derived from the social values of security on the other hand. The analysis of positional games and the visualization of certain network processes is a novel approach in this context of aviation management.

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B-stable subgraphs in undirected graphs

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Abstract

In this article special structures of stable subgraphs and some of their most important properties that will be necessary for calculation of the number of transitive orientations of an undirected graph are described.

Keywords: stable subgraph, B-stable subgraph, transitive orientation, non-triangulated chain.

1 Introduction

In order to solve many theoretical and applicative problems, transitive orientation of the undirected graphs was studied by many mathematicians [1], [2], [3]. There was obtained a number of theoretical results and algorithms related to establish ownership of an arbitrary graph to this class, and building a transitive orientation. This type of graphs can be studied using special structures: stable subgraphs and non-triangulated chain. We will recall that a directed graph $\overrightarrow{G} = (X; \overrightarrow{U})$ transitively orientated if there is satisfied transitive relation $[x, y] \in \overrightarrow{U_G} \& [y, z] \in \overrightarrow{U_G} \Longrightarrow [x, z] \in \overrightarrow{U_G}$ for every three vertexes $x, y, z \in X_G$. An undirected graph for which we assign a certain orientation of the edges so that we get a transitive oriented graph will be called transitive orientable graph.

2 B-stable subgraphs

Definition 1. [4] Subgraph A of the undirected graph $G = (X; U)$ will be called stable subgraph if for every vertex $x \in X_G \setminus X_A$ one of the following conditions will be satisfied: $x \sim y$ or $x \nsim y$ for $\forall y \in X_A$.

Definition 2. Stable subgraph F of the undirected graph $G = (X;U)$ will be called B-stable subgraph if for every stable subgraph M of G one of the following conditions is satisfied:

1. $X_F \cap X_M = \emptyset$;

2. $X_F \subseteq X_M$.

Observation 1. Complete graph K_n does not contain B-stable subgraphs.

This is the consequence of the fact that intersection of every subset of vertexes which defines a stable subgraph from K_n is a nonempty set.

Lemma 1. If A and B are B-stable subgraphs, then $X_A \cap X_B = \emptyset$.
Proof: Suppose contrary. Let $F = A \cap B$. In this case, by lemma 3 from [5] it is obtained that $X_A \setminus X_F$ defines a stable subgraph. It can be observed that $X_A \setminus X_F \subset X_A$. So, there is a contradiction with the fact that A is a B-stable subgraph. This contradiction proves that intersection of the vertexes of two B-stable subgraphs is an empty set.■

Theorem 1. If F is a B-stable subgraph of the graph $G = (X;U)$, and $x \in X_G \setminus X_F$ is a vertex adjacent to the set X_F, then for every transitive orientation $\overrightarrow{G}$ only one of the following holds:

1. $[x, y] \in \overrightarrow{U_G}, \forall y \in X_F$;

2. $[y, x] \in \overrightarrow{U_G}, \forall y \in X_F$.

Proof: Suppose contrary. Let vertexes $y, z \in X_F$, so in the transitive orientation $\overrightarrow{G}$ the following relation is satisfied:

$$[y, x] \in \overrightarrow{U_G} \& [x, z] \in \overrightarrow{U_G}. \tag{1}$$

It can be easily observed that for every vertex $t \in X_G \setminus X_F, t \neq x$ as $[t, x] \in U_G$, implies that there is an edge $[t, w], \forall w \in X_F$. By the observation 1 it is clear that G is not a complete graph.

Lets suppose a vertex $s \in X_G$ for which there exists vertex $v \in X_G$, where the relation $v \neq s$ is satisfied, and $[v, s] \notin \overrightarrow{U_G}$.

There are possible the following situations:

1. $s \in X_F, v \in X_F$;

2. $s \in X_F, v \notin X_F$;

3. $s \notin X_F, v \in X_F$;

4. $s \notin X_F, v \notin X_F$.

Next, every situation will be analyzed particularly.

a) Because $s \in X_F$, it results that $[x, s] \in U_G$. Based on relation (1) at least one of the following relations holds $[z, s] \in U_G$ or $[y, s] \in U_G$. Suppose that

$$[z, s] \in U_G. \tag{2}$$

If $v = y$, because of the fact that every vertex $t \in X_G \setminus X_F$ adjacent to x is also adjacent to every vertex in X_F, including z, it results that the set defined by vertexes $\{x, z\} \subset X_F$, generates a stable subgraph. This fact contradicts assumption that F is a B-stable subgraph.

In the case that $v \neq y$, by relation (1) and (2) it is obtained that $[v, x] \in \overrightarrow{U_G}$. This implies that $[v, z] \in \overrightarrow{U_G}$. It can be easily shown that the set $\{x, z\} \subset X_F$ also generates a stable subgraph. So, there is contradiction of the fact that F is a B-stable subgraph.

b) If $x \in X_G \setminus X_F$ and $[v, s] \notin U_G$, where $s \in X_F$, then by relation (1) it results that $[v, x] \notin U_G$. If there is a vertex $w \in X_G \setminus X_F$ adjacent to the set X_F, then it can be easily observed that set of vertexes $(X_F \setminus \{y, s\}) \cup \{x, w\}$ defines a stable subgraph. It is clear that intersection of this set of vertexes with X_F is an empty set. This fact contradicts that F is a B-stable subgraph.

c) The case, when $s \notin X_F, v \in X_F$, can be expressed in the same way as the case *b)* by replacing name of variables.

d) Because $s, v \in X_G \setminus X_F$ and $[s, v] \notin U_G$, then the following situations are possible:

1. $s \sim u, \forall u \in X_F$;

2. $s \nsim u, \forall u \in X_F$.

1. If $s \sim u, \forall u \in X_F$, then by relation (1) it can obtained that $[y, s] \in \overrightarrow{U_G} \& [s, x] \in \overrightarrow{U_G}$. So, every vertex adjacent to s will be adjacent to $u, \forall u \in X_F$ too.

2. In case that $s \nsim u, \forall u \in X_F$, then by (1) it can be observed that $[s, u] \notin U_G$, where $u \in \Gamma(w), \forall w \in X_F$.

Because $[v, s] \notin U_G$, then the same procedure as for vertex v can be applied. And so, it can be obtained that $(X \setminus \{y\}) \cup A$, where $A = \{x | [x, y] \in U_G, \forall y \in X_F\}$ is a stable subgraph.

All the contradictions mentioned above lead to the proof of the theorem. ■

3 Conclusion

B-stable subgraphs will be used in further studies of transitively orientable graphs, in order to find an algorithm of construction of the transitive orientation and construction of the formula for number of transitive orientations of the graph.

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Probabilistic Approach to Stochastic and Agent-Based Computational Models

Ghennadii Gubceac, Florentin Paladi

Abstract

The application refers to the generic stochastic model for crystal nucleation which is useful to depict the impact of interface between the nucleus considered as a cluster of a certain number of molecules and the liquid phase for the enhancement of the overall nucleation process.

Keywords: Agent-Based Models; Stochastic Processes; Complex Systems.

1 Introduction

In general, agent-based modeling is currently a technique widely used to simulate complex systems in computer science and social sciences. On the other hand, a Markovian process is a stochastic process whose future probabilities are determined by its most recent values. The agent-based computational models (ABM) fits well this description, except the cases when decisions are dependent on the state of the systems of more than one step ago, that is the case when ABM agents experience learning, adaptation, and reproduction [1].

In this study, the application refers to the generic stochastic model for crystal nucleation which is useful to depict the impact of interface between the nucleus considered as a cluster of a certain number of molecules and the liquid phase for the enhancement of the overall nucleation process. It is generally known that first-order phase transitions occur by nucleation mechanism, and both the nucleus, a cluster of atoms or molecules, and the nucleation work, an energy barrier to the phase transition, are basic thermodynamic quantities in the theory of nucleation. However, the critical nucleus formation is statistically a random event with a probability largely determined by the nucleation work which increases with the subnuclei size [2]. The traditional differential equation modeling is not the alternative to agent-based models; only a set of differential equations, each describing the dynamics of one of the system's constituent units, is an agent-based model [3].

2 A probabilistic approach to the crystal nucleation process

The application refers to the nucleation process, a widely spread phenomenon in both nature and technology, which may be considered as a representative of the aggregation phenomena in complex systems. Let's consider N atoms which can be in 3 different states (*cluster*, *liquid* and their *interface*), and can perform 4 possible moves: *liquid* to *interface*, *interface* to *liquid*, *interface* to *cluster*, and *cluster* to *interface*. One can identify 4 different combinations denoted by probabilities $p_1 \dots p_4$. That is, drawing randomly one particle, it will be of type i with probability p_i. Let $N = 1, 2, \dots, \infty$, be the total number of atoms in the system, and $\{n_1, n_2, n_3, n_4\}$ are their partition into 4 subsets. Each subset can be called cluster, and the process itself – clustering. Schematic distribution of particles in clusters is shown in Fig. 1. This diagram illustrates a potential process for the generation of a crystal nucleus in the course of irreversible structural relaxation of the nonequilibrium supercooled liquid. The number of possible partitions in this case is

$$P(N, m = 4) = \frac{1}{3\,!} \prod_{i=1}^{3} (N + i),$$

where $n_i = \overline{0,\, N}$, $i = \overline{1,\, 4}$ and $\sum_{i=1}^{4} n_i = N$. For example, in a system of N=1,000 atoms, $P(N$,m=4) equals to 167,668,501! According-

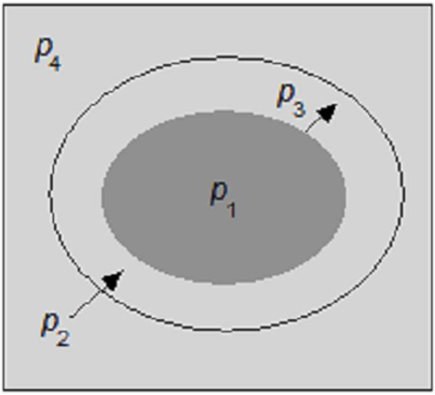

Figure 1. Schematic diagram illustrating the distribution of particles with corresponding share in cluster, p_1, liquid, p_4, and at the interface, p_2 and p_3. The darkness represents the degree of order in the molecular arrangements, and the encircled part stands schematically for a more ordered cluster, namely a crystal nucleus.

ingly, the number of repeated computer runs in an ABM model would be very large due to different possible partitions. But we are able to overcome this problem by developing similar stochastic mathematical models which can describe exactly the results of the agent-based computational models, and, finally, by bridging the gap between ABM modeling and stochastic processes.

3 Conclusion

A useful aggregation procedure for representing the three-dimensional distribution of states, and a general formula that describes clustering process among interacting agents in heterogeneous populations are proposed. In particular, we obtained that different distributions of states can lead to the same point inside the sphere around the origin, i.e. different microscopic partitions can generate the same aggregate result.

It is proven that while the number of particles at the liquid-cluster interface increases, the stability of the entire system decreases simultaneously, and the nucleus formation would be definetely enhanced due to the displacement of the bifurcation point in the region of smaller clusters [4].

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Using INS to correct SAR data on the fly

Maxim Guryanov, Alexander Prokofyev

Abstract

Airplane Synthetic Aperture Radar (SAR) data has phase errors caused by turbulence. These errors affect the resolution of obtained SAR images. In this paper we present a new technique to detect and correct phase errors in SAR data based on inertial navigation system (INS) data and mathematical modelling of the airplane's flight. All computations can be performed in real time. This approach provides better SAR image quality without using autofocus algorithms.

Keywords: Synthetic Aperture Radar (SAR), autofocus.

1 Introduction

The main characteristic of a SAR image is the image resolution. Different autofocus algorithms like Coherence Map Drift (CMD) or Phase Gradient Autofocus (PGA) can improve quality of SAR images using raw SAR data. The problem is computational complexity.

One of the reasons for the low quality of SAR images is the difference between the actual trajectory of flight and the trajectory the mathematical model is based on. The common procedure of image creation is based on a linear trajectory.

We present an algorithm for using INS data to negate geometric distortions of the flight by correcting phase errors in SAR data. After that, regular procedures of image creation based on a linear trajectory can be used.

2 Mathematical model of the flight

The common model with a linear trajectory is used. Let $Z(n)$ be the difference between actual and expected flight altitudes. Let n refer to the n-th range bin. $Y(n)$ is the difference between actual and expected range positions. Let the D_m be the distance from the airplane to scatter at the range bin m. β_m is the angle of sight to the m-th row. The model looks like it is shown in Fig.1.

$Z(n)$ and $Y(n)$ can be obtained from the INS. The main idea is based on the fact that precision of inertial systems is low for big periods of time and high for small periods. Position errors are taken at the rate of pulse repetition frequency (PRF) of SAR. So we have error estimation from INS for every azimuth bin.

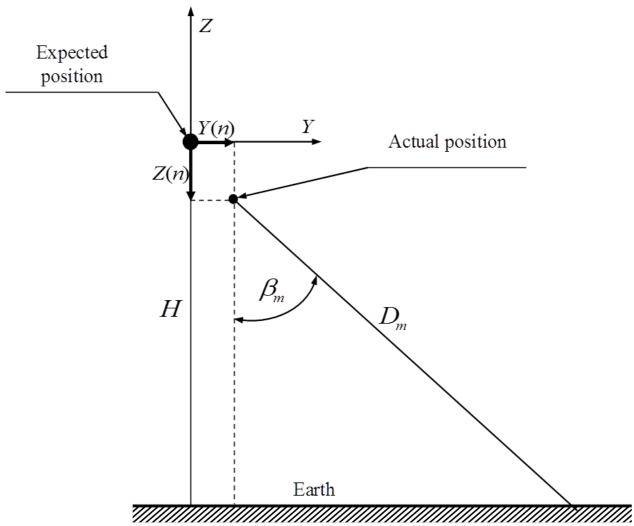

Figure 1. Illustration of positional errors.

3 SAR data correction

We can use $Z(n)$ and $Y(n)$ to calculate the phase error for every range bin. The phase of complex signal obtained from point scatter at azimuth bin n and range bin m is:

$$\Phi_{n,m} = -\left(\frac{4\pi \cdot D_m}{\lambda}\right). \tag{1}$$

The distance D_m has an error:

$$\Delta D(n,m) = -\left(Z(n)\cos(\beta_m) - Y(n)\sin(\beta_m)\right). \tag{2}$$

Distance errors may cause range bin shifting. We can estimate the actual range bin m_{new} for scatter at (n,m) as it is shown in Eq. (3)

$$m_{new} = m + \left(\frac{2 \cdot \Delta D(n,m)}{\lambda}\right). \tag{3}$$

The actual phase for every signal can be found as it is shown in Eq. (4)

$$\Phi_{new}(n,m) = \Phi_{n,m} - \left(\frac{4\pi \cdot \Delta D(n,m)}{\lambda}\right). \tag{4}$$

New SAR data can be formed in real time. All these results can be used only if errors are small:

$$Z(n) \ll H, Y(n) \ll H. \tag{5}$$

4 Conclusion

The proposed method was successfully applied in UAV on-board SAR-processing system. It is possible to obtain poor SAR images in real time with a minimum time delay. The high resolution SAR images can be formed from resulting SAR data with other autofocus algorithms.

The proposed method cannot totally replace regular autofocus algorithms. The modern inertial navigation system does not have sufficient precision. This method cannot remove other reasons for low SAR image resolution, like a high Doppler rate.

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Parallel algorithm to solve two person game with perfect information

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Abstract

In this article for solving the informational extended games we apply the Selten-Harsanyi principle. We elaborate the parallel algorithm to determine the Bayes-Nash equilibrium profile in informational extended games.

Keywords: informational extended games, complete and perfect information, Bayes-Nash equilibrium.

1 Two person game with informational extended strategies

Let $\Gamma = \langle I = \{1, 2\}, X, Y, H_i : X \times Y \to \mathbb{R} \rangle$ be the normal form of the static noncooperative games with **complete and perfect information**, where I is the set of players, X, Y are the sets of available alternatives of the player *1* and player *2*, $H_i : X \times Y \to R, i \in I$ is the payoff function of the player $i \in I$. The players know exactly their and of the other players payoff functions and they know the sets of strategies. Players *1* and *2* know what kind of the strategy will be chosen by each of other. These conditions stipulate that we can use the informational extended strategies generated by a two-directional informational flow[1]. In the general case the set of the informational extended strategies of the player *1* (respectively *2*) is the set of the functions $\Theta_1 = \{\theta_1 : Y \to X\}$ (respectively $\Theta_2 = \{\theta_2 : X \to Y\}$). The game is played as follows: independently and simultaneously each player chooses the informational extended strategy $\theta_1 \in \Theta_1$ and

$\theta_2 \in \Theta_2$, after that the players *1* and *2* calculate the value of the payoff $H_i(\theta_1(y), \theta_2(x))$, and with this the game is finished.

Remark. *The game described above will be denoted by $G(1 \leftrightarrows 2)$ and is the game with **incomplete information** because the player 2 (for example) does not know what kind of the informational extended strategy $\theta_1 \in \Theta_1$ will be chosen by the player 1, and so the player 1 generates the uncertainty of the player 2 about the complete structure of the payoff function $H_2(\theta_1(y), \theta_2(x))$ in the game with non informational extended strategies. Therefore to solve the game $G(1 \leftrightarrows 2)$ the Harsanyi -Selten principle [2] will be used.*

2 Converting the two person game with informational extended strategies to Harsanyi - Selten game

Let $\Delta_1 = \{\delta_1^j, j \in J_1\}$ ($\Delta_2 = \{\delta_2^k, k \in J_2\}$ respectively) be the set of types for the player 1 (player 2) that means: player 1 (player 2) is of the type δ_1^j (of the type δ_2^k) if he, choosing the $\theta_1^j \in \Theta_1$ ($\theta_2^k \in \Theta_2$) informational extended strategy, generates for the player 2 (for player 1) uncertainty in the structure of its payoff function. Denote by $p(\delta_2^k/\delta_1^j)$ ($q(\delta_1^j/\delta_2^k)$ respectively) a probability for the player 1 (for player 2) that means the following: if the player 1 (player 2) chooses the strategy θ_1^j (strategy θ_2^k), than he believes that the player 2 (player 1) with the $p(\delta_2^k/\delta_1^j)$ ($q(\delta_1^j/\delta_2^k)$ respectively) chooses the strategy θ_2^k (strategy θ_1^j). The set of all range of the informational extended strategies of the players will be denoted by $\tilde{X}_j = \left\{\tilde{x}_j \in X : \tilde{x}_j = \theta_1^j(y), \forall y \in Y\right\}$ for all $j \in J_1$ and $\tilde{Y}_k = \left\{\tilde{y}_k \in Y : \tilde{y}_k = \theta_2^k(x), \forall x \in X\right\}$ for all $k \in J_2$. Using these notations we can construct the normal form $\Gamma_B^* = \left\langle J, \{R_j\}_{j \in J}, \{U_j\}_{j \in J}\right\rangle$ of the Selten-Harsanyi game with complete and imperfect information, that is associated with the game $G(1 \leftrightarrows 2)$. Here the set of type-players $J = \left\{j = (i, \delta_i^j), i = 1, 2, j = 1, ..., m_1 + m_2\right\}$ is equal to the sets of all informational extended strategies of the players, $J = J_1 \bigcup J_2$. The sets of the pure strategies are $R_j = \widetilde{X}_j$, for $j \in J_1$ and $R_j = \widetilde{Y}_j$ for $j \in J_2$. The payoff function for all type-players $j = (1, \delta_1^j)$, $j \in J_1$ is $U_j\left(\tilde{x}_j, \{\tilde{y}_k\}_{k\in J_2}\right) = \sum_{k\in J_2} p\left(\delta_2^k|\delta_1^j\right) H_1\left(\tilde{x}_j, \tilde{y}_k\right)\ \forall \tilde{x}_j \in \tilde{X}_j, \tilde{y}_k \in \tilde{Y}_k$. In the similar mode the payoff function for all type-players $j = (2, \delta_2^j)$, $j \in J_2$ is $U_j\left(\{\tilde{x}_k\}_{k\in J_1}, \tilde{y}_j\right) = \sum_{k\in J_1} p\left(\delta_1^k|\delta_2^j\right) H_2\left(\tilde{x}_k, \tilde{y}_j\right)\ \forall \tilde{x}_k \in \tilde{X}_k, \tilde{y}_j \in \tilde{Y}_j$. The game Γ_B^* is played as follows: independently and simultaneously each type-player $j = (i, \delta_i^l)$ chooses the strategy $r_j \in R_j$, after that each player calculates the payoff values $U_j(\cdot)$ and with this the game is finished. Denote by $NE\left[\Gamma_B^*\right]$ the set of all equilibrium profiles in the game Γ_B^*.

Definition. *For all fixed probabilities* $p(\cdot), q(\cdot)$ *strategy profile* $(x^*, y^*) \equiv (x^*(p), y^*(q))$, $x^* \in X$, $y^* \in Y$, *for which the conditions* $\widetilde{x}_j^* = \theta_1^j(y^*)\ \forall j \in J_1$ *and* $\widetilde{y}_k^* = \theta_2^k(x^*)\ \forall k \in J_2$ *are fulfilled, is called the Bayes-Nash equilibrium profile in non informational extended strategies generated by the* $\left(\widetilde{x}_j^*, \widetilde{y}_k^*\right) \in NE\left[\Gamma_B^*\right]$.

Denote by $BN\left[G\left(1 \leftrightarrows 2\right)\right]$ the set of all Bayes-Nash equilibrium profiles in the game $G\left(1 \leftrightarrows 2\right)$. According to [3] we can prove the following theorem.

Theorem. *Let the game* Γ *satisfy the following conditions:*

1) *the* X *and* Y *are non-empty compact and convex subsets of the finite-dimensional Euclidean space;*

2) *the functions* $\theta_1^j, \forall j \in J_1$ *and* θ_2^k, $\forall k \in J_2$ *are continuous on* Y *(on* X *respectively) and functions* H_1, H_2 *are continuous on* $X \times Y$;

3) *the functions* $\theta_1^j, \forall j \in J_1$ *(*θ_2^k, $\forall k \in J_2$ *respectively) are quasi-concave on* Y *(on* X *respectively), the functions* H_1 *(*H_2 *respectively) are quasi-concave on* X *(*Y *respectively) and monotonically increasing on* $X \times Y$.

Then $BN\left[G\left(1 \leftrightarrows 2\right)\right] \neq \emptyset$.

3 Parallel algorithm to solve two person game with perfect information

If in the game Γ the sets of strategies X and Y of the player 1 and 2 are at most countable, H_1 and H_2 are the discrete payoff functions, then, using the mixed Distribute Memory Model and Shared Memory Model parallel cluster, for all fixed probabilities $p_k \equiv p(\delta_2^k/\delta_1^j)$ and $q_j \equiv q(\delta_1^j/\delta_2^k)$ we elaborate the following parallel algorithm to determine the Bayes-Nash equilibrium profile $(x^*(p_k), y^*(q_j))$ in game Γ:

a) using MPI programming model and open source library ScaLAPACK-BLACS, scatter for MPI process (k, j) all of the necessary data to calculate $(x^*(p_k), y^*(q_j))$ for fixed p_k and q_j;

b) for all fixed functions θ_1^j, θ_2^k and probabilities p_k, q_j each (k, j) MPI process using OpenMP programming model and an open source library ScaLAPACK, calculate the Nash equilibrium profile $\left(\widetilde{x}_j^*, \widetilde{y}_k^*\right)$ in the game Γ_B^*, after that calculate the strategy profile $(x^*(p_k), y^*(q_j))$, where $x^*(p_k) = \left[\theta_2^k\right]^{-1}(\widetilde{y}_k^*)$ and $y^*(q_j) = \left[\theta_1^j\right]^{-1}\left(\widetilde{x}_j^*\right)$.

c) using open source library ScaLAPACK-BLACS, the root MPI process is gathering the strategy profiles $(x^*(p_k), y^*(q_j))$ from (k, j) MPI process.

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Probability Distribution for the Sum of Two Independent Telegraph Processes

Alexander D. Kolesnik

Abstract

Closed-form expressions for the transition density $\varphi(x, t)$ and the probability distribution function $\Phi(x, t) = \Pr\{S(t) < x\}, \ x \in \mathbb{R}, \ t > 0$, of the sum $S(t) = X_1(t) + X_2(t)$ of two independent Goldstein-Kac telegraph processes $X_1(t), X_2(t)$ at arbitrary time instant $t > 0$, are presented. A governing third-order partial differential equation for $\varphi(x, t)$ is also given.

Keywords: telegraph process, telegraph equation, transition density, probability distribution function, sum of telegraph processes, hypergeometric functions

Let $X_1(t), X_2(t)$ be two independent Goldstein-Kac telegraph processes on the real line $\mathbb{R}$ that, at the initial time instant $t = 0$, simultaneously start from the origin $0 \in \mathbb{R}$. Both processes are developing with the same speed c and are controlled by two independent Poisson processes of the same rate $\lambda > 0$.

Consider the sum $S(t) = X_1(t) + X_2(t), \ t > 0$, of these processes. Let $\varphi(x, t), \ x \in \mathbb{R}, \ t \geq 0$, denote the transition density of process $S(t)$ treated as a generalized function. Density $\varphi(x, t)$ is concentrated in the close interval $[-2ct, 2ct]$ and consists of two components. The singular component is concentrated at three points $0, \pm 2ct$ and corresponds to the case when no Poisson events occur up to time t. The remaining part $(-2ct, 0) \cup (0, 2ct)$ of the interval $[-2ct, 2ct]$ is the support of the absolutely continuous part of density $\varphi(x, t)$ corresponding to the case when at least one Poisson event occurs up to time instant t.

The explicit form of $\varphi(x, t)$ is given by the following theorem.

Theorem 1. *The density* $\varphi(x,t)$, $x \in \mathbb{R}$, $t \geq 0$, *has the form:*

$$\varphi(x,t) = \frac{e^{-2\lambda t}}{2}\,\delta(x) + \frac{e^{-2\lambda t}}{4}\,[\delta(2ct+x) + \delta(2ct-x)]$$
$$+ \frac{e^{-2\lambda t}}{2c}\left[\lambda I_0\left(\frac{\lambda}{c}\sqrt{4c^2t^2-x^2}\right) + \frac{1}{4}\,\frac{\partial}{\partial t} I_0\left(\frac{\lambda}{c}\sqrt{4c^2t^2-x^2}\right)\right.$$
$$\left. + \frac{\lambda^2}{2c}\int_{|x|}^{2ct} I_0\left(\frac{\lambda}{c}\sqrt{\tau^2-x^2}\right)d\tau\right]\Theta(2ct-|x|),$$

where $\delta(x)$ *is the Dirac delta-function,* $\Theta(x)$ *is the Heaviside unit step function and*

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{z}{2}\right)^{2k}$$

is the modified Bessel function of zero order.

Consider the function

$$g(x,t) = \left(\frac{\partial}{\partial t} + 2\lambda\right)\varphi(x,t). \tag{1}$$

Here $\partial/\partial t$ means differentiation in t of the generalized function $\varphi(x,t)$. The unexpected and amazing fact is that this function satisfies the Goldstein-Kac telegraph equation with doubled parameters $2c$ and 2λ.

Theorem 2. *Function* $g(x,t)$ *defined by Eq.*(1) *satisfies the telegraph equation*

$$\left(\frac{\partial^2}{\partial t^2} + 4\lambda\frac{\partial}{\partial t} - 4c^2\frac{\partial^2}{\partial x^2}\right)g(x,t) = 0. \tag{2}$$

From Eq.(1) and Eq.(2) it follows that the transition density $\varphi(x,t)$ of process $S(t)$ satisfies the third-order hyperbolic partial differential equation

$$\left(\frac{\partial}{\partial t} + 2\lambda\right)\left(\frac{\partial^2}{\partial t^2} + 4\lambda\frac{\partial}{\partial t} - 4c^2\frac{\partial^2}{\partial x^2}\right)\varphi(x,t) = 0. \tag{3}$$

Note that differential operator in Eq.(3) represents the product of the Goldstein-Kac telegraph operator with doubled parameters $2c,\ 2\lambda$ and shifted time differential operator. This interesting fact means that, while the densities of two independent telegraph processes $X_1(t)$ and $X_2(t)$ satisfy the second-order telegraph equation (see [1, Section 2.3]), their convolution (that is, the density $\varphi(x,t)$ of the sum $S(t) = X_1(t) + X_2(t)$) satisfies third-order equation Eq.(3). One can, however, check that $\varphi(x,t)$ is not the fundamental solution of equation Eq.(3).

Under Kac's scaling condition $c \to \infty,\ \lambda \to \infty,\ (c^2/\lambda) \to \rho,\ \rho > 0$, equation Eq.(3) transforms into the heat equation

$$\left(\frac{\partial}{\partial t} - \rho \frac{\partial^2}{\partial x^2} \right) u(x,t) = 0.$$

This means that $S(t)$ is asymptotically a homogeneous Wiener process with zero drift and diffusion coefficient $\sigma^2 = 2\rho$.

In the next theorem we present the closed-form expression for the probability distribution function

$$\Phi(x,t) = \Pr \{ S(t) < x \}, \qquad x \in \mathbb{R}, \quad t > 0,$$

of process $S(t)$.

Theorem 3. *The probability distribution function $\Phi(x,t)$ has the form:*

$$\Phi(x,t) = \begin{cases} 0, & \text{if } x \in (-\infty,\ -2ct], \\ G^-(x,t), & \text{if } x \in (-2ct,\ 0], \\ G^+(x,t), & \text{if } x \in (0,\ 2ct], \\ 1, & \text{if } x \in (2ct,\ +\infty), \end{cases} \qquad t > 0,$$

where functions $G^{\pm}(x,t)$ are given by the formula:

$$
\begin{aligned}
G^{\pm}(x,t) = \frac{1}{2} \pm \frac{e^{-2\lambda t}}{4} \cos\left(\frac{\lambda x}{c}\right) \\
&+ \frac{\lambda x e^{-2\lambda t}}{2c} \left[\sum_{k=0}^{\infty} \frac{(\lambda t)^{2k}}{(k!)^2} \left(1 + \frac{\lambda t}{2k+2}\right) F\left(-k, \frac{1}{2}; \frac{3}{2}; \frac{x^2}{4c^2t^2}\right) \right. \\
&\left. + \sum_{k=0}^{\infty} \frac{(\lambda t)^{2k+1}}{(k!)^2\ (2k+1)}\ {}_3F_2\left(-k, -k-\frac{1}{2}, \frac{1}{2};\ -k+\frac{1}{2}, \frac{3}{2};\ \frac{x^2}{4c^2t^2}\right)\right].
\end{aligned}
$$

Here

$$F(\alpha,\beta;\gamma;z) \equiv\ {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k\ (\beta)_k}{(\gamma)_k}\ \frac{z^k}{k!}$$

is the Gauss hypergeometric function and

$${}_3F_2(\alpha,\beta,\gamma;\xi,\zeta;z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k\ (\beta)_k\ (\gamma)_k}{(\xi)_k\ (\zeta)_k}\ \frac{z^k}{k!}$$

is the generalized hypergeometric function.

The probability distribution function $\Phi(x,t)$ is left-continuous with jumps at the origin $x=0$ and at the terminal points $x=\pm 2ct$ determined by the singularities concentrated at these three points.

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Optimization of Markov Processes with Final Sequence of States and Unitary Transition Time

Alexandru Lazari

Abstract

In this paper the Markov processes with final sequence of states and unitary transition time are studied. These stochastic systems represent a generalization of zero-order Markov processes studied in [1]. The evolution time of these systems, as a function of distribution of the states and transit matrix, is minimized using signomial and geometric programming approaches.

Keywords: Markov Process, Final Sequence of States, Evolution Time, Geometric Programming, Signomial Programming, Posynomial Function.

1 Introduction

The stochastic systems with final sequence of states represent an extension of classical Markov processes. Various kinds of these systems were studied in [3]. In this thesis the efficient polynomial algorithms for determining the main probabilistic characteristics (expectation, variance, mean square deviation, n-order moments) of evolution time of the given stochastic systems were proposed.

Another interpretations of these Markov processes were analyzed in 1981 by Leo J. Guibas and Andrew M. Odlyzko in [6] and G. Zbaganu in 1992 in [5]. First article considers the evolution of these stochastic systems as a string, composed from the states of the systems and studies the periods in this string. In the second paper the author considers that the evolution of Markov process is similar with a poem written by an ape. The evolution time of the system is associated with the time that needs for the ape to write that poem (the final sequence of states of the system).

Based on the results mentioned above, efficient methods for minimizing the expectation of evolution time of zero-order Markov processes with final sequence of states and unitary transition time were obtained in [1]. The main idea was that the expectation of the evolution time can be written as a posynomial plus one unit. The geometric programming approach was applied and the problem was reduced to the case of convex optimization using the interior-point methods.

Next, in this paper, a generalization of this problem is considered. The Markov processes (of order 1) are analyzed and the evolution time is minimized. The signomial and geometric progamming (described in [4]) are applied.

2 Statement of the problem

Let us consider the discrete stochastic system L (defined in [2]) with the set of possible states denoted by V, such that $|V| = \omega < \infty$. The state of the system at every moment of time $t \in \mathbb{N}$ is denoted by $v(t) \in V$.

Let $p^*(v)$ represents the probability that $v(0) = v$, $v \in V$, and $p(u, v)$ represents the probability of transition from the state $u \in V$ to $v \in V$.

Let $x_1, x_2, \ldots, x_m \in V$ be fixed. We say that the system stops, when it passes through all the states $x_1, x_2, \ldots, x_m$, consecutively. The stopping time T represents the evolution time of the given system L.

Next, we consider that the distributions p and p^* are not fixed. So, we have the Markov process $L(p^*, p)$ with final sequence of states X, distribution of the states p^* and transition matrix p, for every parameters p and p^*. The problem is to determine the optimal distribution $p^* = \overline{p}^*$ and optimal transition matrix $p = \overline{p}$, that minimize the expectation of the evolution time $T(p^*, p)$ of the stochastic system $L(p^*, p)$.

3 Main results overview

Theorem 1. *The optimal distribution of the states is* $\overline{p}^*$*, where* $\overline{p}^*(x_1) = 1$ *and* $\overline{p}^*(x) = 0$*,* $\forall x \in V \backslash \{x_1\}$.

Theorem 2. *We consider the sets* $\overline{X} = \{x_1, x_2, \ldots, x_{m-1}\}$ *and* $\overline{Y} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)\}$*. The optimal transition matrix* $\overline{p}$ *has the following properties:*

1. $\overline{p}(x, y) = 0$*, if* $(x, y) \in V^2 \backslash \overline{Y}$ *and* $x \in \overline{X}$*;*
2. $\overline{p}(x, y) = 1$*, if* $(x, y) \in \overline{Y}$ *and* $(x, z) \in V^2 \backslash \overline{Y}$*,* $\forall z \in V \backslash \{y\}$*;*
3. $\overline{p}(x, x_1) = 1$ *and* $\overline{p}(x, y) = 0$*,* $\forall y \in V \backslash \{x_1\}$*, if* $x \in V \backslash \overline{X}$*;*
4. $\overline{p}(x, y) > 0$*,* $\forall (x, y) \in \overline{Y}$*.*

Theorem 3. *The optimal transition matrix can be determined solving the following signomial programs:*

$$\mathbb{E}(T(\overline{p})) = d_1 d_2^{-1} \to \min \qquad\qquad \mathbb{E}(T(\overline{p})) = d_1 d_2^{-1} \to \min$$

$$\begin{cases} \sum\limits_{(x,y) \in \overline{Y}} \overline{p}(x, y) = 1, \ \forall x \in \overline{X} \\ d_{i,1}^{-1} d_i + d_{i,1}^{-1} d_{i,2} = 1, \ i = \overline{1, 2} \\ d_{i,j}^{-1} \delta_{i,j}(\overline{p}) = 1, \ i, j = \overline{1, 2} \\ d_i > 0, \ d_{i,j} > 0, \ i, j = \overline{1, 2} \\ \overline{p}(x, y) > 0, \ \forall (x, y) \in \overline{Y}. \end{cases} \qquad \begin{cases} \sum\limits_{(x,y) \in \overline{Y}} \overline{p}(x, y) = 1, \ \forall x \in \overline{X} \\ d_{i,1}^{-1} d_i + d_{i,1}^{-1} d_{i,2} = 1, \ i = \overline{1, 2} \\ d_{i,j}^{-1} \delta_{i,3-j}(\overline{p}) = 1, \ i, j = \overline{1, 2} \\ d_i > 0, \ d_{i,j} > 0, \ i, j = \overline{1, 2} \\ \overline{p}(x, y) > 0, \ \forall (x, y) \in \overline{Y}. \end{cases}$$

according to the properties described by Theorems 1 and 2, where $\delta_{i,j}(\overline{p})$*,* $i, j = \overline{1, 2}$*, are the posynomials from the decomposition* $\mathbb{E}(T(\overline{p})) = (\delta_{1,1}(\overline{p}) - \delta_{1,2}(\overline{p}))(\delta_{2,1}(\overline{p}) - \delta_{2,2}(\overline{p}))^{-1}$ *that follows from the algorithm developed in [3]. These signomial programs represent geometric programs with posynomial equality constraints and can be handled as geometric programs using the way followed in [4]. If* $\overline{p}^1$ *and* $\overline{p}^2$ *are the solutions of the problems described above, then the optimal transition matrix is* $\overline{p} \in \{\overline{p}^1, \overline{p}^2\}$ *for which* $\mathbb{E}(T(\overline{p}))$ *is minimal.*

4 Conclusion

In this paper it is presented how to optimize the evolution time of Markov processes with final sequence of states and unitary transition time. To achieve this goal, the author successfully uses signomial and geometric programming methods.

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Determining Nash Equilibria for Stochastic Positional Games with Average Payoffs

Dmitrii Lozovanu, Stefan Pickl

Abstract

Stochastic positional games with average payoffs of the players are studied. Nash equilibria conditions for the considered class of games are formulated and algorithms for determining the optimal stationary strategies of the players are proposed. The obtained results are applied for analyzing cyclic games and Shapley stochastic games.

Keywords: Markov decision processes, Optimal stationary strategy, Shapley stochastic games, Stochastic positional games, Nash equilibria.

1 Introduction

We consider a class of stochastic positional games with average payoffs. Necessary and sufficient conditions for the existence of Nash equilibria in the considered class of games are formulated and proven. Based on these results we develop algorithms for determining the optimal stationary strategies of the players. Moreover, we show that for the antagonistic positional games saddle points always exist and efficient iterative procedure for determining the saddle points can be derived. The obtained results generalize Nash equilibria conditions for deterministic positional games and they can be used for studying and solving some classes of Shapley stochastic games with average payoffs.

2 Formulation of the Basic Game Model

A stochastic positional game with m players and average payoffs of the players consists of the following elements:

- a finite set of states X that is divided into m disjoint subsets $X = X_1 \cup X_2 \cup \cdots \cup X_m$, where each subset X_i represents the position set of player $i \in \{1, 2, \ldots, m\}$;
- a finite set of actions $A = \cup_{x \in X} A(x)$, where $A(x)$ represents the set of actions in the state $x \in X$, and in the state $x \in X_i$ of the position set of player $i \in \{1, 2, \ldots, m\}$ the action $a \in A(x)$ can be chosen only by the corresponding player i;
- a transition probability function $p : X \times A \times X \to [0, 1]$ that gives the probability transitions $p^a_{x,y}$ from an arbitrary state $x \in X$ to an arbitrary state $y \in Y$ for a fixed action $a \in A$, where $\sum_{y \in X} p^a_{x,y} = 1, \forall x \in X,\ a \in A$;
- m transition cost functions $c^i : X \times X \to [0, 1]$ that give the transition costs $c^i_{x,y}$ from an arbitrary state $x \in X$ to an arbitrary state $y \in X$ for each player $i \in \{1, 2, \ldots, m\}$;
- a starting state $x_0 \in X$.

The game starts at a given state x_0 of the dynamical system at the time moment $t = 0$. In this state the move is made by player $i \in \{1, 2, \ldots, m\}$ if $x_0 \in X_i$. The move means that player i fixes an action $a_0 \in A(x_0)$ in the state x_0 and the dynamical system passes randomly to a state $x_1 \in X$ according to transition probabilities $p^{a_0}_{x,y}$. At this stage in the state x_0 the immediate cost $\mu^j(x_0, a_0) = \sum_{y \in X} p^{a_0}_{x_0,y} c^j_{x_0,y}$ for each player $j \in \{1, 2, \ldots, m\}$ is determined. After that, if $x_1 \in X_i$, then player $i \in \{1, 2, \ldots, m\}$ fixes an action $a_1 \in A(x_1)$ and the system passes randomly to the next state x_2 according to transition probabilities $p^{a_1}_{x_1,y}$. At this stage in the state x_1 the immediate cost $\mu^j(x_1, a_1) = \sum_{y \in X} p^{a_1}_{x_1,y} c^j_{x_1,y}$ for each player $j \in \{1, 2, \ldots, m\}$ (and so on indefinitely) is again determined. In such a way a play of the stochastic positional game $x_0, a_0, x_1, a_1, \ldots, x_t, a_t, \ldots$ defines a stream of immediate costs $\mu_0^j, \mu_1^j, \mu_2^j, \ldots$, where $\mu_t^j = \mu^j(x_t, a_t)$, $t = 0, 1, 2, \ldots$. The average stochastic positional game is an infinite game with the following payoffs of the players: $\omega^j = \lim_{t\to\infty} \frac{1}{t}\sum_{\tau=1}^{t-1} \mu_\tau^j$, $\quad j = 1, 2, \ldots, m$. In [1, 2] it is shown that this game can be formulated in the terms of stationary strategies. The stationary strategies of the players are defined as m maps $s^i : x \to a \in A(x)$ for $x \in X_i$, $i = 1, 2, \ldots, m,$. If the players $1, 2, \ldots, m$ fix their stationary strategies $s^1, s^2, \ldots, s^m$, respectively, then we obtain a situation $s = (s^1, s^2, \ldots, s^m)$. This situation corresponds to a simple Markov process determined by the probability distributions $p_{x,y}^{s^i(x)}$ in the states $x \in X_i$ for $i = 1, 2, \ldots, m$. We denote by $P^s = (p_{x,y}^s)$ the matrix of probability transitions of this Markov process. If the starting state x_0 is given, then for the Markov process with the matrix of probability transitions P^s we can determine the average cost per transition $\omega_{x_0}^i(s^1, s^2, \ldots, s^m)$ with respect to each player $i \in \{1, 2, \ldots, m\}$ taking into account the corresponding cost matrix $C^i = (c_{x,y}^i)$. So, on the set of situations we can define the payoff functions of the players as follows: $F_{x_0}^i(s^1, s^2, \ldots, s^m) = \omega_{x_0}^i(s^1, s^2, \ldots, s^m)$, $\quad i = 1, 2, \ldots, m$. In this game we are seeking for a *Nash equilibrium.* We denote this game by $(X,\, A,\, \{X_i\}_{i=\overline{1,m}},\, \{c^i\}_{i=\overline{1,m}},\, p,\, x_0)$. In the case $m = 2, c^2 = -c^1$ we obtain an antagonistic stochastic positional game. If $p_{x,y}^a = 0 \vee 1,\ \forall x, y \in X,\ \forall a \in A$, the game is transformed into the cyclic game [2].

3 The Main Result

Theorem 1. *Let* $(X, A, \{X_i\}_{i=\overline{1,m}}, \{c^i\}_{i=\overline{1,m}}, p, x)$ *be an average stochastic positional game. Then in this game there exists a Nash equilibrium for an arbitrary starting position* $x \in X$ *if and only if there exist the functions* $\varepsilon^i : X \to R$, $i = 1, 2, \ldots, m$ *and the values* $\omega_x^1, \omega_x^2, \ldots, \omega_x^m$ *for* $x \in X$ *that satisfy the following conditions:*

1) $\mu_{x,a}^i + \sum\limits_{y\in X} p_{x,y}^a \varepsilon_y^i - \varepsilon_x^i - \omega_x^i \geq 0, \quad \forall x \in X_i,\ \forall a \in A(x),$

$i = 1, 2, \dots, m,$ *where* $\mu^i_{x,a} = \sum\limits_{y \in X} p^a_{x,y} c^i_{x,y};$

2) $\min\limits_{a \in A(x)} \{\mu^i_{x,a} + \sum\limits_{y \in X} p^a_{x,y}\varepsilon^i_y - \varepsilon^i_x - \omega^i_x\} = 0, \quad \forall x \in X_i,\ i = 1, 2, \dots, m;$

3) *on each position set* X_i, $i \in \{1, 2, \dots, m\}$ *there exists a map* $s^{i^*} : X_i \to A$ *such that*
$s^{i^*}(x) = a^* \in Arg \min_{a \in A(x)} \left\{\mu^i_{x,a} + \sum\limits_{y \in X} p^a_{x,y}\varepsilon^i_y - \varepsilon^i_x - \omega^i\right\}$ *and*
$\mu^j_{x,a^*} + \sum\limits_{y \in X} p^{a^*}_{x,y}\varepsilon^j_y - \varepsilon^j_x - \omega^j = 0, \quad \forall x \in X_i, \quad j = 1, 2, \dots, m.$

If such conditions hold, then the set of maps $s^{1^*}, s^{2^*}, \dots, s^{m*}$ *determines a Nash equilibrium of the game for an arbitrary starting position* $x \in X$ *and* $F^i_x(s^{1^*}, s^{2^*}, \dots, s^{m*} = \omega^i_x$, $i = 1, 2, \dots, m.$

Based on this theorem and the results from [1, 2] we have elaborated algorithms for determining the optimal stationary strategies of the players (in the case when a Nash equilibrium for the game exists).

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Methodology of matrix representation of higher order elasticity constants. Fourth order tensor

Vasile Marina, Viorica Marina

Abstract

The constitutive nonlinear equations of anisotropic materials are examined in reversible deformation area. The constitutive equations of the second order, in which the tensors of elastic constants of forth order listed, are analyzed in detail. The matrix representation of these tensors and analysis of independent constants of elasticity in function of material symmetry and type of atoms interactions is given.

Keywords: tensor, stress, strain, symmetry, constant elasticity

1 Introduction

With superior order tensors (four, six, eight) we meet at studying the relations between stress and strain. At reversible process the governing equations are written in the form

$$t_{ij} = c_{ijnm} d_{nm} + c_{ijnmpq} d_{nm} d_{pq} + c_{ijnmpqkl} d_{nm} d_{pq} d_{kl} + ...,$$

where by t_{ij}, d_{ij} – the stress and strain tensors are denoted respectively, but by c_{ijnm}, c_{ijnmpq}, $c_{ijnmpqkl}$ – tensors of elasticity constants of order four, six and eight. From symmetry of stress, strain tensors and thermodynamic lows, for tensors of elastic constants the following symmetry relations result:

$$c_{ijnm} = c_{jinm} = c_{ijmn} = c_{nmij} \tag{1}$$

$$c_{ijnmpq} = c_{jinmpq} = c_{ijmnpq} = c_{ijmnqp} = c_{nmijpq} = c_{pqnmij} = c_{ijpqnm}$$

$$c_{ijnmpqkl} = c_{jinmpqkl} = c_{ijmnpqkl} = c_{ijnmqpkl} =$$

$$= c_{ijnmpqlk} = c_{nmijpqkl} = c_{k\ln mpqij}. \tag{2}$$

In function of type of interaction between atoms or molecules, the extra relations can be added to the relationships (1), (2). If the interactions between atoms or molecules are central (the ionic contact), then tensors of elasticity constants of any order are totally symmetric. Remember, that one tensor is totally symmetric if it is symmetric in order with all pairs of indices. In case of forth order tensor, there exists one more relation $c_{ijnm} = c_{injm}$. The material symmetry, which is quantitatively expressed by symmetry plans and symmetry axes of different order, leads to reduction of number of independent constants of elasticity.

2 A matrix representation of a fourth order tensor

The experimental dates for components of tensors of elasticity constants are presented in crystallographic system of coordinates, in which there are sizes only of the independents. Calculation of elasticity constants in arbitrary system is considerably simplified, if the superior order tensors are represented by composite matrix [1]. So, the fourth order tensor can be presented under the shape $c_{ijnm} = (c_{ij})_{nm}$, where, $(c_{ij})_{nm}$ – is square composite matrix of the second order, every element of which represents also a square matrix (3).

For the fourth order tensor, which enjoys the symmetry properties (1), the components of the composite matrix are expressed only by 21 independent constants. The 21 independent constants can be presented as a column matrix 21x1, the elements of which we denote by a_I, where $I = 1, 2, \ldots, 21$.

$$C := \begin{bmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{11} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{12} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{13} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{21} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{22} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{23} \\ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{31} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{32} & \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}_{33} \end{bmatrix} \quad (3)$$

So, the tensor of elasticity constants will be expressed in the following way:

$$C := \begin{bmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & a_5 \\ a_3 & a_5 & a_6 \end{pmatrix} & \begin{pmatrix} a_2 & a_7 & a_8 \\ a_7 & a_9 & a_{10} \\ a_8 & a_{18} & a_{11} \end{pmatrix} & \begin{pmatrix} a_3 & a_8 & a_{12} \\ a_8 & a_{13} & a_{14} \\ a_{12} & a_{14} & a_{15} \end{pmatrix} \\ \begin{pmatrix} a_2 & a_7 & a_8 \\ a_7 & a_9 & a_{10} \\ a_8 & a_{18} & a_{11} \end{pmatrix} & \begin{pmatrix} a_4 & a_9 & a_{13} \\ a_9 & a_{16} & a_{17} \\ a_{13} & a_{17} & a_{18} \end{pmatrix} & \begin{pmatrix} a_5 & a_{10} & a_{14} \\ a_{10} & a_{17} & a_{19} \\ a_{14} & a_{19} & a_{20} \end{pmatrix} \\ \begin{pmatrix} a_3 & a_8 & a_{12} \\ a_8 & a_{13} & a_{14} \\ a_{12} & a_{14} & a_{15} \end{pmatrix} & \begin{pmatrix} a_5 & a_{10} & a_{14} \\ a_{10} & a_{17} & a_{19} \\ a_{14} & a_{19} & a_{20} \end{pmatrix} & \begin{pmatrix} a_6 & a_{11} & a_{15} \\ a_{11} & a_{18} & a_{20} \\ a_{15} & a_{20} & a_{21} \end{pmatrix} \end{bmatrix} \quad (4)$$

If tensor is totally symmetric, the following relations can be $a_8 = a_5,\ a_7 = a_4,\ a_{12} = a_6,\ a_{19} = a_{18},\ a_{13} = a_{10},\ a_{14} = a_{11}$. In the case of orthotropic material and for material with cubic symmetry the matrix of elasticity constants is expressed by (5). The relations between stress

$$C := \begin{bmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_6 \end{pmatrix} & \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & 0 & 0 \\ a_{12} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_{16} & 0 \\ 0 & 0 & a_{18} \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{19} \\ 0 & a_{19} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_{12} \\ 0 & 0 & 0 \\ a_{12} & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{19} \\ 0 & a_{19} & 0 \end{pmatrix} & \begin{pmatrix} a_6 & 0 & 0 \\ 0 & a_{18} & 0 \\ 0 & 0 & a_{21} \end{pmatrix} \end{bmatrix} \quad C := \begin{bmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_4 \end{pmatrix} & \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & a_7 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & a_7 & 0 \\ a_7 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_4 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_7 \\ 0 & a_7 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & a_7 \\ 0 & 0 & 0 \\ a_7 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_7 \\ 0 & a_7 & 0 \end{pmatrix} & \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_4 & 0 \\ 0 & 0 & a_1 \end{pmatrix} \end{bmatrix} \quad (5)$$

and strain in arbitrary system of coordinates in linear approximation is determined by relation

$$d_{in} = \sum_{k=1}^{3} \sum_{q=1}^{3} \left[\sum_{c=1}^{3} \sum_{l=1}^{3} \sum_{m=1}^{3} \sum_{j=1}^{3} \left[r_{ij} r_{nm} r_{kl} r_{qc} (C_{jm})_{lc} \right] t_{kq} \right], \qquad (6)$$

where by r_{ij} the matrix of rotation is denoted, in which the base position of this coordinate system is determined from crystallographic system.

3 Conclusions

The possibility of matrix presentation by superior order tensors essentially simplifies the mathematic modelling of nonlinear behavior of anisotropic materials. It has been found, that governing equations of second order in general case of anisotropy are expressed by 77 independent constants of elasticity.

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Methodology of matrix representation of higher order elasticity constants. Six order tensor

Viorica Marina, Vasile Marina

Abstract

The constitutive nonlinear equations of anisotropic materials are examined in reversible deformation area. The constitutive equations of the second order, in which the tensors of elastic constants of forth order listed, are analyzed in detail. The matrix representation of these tensors and analysis of independent constants of elasticity in function of material symmetry and type of atoms interactions is given.

Keywords: tensor, stress, strain, symmetry, constant elasticity.

1 A matrix reprezentation of a six order tensor

In the case of nonlinear relations between stress and strain the six and eight order tensors are intervened, and these tensors can be presented in the form of composite matrix. On basis of symmetric relations it is possible to pass from two indexes notations to one index after Viogt [1-3] convention $11 \sim 1, 22 \sim 2, 33 \sim 3, 23 \sim 4, 13 \sim 5, 12 \sim 6$. Adopting this convention, we will write $c_{ijnm} = c_{KM}$, $c_{ijmnrs} = c_{KMF}$, $c_{ijmnrskq} = c_{KMFL}$, where the small letters have the values 1,2,3, but big 1,2,...,6. Additionally we have

$$c_{KM} = c_{MK},\ \ c_{KMF} = c_{MKF} = c_{FMK} = c_{KFM}$$

$$c_{KMFL} = c_{MKFL} = c_{KMLF} = c_{MKLF} = c_{LFKM} = c_{LFMK} =$$

$$= c_{FLKM} = c_{FLMK} = c_{FKLM} = \dots$$

Matrixes $c_{KM}, c_{KMF}, c_{KMFL}$ don't represent the tensor in the obtained meaning. Therefore, in the rule of components transformation at rotation of reference system there is not directly given the rotation matrix r. It can be demonstrated, that for these matrixes the known rules of components transformation can be used, so

$$c'_{KM} = R_{KI} R_{MJ} c_{IJ}, c'_{KMF} = R_{KI} R_{MG} R_{FT} c_{IGT},$$

$$c'_{KMFL} = R_{KI} R_{MG} R_{FT} R_{LU} c_{IGTU}.$$

$(C_I)_{GT}$ matrix takes the form (1), i.e. $C_1 = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$.

$$C_1 := \begin{bmatrix} \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_2 & c_7 & c_8 & c_9 & c_{10} & c_{11} \\ c_3 & c_8 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_4 & c_9 & c_{13} & c_{16} & c_{17} & c_{18} \\ c_5 & c_{10} & c_{14} & c_{17} & c_{19} & c_{20} \\ c_6 & c_{11} & c_{15} & c_{18} & c_{20} & c_{21} \end{pmatrix} \\ \begin{pmatrix} c_2 & c_7 & c_8 & c_9 & c_{10} & c_{11} \\ c_7 & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_8 & c_{23} & c_{27} & c_{28} & c_{29} & c_{30} \\ c_9 & c_{27} & c_{28} & c_{31} & c_{32} & c_{33} \\ c_{10} & c_{25} & c_{29} & c_{32} & c_{34} & c_{35} \\ c_{11} & c_{16} & c_{30} & c_{33} & c_{35} & c_{36} \end{pmatrix} \\ \begin{pmatrix} c_3 & c_8 & c_{12} & c_{13} & c_{14} & c_{15} \\ c_8 & c_{23} & c_{27} & c_{28} & c_{29} & c_{30} \\ c_{12} & c_{13} & c_{37} & c_{38} & c_{39} & c_{40} \\ c_{13} & c_{28} & c_{38} & c_{41} & c_{42} & c_{43} \\ c_{14} & c_{29} & c_{39} & c_{42} & c_{44} & c_{45} \\ c_{15} & c_{30} & c_{40} & c_{43} & c_{45} & c_{46} \end{pmatrix} \end{bmatrix} \qquad C_2 := \begin{bmatrix} \begin{pmatrix} c_4 & c_9 & c_{13} & c_{16} & c_{17} & c_{18} \\ c_9 & c_{24} & c_{28} & c_{31} & c_{32} & c_{33} \\ c_{13} & c_{28} & c_{38} & c_{41} & c_{42} & c_{43} \\ c_{16} & c_{31} & c_{41} & c_{47} & c_{48} & c_{49} \\ c_{17} & c_{32} & c_{42} & c_{48} & c_{50} & c_{51} \\ c_{18} & c_{33} & c_{43} & c_{49} & c_{51} & c_{52} \end{pmatrix} \\ \begin{pmatrix} c_5 & c_{10} & c_{14} & c_{17} & c_{19} & c_{20} \\ c_{10} & c_{25} & c_{29} & c_{32} & c_{34} & c_{35} \\ c_{14} & c_{29} & c_{39} & c_{42} & c_{44} & c_{45} \\ c_{17} & c_{32} & c_{44} & c_{48} & c_{50} & c_{51} \\ c_{19} & c_{34} & c_{44} & c_{50} & c_{53} & c_{54} \\ c_{20} & c_{35} & c_{45} & c_{51} & c_{54} & c_{55} \end{pmatrix} \\ \begin{pmatrix} c_6 & c_{11} & c_{15} & c_{18} & c_{20} & c_{21} \\ c_{11} & c_{26} & c_{30} & c_{33} & c_{35} & c_{36} \\ c_{15} & c_{30} & c_{40} & c_{43} & c_{45} & c_{46} \\ c_{18} & c_{35} & c_{43} & c_{49} & c_{51} & c_{52} \\ c_{20} & c_{35} & c_{45} & c_{51} & c_{54} & c_{55} \\ c_{21} & c_{36} & c_{46} & c_{52} & c_{55} & c_{56} \end{pmatrix} \end{bmatrix} \qquad (1)$$

So, the tensor of elastic constants of fourth order is expressed by 21 independent components, but six order tensor form 56. These 56 components are presented in the form of column matrix with 56×1 dimensions.

If materials have other elements of symmetry too, the number of independent constants of elasticity is reduced. So, it can be proved, that for materials with cubic symmetry the number of elasticity constants of stress tensor of sixth order is decreased down to six.

The only non-zero constants of elasticity tensor of sixth order are

$$c_1 = C_{111} = c_{22} = C_{222} = c_{37} = C_{333},$$

$$c_2 = C_{112} = c_3 = C_{113} = c_7 = C_{122} = c_{23} =$$

$$= C_{223} = c_{12} = C_{133} = c_{27} = C_{233},$$

$$c_8 = C_{123}, \quad c_{51} = C_{456}, \quad c_{16} = C_{144} = c_{34} = C_{255} = c_{46} = C_{366},$$

$$c_{19} = C_{155} = c_{21} = C_{166} = c_{31} = C_{244} = c_{36} =$$

$$= C_{266} = c_{41} = C_{344} = c_{44} = C_{355}.$$

Therefore, the elastic behavior of material of cubic symmetry in approximation $t_{ij} = c_{ijnm}d_{nm} + c_{ijnmpq}d_{nm}d_{pq}$ is described by 9 independent constants; 3 components of forth order tensor a_1, a_4, a_7 and six independent components of sixth order tensor $c_1, c_2, c_8, c_{16}, c_{19}, c_{51}$. In the case of isotropic material, between independent constants of forth order tensor the relationship takes place $a_4 = \frac{a_1 - a_7}{2}$, but for elasticity constants of sixth order tensor tree more relations are obtained $c_{16} = \frac{1}{2}(c_2 - c_8)$, $c_{19} = \frac{1}{4}(c_1 - c_2)$, $c_{51} = \frac{1}{8}(c_1 - 3c_2 + 2c_8)$. Therefore, the governing equations of second order in case of isotropic materials are expressed from only 5 independent constants. If interaction between atoms is central, than the following relations exist $a_7 = a_4 = \frac{a_1}{3}$, $c_8 = c_{16} = c_{51} = \frac{7c_2 - c_1}{6}$, so, in case of one isotropic material with central interactions, the governing equations of the second order are expressed only by tree independent constants. In case of governing equations of tree order may interfere the eight order tensors. These tensors are expressed by square matrix of sixth order, each element is represented by the sixth order matrix.

2 Conclusions

For cubic symmetry materials the number of independent constants of elasticity is reduced down to 9. In the case of isotropic materials the number of independent constants of elasticity is reduced down to 5, if interaction between atoms is central, than the number of independent constants is reduced down to 3.

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Inverse Problem of Adequate Mathematical Description Synthesis

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Abstract

The main problem of mathematical simulation is the construction (synthesis) of mathematical model of motion of real dynamic system which in aggregate with model of external load gives results of mathematical simulation adequate to experimental observations. Two basic approaches to this problem are selected. These problems are incorrect problems by their nature and so, for their solution the regularization methods are being used.

Keywords: mathematical simulation, adequate mathematical description, inverse problems.

1 Introduction

It is supposed that real physical process is observed and the records of experimental measurements of some characteristics of this process are given. If results of simulation of this physical process with some mathematical description coincide with given experimental measurements characteristics with experiment accuracy, this mathematical description will be named as *an adequate mathematical description*. It is necessary to construct the algorithm of adequate mathematical description of this process for further use. In the given work the mathematical models of physical processes described only by linear system of the ordinary differential equations will be examined [1]:

$$\dot{x}(t) = \tilde{C}x(t) + \tilde{D}z(t), \tag{1}$$

with the equation of observation

$$y(t) = \tilde{F}x(t), \tag{2}$$

where $x(t) = (x_1(t), x_2(t), .., x_n(t))^T$ is vector-function variables characterizing the state of process, $z(t) = (z_1(t), z_2(t), .., z_l(t))^T$ is vector-function of unknown external loads, $y(t) = (y_1(t), y_2(t), .., y_m(t))^T$; $\tilde{C}, \tilde{D}, \tilde{F}$ are matrices of the appropriate dimension with constant coefficients which are given approximately, $\tilde{F}$ is nonsingular matrix dimension $m \times n$ and $rang\tilde{F} = m$, ($(.)^T$ is a mark of transposition). If the part of external loads of real process is known, this case can be reduced to the one which was examined earlier with using the linearity of initial dynamic system. We assume that state variables $x_i(t), 1 \le i \le n$ of system (1) correspond to some real characteristics $\tilde{x}_i(t), 1 \le i \le n$ of process which are being investigated, and that the vector function $\tilde{y}(t) = \tilde{F}\tilde{x}(t)$ is obtained from experimental measurements.

Two basic approaches to synthesis of adequate mathematical description exist [1]. The *problem of synthesis of adequate mathematical description* with the use of system (1) can be formulated as follows: it is necessary to find unknown vector function of external loads $z(t)$ in such a way that the vector function $y(t)$, which is obtained from system (1),(2) under this external load $z(t)$, coincides with experimental data $\tilde{y}(t)$ with a given accuracy of experimental measurements in chosen functional metrics. The adequate mathematical descriptions first of all are aimed at the forecast of motion of real processes. With the help of adequate mathematical description it is possible to predict the motion of real process in new conditions of operation.

2 Statement of problem

Let's assume that external load $z(t)$ is unknown and vector function $\tilde{y}(t)$ in the equations (2) is measured by an experimental way. The part of state variables $\tilde{x}_1(t), \tilde{x}_2(t), ..., \tilde{x}_m(t)$ can be obtained by an inverse of equation (2) with function $\tilde{y}(t)$, as $\tilde{F}$ is nonsingular matrix.

After simple transformations, in some cases it can be obtained several integral equations of the first kind for all components of the unknown function of external load $z(t)$:

$$\int_0^t K_1(t-\tau)z_i(\tau)d\tau = P_i(t), i = 1,..,l, \tag{3}$$

where $K_i(t-\tau), P_i(t)$ are known functions.

Rewrite the equation (3) in the form [1]:

$$A_p z = u_\delta, \tag{4}$$

where A_p is completely continuous operator, $A_p : Z \to U, z \in Z, u_\delta \in U, u_\delta$ is initial experimental data (graphic), z is unknown function, (Z, U are functional spaces). Further, we shall suppose that the element u_δ in the equation (4) is given with a known error: $\|u_\delta - u_{ex}\| \leq \delta$, where δ is const, $\delta > 0$, u_{ex} is exact initial data.

Let's denote by $Q_{\delta,p}$ the set: $Q_{\delta,p} = z : \|A_p z - u_\delta\| \leq \delta$. The set $Q_{\delta,p}$ is not bounded at any δ. Thus, the initial mathematical model of physical process and any function from the set $Q_{\delta,p}$ provide an adequacy of results of mathematical simulation with accuracy δ. Besides, the exact solution $z_{ex}(A_{ex}z_{ex} = u_{ex})$ of the equation (4) may not belong to the set of possible solutions $Q_{\delta,p}$, as the operator A_p is describing inexactly the real physical process [1].

3 Features and Method of Solution

The inverse problems of synthesis have a number of features by virtue of this quality of exact solution:

- the size of an error of the approximate solution in relation to the exact solution z_{ex} has no importance for further use of the approximate solution as we will use the inexact operator A_p for forecast of real process motion;

- there is no sense to study behavior of the approximate solution of an inverse problem depending on the reduction of an error of the initial data δ. In other words, the approximate solution cannot have properties of regularization.

For definition of the unique solution of an inverse problem (4) from set of the possible solutions it is possible to use approach which was offered by Phillips and Tikhonov [2]. Consider now the following extreme problem:

$$\Omega[z_{\delta,p}] = \inf_{z \in Q_{\delta,p}} \Omega[z], \tag{5}$$

where functional $\Omega[z]$ has been defined on set Z [7].

Theorem 1. *If the functional space Z is reflex Banach space, the functional $\Omega[z]$ is convex and lower semi-continuity on Z, Lebesgue set for some function from $Q_{\delta,p}$ is bounded, then the function $z_{\delta,p} \in Q_{\delta,p}$ exists.*

The examples are given [1].

4 Conclusion

The algorithms of synthesis of the mathematical description of real process (for example the motion of some dynamic system) are considered which allows receiving adequate results of mathematical simulation. Two basic approaches to this problem are selected. Within the framework of one of these approaches some algorithms are offered.

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Performance characteristics for semi-Markov polling models with exhaustive service

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Abstract

Some performance characteristics for exhaustive polling models with semi-Markov delays, such as distribution of busy period, non-stationary queue lengths, probability of states are presented.

Keywords: polling system, queue lengths, busy period, probability of states.

1 Introduction

It is well known that polling models play an important role in analysis and designing wireless nertworks [1]. A polling model is a system of multiple queues accessed by a single server in a given order. Among important characteristics of these systems are the k-busy period, probability of states and queueing length [2]. We consider a queueing system of polling type with semi-Markov delays. Handling mechanism for this system is given by polling table $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, r\}$, where the function shows that at the stage j, $j = \overline{1, n}$, user number k, $k = \overline{1, r}$, $r \leq n$ is served (more details see in [1]). The items (messages) of the user k, according to Poisson distribution with parameter λ_k arrive. The service time for the items of class k is a random variable B_k with distribution function $B_k(x) = P\{B_k < x\}$. Duration of the orientation from one user to user k is a random variable C_k with distribution function $C_k(x) = P\{C_k < x\}$. Thus C_k can be interpreted as a loss of time in preparing the service process for user of class k.

2 The distribution of k-busy period

Definition 1. *The k-busy period is a measure of the time that expires from when a server begins to process, after an empty queue, to when the k-queue becomes empty again for the first time [3].*

Denote by $\Pi_k^\delta(x)$ distribution function of the k-busy period, and by $\pi_k^\delta(s)$ it's Laplace-Stieltjes transform.

Theorem 1. *Function $\pi_k^\delta(s)$ is determined from equation*

$$\pi_k^\delta(s) = c_k(s + \lambda_k - \lambda_k \pi_k(s))\pi_k(s), \tag{1}$$

where

$$\pi_k(s) = \beta_k(s + \lambda_k - \lambda_k \pi_k(s)) \tag{2}$$

and by $c_k(s)$ and $\beta_k(s)$ the Laplace-Stieltjes transforms of distribution functions $C_k(x)$ and $B_k(x)$ are denoted,

$$c_k(s) = \int_0^\infty e^{-sx} dC_k(x), \;\; \beta_k(s) = \int_0^\infty e^{-sx} dB_k(x).$$

Remark 1. *From Theorem 1 the first moment of the k-busy period can be obtained.*

3 Probability of states

Denote $P_{B_k}(x), P_{C_k}(x)$ and $P_0(x)$ the probabilities that at instant x the system is busy by service of k-messages, switching to k-messages and system is free, respectively.

Theorem 2. *The Laplace-Stieltjes transforms of $P_{B_k}(x)$, $P_{C_k}(x)$ and $P_0(x)$ are determined from*

$$p_{B_k}(s) = \frac{\lambda_k[1 - \pi_k(s)]}{s[s + \lambda_k - \lambda_k \pi_k^\delta(s)]}, \tag{3}$$

$$p_{C_k}(s) = \frac{\lambda_k[1 - c_k(s)]}{s[s + \lambda_k - \lambda_k \pi_k^\delta(s)]}, \tag{4}$$

$$p_0(s) = \frac{1}{s} - [p_{B_k}(s) + p_{C_k}(s)]), \tag{5}$$

where $\pi_k^\delta(s)$ and $\pi_k(s)$ are determined from (1) and (2).

4 Distribution of virtual queue length

Let $P_m(x)$ be the probability that at the instant x there are m messages in the k-queue. Denote

$$P_k(z, x) = \sum_{m=1}^{\infty} P_m(x) z^m, 0 \le z \le 1,$$

the generating function of queue length distribution and by $p_k(z, s)$, the Laplace transform of function $P_k(z, s)$.

Theorem 3.

$$p_k(z, s) = \frac{1 + \lambda_k \pi_k^\delta(z, s)}{s + \lambda_k - \lambda_k z}, \tag{6}$$

$$\pi_k^\delta(z, s) = \frac{1 - c_k(s + \lambda_k - \lambda_k z)}{s + \lambda_k - \lambda_k z} + \frac{\beta_k(z, s)}{z - \beta_k(s + \lambda_k - \lambda_k z)} \times$$

$$\times [z c_k(s + \lambda_k - \lambda_k z) - \pi_k^\delta(s)], \tag{7}$$

$$\beta_k(z, s) = \frac{1 - \beta_k(s + \lambda_k - \lambda_k z)}{s + \lambda_k - \lambda_k z}. \tag{8}$$

Remark 2. *From Theorem 3 the stationary queue length can be obtained.*

5 Conclusion

The main purpose of research of polling systems is to determine the characteristics of systems development. In this paper there were presented some performance characteristics for exhaustive polling model with semi-Markov delays, such as the distribution of k-period, probability of states, queue lengths distribution, that can be used in analysis of different type of systems from real life.

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Numerical simulation MHD problems in astrophysics

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Abstract

In the paper we discuss problems of numerical simulation of magnetohydrodynamical (MHD) astrophysical problems.

Majority of astrophysical problems are characterized by wide variation of parameters, often by many orders of magnitude. The numerical methodology which we use for the simulation is based on finite volume approach. The operator-difference scheme is implicit and completely conservative. As examples of successful application we describe simulation results of cold protostellar cloud collapse and magnetorotational supernova explosion.

Keywords: numerical methods, astrophysics, magnetohydrodynamics.

1 Introduction

Modern astrophysics is a science where significant part of problems can be solved only by mathematical simulation. The only possibility to make experiment in astrophysics except observations is to make numerical simulations. A lot of astrophysical problems can be described in the frame of gas dynamics or magnetohydrodynamics.

For the simulation of such kind of problems we use numerical method based on finite volume approach. Completely conservative implicit operator-difference scheme on triangular Lagrangian grid of variable structure was applied to the simulation of such astrophysical problems as cold protostellar rapidly rotating cloud collapse, magnetorotational supernova explosion.

2 Basic equations and numerical method

For the simulations we consider a set of magnetohydrodynamical equations with selfgravitation and with infinite conductivity [1].

We have used a numerical technique, based on a conservative (in the absence of gravitation), implicit first order of accuracy in space and time operator difference scheme on a triangular grid with a grid reconstruction, developed and described in [2,3].

For a numerical solutions, we introduce a triangular grid, covering the restricted domain in r, z coordinates.

We suppose that components of the velocity vector $\mathbf{u}$ and gravitational potential Φ will be defined in all knots of the grid. The density ρ, pressure p, components of magnetic field vector $\mathbf{H}$ will be defined in the cells and in the boundary knots of the grid.

For the numerical simulation of the system of gravitational MHD equations the following method has been used. Instead of differential operators (div, grad, rot) we introduce their finite difference analogues. On the base of such operators a completely conservative scheme has been constructed. The scheme is implicit for all velocity components v_r, v_φ, v_z and for the toroidal component of the magnetic field H_φ. The scheme is explicit for the poloidal magnetic field H_r, H_z. The explicitness of the scheme for H_r, H_z does not introduce strong restriction on the time step, because during the evolution of the magnetic field its poloidal values do not change strongly, while the toroidal component appears and increases significantly with time. The scheme is explicit for the gravitational potential, but it was shown in [2] that this explicitness does not introduce significant restrictions on the time step.

3 Results of simulations

The method briefly described in the previous section was applied to the calculations of the collapse of rapidly rotating gas cloud. The problem with the same physical conditions have been investigated by many authors. We have obtained the results for the stages of the collapse up to the secondary compression and discuss the reliability of the results in comparison with previous authors [4]. Our results are in qualitative accordance with other ones got by Lagrangian methods, while we managed to prolong the calculations farther, and we do not confirm the formation of the ring structure, obtained by using of the Eulerian schemes.

We present the results of 2D simulations of the magnetorotational model of the supernova explosion. The idea of the magnetorotational mechanism of supernova explosion was suggested by G.S.Bisnovatyi-Kogan in 1970 [5]. After the core collapse the core consists of rapidly rotating proto-neutron star and differentially rotating envelope. The generated by the differential rotation toroidal part of the magnetic energy grows linearly with time at the initial stage of the evolution of the magnetic field. The linear growth of the toroidal magnetic field is terminated by the development of magnetohydrodynamic instability, leading to the drastic acceleration of the growth of magnetic energy. At the moment, when the the magnetic pressure becomes comparable with the gas pressure at the periphery of the proto-neutron star $\sim 10-15$km from the star center, the MHD compression wave appears and goes through the envelope of the collapsed iron core. It transforms soon to the fast MHD shock and produces supernova explosion. Our simulations give the energy of the explosion $0.6 \cdot 10^{51}$ ergs. The amount of the ejected by the explosion mass $\sim 0.14 M_\odot$ [6].

4 Conclusion

The completely conservative implicit operator-difference scheme was successfully applied to the simulation of the problem of rapidly rotating cold protostellar cloud collapse and magnetorotational supernova explosion. The simulations were made in 2D, while generalization of the numerical method on 3D case can be done in a natural way [7].

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Stochastic Modeling of Economic Growth

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Abstract

In this paper stochastic approach to the economic growth modeling will be proposed.The problem of how government policy, especially in domain of corruption, influenced economic growth will be examined. For this problem solving stochastic maximum principle will be applied.

Keywords: economic growth, stohcastic modeling, stochastic maximum principle, government policy, corruption.

1 Introduction

Economic growth is the basic component of the sustainable socio economic development in any country, especially in the country in transition like Republic of Moldova. Therefore, optimal utilization of the government expenditure in order to minimize individual theft from it is a main problem to solve. In this direction some applications referred to the Republic of Moldova economic growth were examined [1]. In present paper the same problem at the branches level will be discussed, and for its solving the method [3] will be applied.

2 Formulation of the problem

As in [2] the model of multiple equilibrium in corruption and economic growth in stochastic formulation will be examined. Suppose that a N

individuals denoted by $i \in [0, N]$ are aimed to maximize an objective function

$$\max_{c_i} L = E\Big[\int_0^\infty e^{-\rho t}(\frac{c_{it}^{1-\sigma}}{1-\sigma})dt\Big], \tag{1}$$

where E is an expectation operator, and c_i is particular consumption of the individual i, and σ is the value inverse to the intertemporal elasticity of substitution. It is assumed that population is constant over time and normalized to 1. At every moment of time one unit of labor services of each individual are allocated between productive work, Lp, and theft from the government, S: $Lp_i + S_i = 1, \quad L_i \geq 0, \quad S_i \geq 0 \quad \forall i$. Government expenditure takes part from production function as in [2] and may be suited by rent seekers, who are able either to consume the proceeds or to invest them in their own firms.

The total amount of resources that are extracted by individual i is $S_i G\phi(\bar{S})$, $\phi'(.) > 0$, $0 < \phi(\bar{S}) < 1 \ \forall \bar{S} \in (0,1)$, here $\bar{S} = \int_0^1 S_i di$. So, this amount depends on the time that i spends stealing, S_i, and the amount of productive government expenditure available, G. $\phi(\bar{S})$ represents the proportion of stolen resources actually kept by the rent seeker. It is assumed that ϕ is a positive function in respect with $\bar{S}$, the total rent steeling activity in the economy. The production function for firm i is

$$Y_i = K_i^{1-\alpha} L_i^{\alpha}\{G[1 - \bar{S}\phi(\bar{S})]\},$$

where G is government expenditure, K_i is the capital stock belonging to firm i, $\bar{S}\phi(\bar{S})$ is the amount stolen that fails to reach the production processes as an input. It is assumed [2] that $\tau = G/Y = constant$, where τ is the tax rate. In per capita indicators, production function looks as:

$$y_i = k_i^{1-\alpha} l_i^{\alpha} g[1 - \bar{S}\phi(\bar{S})].$$

So, capital per capita belonging to individual i in the stochastic case evolves according to

$$dk_i = f(k_i, c_i)dt + qdz, \tag{2}$$

where dz is a stochastic Wiener processes. Now, the stochastic optimal control problem may be formulated as maximization of the objective function Eq. (1) subjected to the following restriction Eq. (2):

$f(k_i, c_i) = (1-\tau)wl_i + rk_i - c_i + S_i g_i \phi(\bar{S})$, and vector function u is given. The corresponding optimality conditions are:

$$0 = \max_{k_i}[u(k,c) + \frac{1}{dt}E(dL)], \tag{3}$$

and the Hamton Jacoby Bellman (HJB) equation becomes:

$$0 = \max_{k_i}[u(k,c) + L_t + L_k f + \frac{1}{2}q^2 L_{kk}]. \tag{4}$$

The Hamiltonian function H, for the stochastic case is given as: $H = u + L_t + L_k f + \frac{1}{2}q^2 L_{kk}$. Taking derivatives from HJB equation with respect to k we obtain

$$L_{kt} + L_{kk}f + \frac{1}{2}q^2 L_{kkk} = -u_k - f_k L_k - \frac{1}{2}(q^2)_k L_{kk}. \tag{5}$$

Using chain rule and considering second order contributions of the derivatives with respect to k (Ito's Lemma) reduces to:

$$dL_k = \frac{\partial L_k}{\partial k}\frac{dk}{dt}dt + \frac{1}{2}(q^2)_k L_{kk}. \tag{6}$$

Because from Ito's Lemma, $E[d(k)^2] = q^2 dt$ and substituting previous equation in the last one we obtain

$$\frac{dL_k}{dt} = L_{kt} + L_{kk}f + \frac{1}{2}L_{kkk}q^2 \tag{7}$$

is obtained:

$$\frac{dL_k}{dt} = -u_k - f_k L_k - \frac{1}{2}(q^2)_k Lkk. \tag{8}$$

Deriving previous equation with respect to k, using chain rule and considering second order contributions in the derivatives with respect to k (Ito's Lemma) is obtained:

$$dL_{kk} = \frac{\partial L_{kk}}{\partial t}dt + \frac{\partial}{L_{kk}}\partial k\frac{dk}{dt}dt + \frac{1}{2}\frac{\partial^2 L_{kk}}{(\partial k)^2}dk^2. \tag{9}$$

Then, Ito's Lemma, chain rule application and substituting Eq. (7) in Eq. (9) get:

$$\frac{dL_{kk}}{dt} = -u_{kk} - 2L_{kk}f_k - L_k f_{kk} - (q^2)_k L_{kkk} - \frac{1}{2}(q^2)_{kk}L_{kk}. \tag{10}$$

Equating adjoint variable, μ to the first derivatives of the objective function L with respect to state variable k and ω as the second derivatives with respect to the state variable, the following is obtained:

$$\frac{d\mu}{dt} = -u_k - f_k\mu - \frac{1}{2}(q^2)_k\omega, \tag{11}$$

$$\frac{d\omega}{dt} = -u_{kk} - 2\omega f_k - \mu f_{kk} - \frac{1}{2}(q^2)_{kk}\omega. \tag{12}$$

Here $\mu = (\mu_1, \mu_2, ..., \mu_n)$ and $\omega = (\omega_1, \omega_2, ..., \omega_n)$ are two n-dimensional vectors. Notice that the resulting problem is a $2n$ point boundary value problem. Summarizing for the stochastic case, the Hamiltonian function and the adjoint equations to be solved are

$$H = u + L_k f + \frac{1}{2}(q^2)L_{kk}, \tag{13}$$

$$\frac{d\mu}{dt} = -u_k - f_k\mu - \frac{1}{2}(q^2)_k\omega, \tag{14}$$

$$\frac{d\omega}{dt} = -u_{kk} - 2\omega f_k - \mu f_{kk} - \frac{1}{2}(q^2)_{kk}\omega. \tag{15}$$

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Application of HPC technologies for mathematical modeling of seismic impact on underground structures

Boris Rybakin, Peter Bogatencov,
Grigore Secrieru, Elena Gutuleac

Abstract

The paper describes the mathematical model and numerical results of the study of seismic impact on the buried in the ground projectile with fuse.

Keywords: mathematical modeling, High Performance Computing, seismic effects modeling.

1 Introduction

Today the new knowledge retrieval in the area of technological risks assessment and the creation of new technologies, services and products have directly related to the application of mathematical modeling and use of multiprocessor computer systems. The creation of high-performance computing environment and effective software applications are important scientific and practical results that allow high-precision simulation and visualization of complex physical processes and objects by mathematical modeling without full-scale experiments [1]. It is fully applied to a wide range of problems of solid mechanics with the influence of various physical effects and properties characteristics of construction materials [2, 3, 4]. Modeling of such problems is characterized by high demands to computing resources and requires using of specialized HPC methods and technologies.

2 Problem formulation and mathematical model

The two-dimensional model of elastic-plastic medium has been chosen to describe the behaviour of the structural material and explosive substance (ES). The given model belongs to the class of models with internal state parameters and bases on thermodynamic principles of continuum mechanics. The equations of state of the medium allow closing the system of determining equations of motion, then select the numerical method for the developing of numerical solution algorithm. The relevant equations of state have been chosen for description of the structural material, explosive substance, detonation products and ground [2, 4].

Seismic wave has generated as the function of pressure at the vertical boundary of the computational domain: $P(t) = (P_0 e^{-\alpha t} + P_{0C})H(t)$, where $P_0 + P_{0C} = P_{0S}$ is peak impact pressure, P_{0C} is constant pressure, α is attenuation constant, $H(t)$ is unit step function [4].

3 Numerical calculations and results

At the stage of computational experiment designing for solving such problems it is necessary to specify the initial and boundary conditions, as well as the values of various parameters and constants of materials for realistic equations of state. At time $t = 0$ microseconds (μs) the computational domain is a rectangular area in a two-dimensional Lagrangian coordinate system, where the initial state of relevant system components is simulated. The projectile is modeled as aluminium construction (width is R, length is $3R$, wall thickness is $R/10$), filled with explosive substance (TNT). In the end of the structure there is the fuse (TNT with 4% higher sensitivity than the filler). The following numerical values of the constants are used in the calculations:

structural parameters (aluminium) are $\rho_0 = 2.7 g/cm^3$, $\mu_0 = 0.276\ Mbar$, $Y_0 = 0.029\ Mbar$, $Y_{max} = 0.068\ Mbar$, $b = 3.0$, $n = 0.35$, $h = 0.62$, $k_1 = 7.906 \cdot 10^{-1}$, $k_2 = 1.325$, $k_3 = 2.13$, $\gamma_0 = 2.0$, $s =$

1.34, $C = 0.525$, $\beta = 1.25 \cdot 10^{-1}$;
explosive substance (TNT) parameters are $\rho_0 = 1.72\ g/cm^3$, $k = 0.123$, $n = 3.0$, $A = 0.0764$, $\gamma = 3.0$.
The whole construction is surrounded by the water-saturated ground, including 10% of air, 30% of water, 60% of quartz. Seismic parameters are $\alpha = 0.02$, $P_{0C} = 2.8 \cdot 10^{-6}\ Mbar$.

Two series of calculations for two values of the initial pressure $P_0(17.0 \cdot 10^{-4} Mbar$ and $19.2 \cdot 10^{-4} Mbar)$ were conducted. The general computation time was 180 μs. The plane wave from the vertical boundary of the computational domain began propagation in time $t = 0\ \mu s$, reached the end of construction and partially reflected from it. In both cases the shock wave came to the fuse in the same time ($t = 50\ \mu s$). The fuse did not blast out for the value of the initial pressure $P_0 = 17.0 \cdot 10^{-4} Mbar$, but it detonated in $t = 51\ \mu s$ for $P_0 = 19.2 \cdot 10^{-4} Mbar$). The numerical results show that minimum pressure to initiate detonation of fuse is $P = 0.0054\ Mbar$.

4 Conclusion

In the paper the examples of results of numerical calculations for solving solid mechanics problems that consider physical effects and material properties are presented. The numerical analysis shows that for the given initial and boundary conditions projectile detonation primarily depends on the initial pressure value in the seismic waves function. The critical pressure to initiate fuse detonation is $P = 0.0054\ Mbar$.

It should be noted that the proposed algorithm is quite effective to be executed using HPC technology. Several international projects contributed to the creation of the own high-performance computing resources in Moldova. In particular, the computing clusters with parallel architecture were installed in the Moldova State University, and in the Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova. Development of the HPC and distributed computing infrastructure in Moldova at present is supported by European Commission and by bilateral project STCU-ASM. Existing computer resources are used for testing technologies and development of parallel applications of mathematical modeling, including modeling of seismic effects on underground structures.

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Explicit Thermal Stresses Within a Thermoelastic Half-Strip and Their Graphical Presentation Using Maple – 15 Soft

Victor Şeremet, Ion Creţu, Dumitru Şeremet

Abstract

This article presents a closed form of new solution of a particular boundary value problem (BVP) of thermoelasticity for a half-strip. The thermoelastic displacements and thermal stresses are created by an interior temperature gradient given within a rectangle of the thermoelastic half-strip. To solve this BVP we use Maysel's integral formula for thermoelastic displacements, Duhamel-Neumann law for thermal stresses and the obtained before influence functions for volume dilatation $\Theta^{(i)}(x,\xi); i = 1,2$. Graphics of the derived in elementary functions thermal stresses are plotted using soft Maple 15.

Keywords: Green's functions, temperature gradient, thermal stresses, volume dilatation.

1 Calculation of the thermal stresses σ_{ij}

Suppose we want to determine the thermal stresses $\sigma_{ij}(\xi); i,j = 1,2$ in the half-strip $V \equiv (0 \leq x_1 < \infty, 0 \leq x_2 \leq a_2)$, caused by the following interior temperature gradient $\Delta T(x)$ given within the rectangle $V^{\shortmid} \equiv [a \leq x_1 \leq b, c \leq x_2 \leq d] \in V$:

$$\Delta T(x) = \begin{cases} T_0 = const., & x \equiv (x_1, x_2) \in V^{\shortmid} \in V, \\ & a \geq 0, b \geq 0, c \geq 0, d \geq 0; \\ 0, & x \equiv (x_1, x_2) \in \Omega \equiv V \setminus V^{\shortmid}. \end{cases} \tag{1}$$

at the homogeneous mechanical boundary conditions:

$$\begin{aligned} &u_1 = \sigma_{12} = 0; \xi_1 = 0, 0 \leq \xi_2 \leq a_2; \\ &\sigma_{22} = u_1 = 0; \xi_2 = 0, 0 \leq \xi_1 < \infty; \\ &u_2 = \sigma_{21} = 0; \xi_2 = a_2, 0 \leq \xi_1 < \infty. \end{aligned} \tag{2}$$

To solve this BVP we have to use Maysel's formula [2]:

$$u_i(\xi) = \gamma \int\limits_V \Delta T(x) \Theta^{(i)}(x, \xi) dx_1 dx_2; i = 1, 2, \tag{3}$$

where $\Theta^{(i)}(x, \xi)$ are the influence functions of an inner unit point force on the volume dilatation and $\gamma = \alpha(2\mu + 3\lambda)$ is a thermoelastic constant; α is coefficient of linear thermal expansion; λ, μ are Lame's constants of elasticity. Functions $\Theta^{(i)}(x, \xi)$ were derived in the handbook [1] (see problem 12.L. 8 and the answer to it).

So, Maysel's formula (3) in our case can be rewritten as follows:

$$u_i(\xi) = \gamma T_0 \int\limits_a^b dx_1 \int\limits_c^d \Theta^{(i)}(x, \xi) dx_2 =$$

$$= -\frac{\gamma T_0}{4\pi(\lambda + 2\mu)} \frac{\partial}{\partial \xi_i} \int\limits_a^b dx_1 \int\limits_c^d \ln \frac{\bar{E}\bar{E}_1\tilde{E}_2\tilde{E}_{12}}{\tilde{E}\tilde{E}_1\bar{E}_2\bar{E}_{12}} dx_2, \tag{4}$$

where the functions $\bar{E}, \bar{E}_1, \tilde{E}_2, \tilde{E}_{12}, \tilde{E}, \tilde{E}_1, \bar{E}_2, \bar{E}_{12}$ are determined by the expressions:

$$\begin{aligned} &\bar{E} = 1 + 2e^{(\pi/2a_2)(x_1 - \xi_1)} \cos(\pi/2a_2)(x_2 - \xi_2) + e^{(\pi/a_2)(x_1 - \xi_1)}; \\ &\bar{E}_1 = \bar{E}(x; -\xi_1, \xi_2); \bar{E}_2 = \bar{E}(x; \xi_1, -\xi_2); \bar{E}_{12} = \bar{E}(x; -\xi_1, -\xi_2); \\ &\tilde{E} = 1 - 2e^{(\pi/2a_2)(x_1 - \xi_1)} \cos(\pi/2a_2)(x_2 - \xi_2) + e^{(\pi/a_2)(x_1 - \xi_1)}; \\ &\tilde{E}_1 = \tilde{E}(x; -\xi_1, \xi_2); \tilde{E}_2 = \tilde{E}(x; \xi_1, -\xi_2); \tilde{E}_{12} = \tilde{E}(x; -\xi_1, -\xi_2). \end{aligned}$$

Next, substituting in Duhamel-Neumann law [2]:

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \delta_{ij}(\lambda u_{k,k} - \gamma T); i, j = 1, 2, \tag{5}$$

Eq. (4) and taking the respective integrals, we obtain the final expressions for thermal stresses:

$$\sigma_{11}(\xi) = \frac{\mu\gamma T_0}{\pi(\lambda+2\mu)} F(\xi) + \begin{cases} -\gamma T_0; & \xi \in V^{\prime} \\ 0, & \xi \in \Omega \end{cases}; \tag{6}$$

$$\sigma_{22}(\xi) = -\frac{\mu\gamma T_0}{\pi(\lambda+2\mu)} F(\xi) + \begin{cases} -\gamma T_0; & \xi \in V^{\prime} \\ 0, & \xi \in \Omega \end{cases}; \tag{7}$$

$$\sigma_{12}(\xi) = -\frac{\mu\gamma T_0}{4\pi(\lambda+2\mu)} \ln \frac{\bar{E}\tilde{E}_1\bar{E}_2\tilde{E}_{12}}{\tilde{E}\bar{E}_1\tilde{E}_2\bar{E}_{12}} \Bigg|_{x_2=c}^{x_2=d} \Bigg|_{x_1=a}^{x_1=b}. \tag{8}$$

In Eqs. (6) and (7) the function $F(\xi)$ is defined by the following expression:

$$F(\xi) = \left[-\bar{f} + \tilde{f}_1 + \bar{f}_2 - \tilde{f}_{12} - \tilde{f} + \bar{f}_1 + \tilde{f}_2 - \bar{f}_{12} \right]_{x_1=a;x_2=c}^{x_1=b;x_2=d}, \tag{9}$$

where the functions $\bar{f}, \tilde{f}_1, \bar{f}_2, \tilde{f}_{12}, \tilde{f}, \bar{f}_1, \tilde{f}_2, \bar{f}_{12}$ are determined as follows:

$$\bar{f} = \arctan \frac{e^{(\pi/2a_2)(x_1-\xi_1)} + \cos(\pi/2a_2)(x_2-\xi_2)}{\sin(\pi/2a_2)(x_2-\xi_2)};$$
$$\bar{f}_1 = \bar{f}(x;-\xi_1,\xi_2); \bar{f}_2 = \bar{f}(x;\xi_1,-\xi_2); \bar{f}_{12} = \bar{f}(x;-\xi_1,-\xi_2);$$
$$\tilde{f} = \arctan \frac{e^{(\pi/2a_2)(x_1-\xi_1)} - \cos(\pi/2a_2)(x_2-\xi_2)}{\sin(\pi/2a_2)(x_2-\xi_2)};$$
$$\tilde{f}_1 = \tilde{f}(x;-\xi_1,\xi_2); \tilde{f}_2 = \tilde{f}(x;\xi_1,-\xi_2); \tilde{f}_{12} = \tilde{f}(x;-\xi_1,-\xi_2).$$

2 Graphical presentation of the thermal stresses σ_{ij}

Graphics of thermal stresses $\sigma_{11}, \sigma_{22}, \sigma_{12}$ caused by the following interior temperature gradient $T_0 = 50K$ given within the rectangle

$V^{\shortmid} \equiv [6 \leq x_1 \leq 10, 4 \leq x_2 \leq 6] \in V$ were constructed at the following values of elastic and thermal constants: Poisson ratio $\nu = 0.3$, the modulus of elasticity $E = 2.1 \cdot 10^5 MPa$ and $\alpha = 1.2 \cdot 10^{-5}(K)^{-1}$. The graphics constructed by using computer program Maple 15 are presented in the Fig. 1.

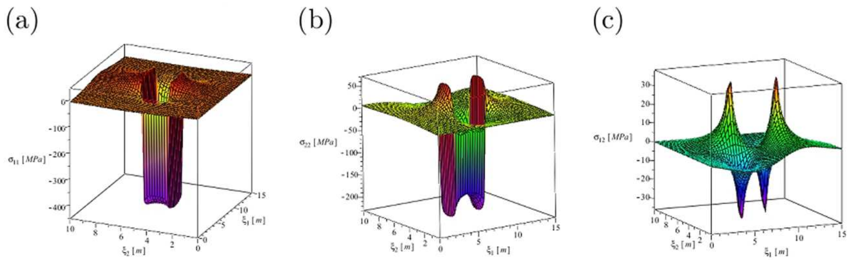

Figure 1. Graphics of normal σ_{11}, σ_{22} and tangential thermal stresses σ_{12} (figures 1(a), 1(b)) and (figure 1(c))

3 Conclusion

Analyzing the graphics of the thermal stresses (see Fig. 1), it should be noted that all boundary conditions are satisfied. Also, in the corner points of the inner rectangle the thermal stresses have some jumps (singularities).

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Queuing models in the port activity

Ionela Rodica Ţicu

Abstract

The methodological support of the research is based on some concepts from the probability theory, queuing systems theory, methods used in the theory of random processes used to get more efficient the port activity.

Keywords: Polling systems, probability theory, mathematical model, random processes, port activity

1 Introduction

A class of models is represented by the priority queuing models, the study of which is an important section of the theory of queuing systems that has developed intensively in recent years. In priority queuing systems, the serving station often needs random time to change the current orientation towards queues. Thus, a new class of queuing systems called generalized systems appears; from mathematical point of view, they are more advanced than the traditional models and more appropriate to the real processes. The results obtained for generalized models of priority queuing systems are extremely important both from a theoretical and applicative point of view. These models play a crucial role in the analysis and design of regional wireless broadband computer networks.

2 Polling Systems

The Polling systems have been introduced in the early '70s as time-sharing models for computer systems.

The Polling systems are queuing systems with a single server that "scans" the various classes of requests (different queues of requests), serving the requests of each class for a certain length of time, then they reorient towards another class, doing so until the queue is empty, or by another rule.

3 Polling Models

The Polling models are queuing models with a single node where there are several waiting areas for different queues of messages, and a server that serves each waiting queue based on a set rule (the so-called Polling table). Queuing models with a single node are very important in the queuing theory, as they offer very good prospects for the study of complex queues with multiple nodes.

3.1 The General Type Polling Model

We shall consider the Polling queuing system with semi-Markov exchange. The mechanism of service is given by the Polling Table $f : \{1, 2, k, n\} \rightarrow \{1, 2, k, r\}$, where the function f indicates that in the phase $j, j = 1, n$, user $k, k = 1, r$ is served. The items (posts) of user k arrive after the Poisson distribution law with the parameter $\tilde{\lambda}_k$. The serving time for the class item k is given by the random variable B_k with the distribution function $B_k(x) = P\{B_k < x\}$.The guidance duration from a user to user k is a random variable $B_k(x) = P\{B_k < x\}$ with the following distribution function $C_k(x) = P\{C_k < x\}$.

The main goal of studying the Polling system is to determine the important characteristics of the system, for example: the busy period, the probability of states, the queue length, etc. But not always analytical formulas can be directly used to determine these characteristics, therefore, the development of new numerical methods and algorithms based on these methods are very important.

3.2 Research Methods and Algorithms of Polling systems

Currently, several methods are proposed for the research of Polling systems. We shall briefly stop at some of them.

3.2.1 The method of semi-regenerate processes

The method of semi-regenerate processes was developed by the teachers V. Rikov and Gh. Mishkcoy. This method supplemented by the notion of generalized model allowed obtaining new analytical results for a broad class of Polling models: models with nonzero exchange (semi-Markov type) of states. A number of features were obtained, for example: for k-busy period, the probabilities of states, the allocation of the queue both for the stationary state and for the virtual state etc.

3.2.2 The averages method

The averages method is extensively described and is designed to calculate the average length of the queues in an arbitrary time point from systems, for which the average length of queues attendance, particularly in systems with M/G/1 cyclic survey and comprehensive and access service can be obtained. Based on the average time of queue attendance and of the average remaining value, the average number of requests in the queues of systems is calculated, as solution of the systems of linear equations. It is known that the averages method can be extended for the following Polling systems: grouped Poisson flow systems, systems with periodic survey, discrete-time systems, also the application of the method to the approximate analysis of other Polling models. The averages method is applied to calculate the approximate average of the waiting time in systems with limited serving queues.

4 Conclusion

In the theory of queuing systems, great attention is currently paid to the development of numerical algorithms for finding basic probabilistic characteristics of the investigated queuing systems. In order to numerically solve one of these probabilistic features, numerical algorithms are proposed in this paper and new results are presented; based on these results, matrix algorithms are developed and justified in order to determine the distribution and the busy period in terms of Laplace-Stieltjes transform, for Polling systems with semi-Markov delays, respectively for generalized priority queuing systems.

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Iterative approach for solving fuzzy multi-criteria transportation problem of "bottleneck" type

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Abstract

In the paper an iterative fuzzy programming approach is developed for solving the multi-objective transportation problem of "bottleneck" type with some imprecise data. Minimizing iteratively the worst upper bound to obtain an efficient solution which is close to the best lower bound for each objective function, we find the set of efficient solutions for all time levels.

Keywords: fuzzy programming, fuzzy model, transportation problem, efficient solution.

1 Introduction

It is well known, the increasing of criteria number and imposing of minimal time to realize the model solution leads only to increasing of solution accuracy for optimal decision making problems. There are many efficient algorithms that solve such models with deterministic data [2]. Since in real life, often some parameters are of fuzzy type, in the proposed work this case is studied.

2 Problem formulation

Because in any optimization model, objective function coefficients have the largest share in the objective function variations, we shall consider these of fuzzy type and develop the next multi- criteria transportation problem of "bottleneck" type with fuzzy costs coefficients:

$$\min Z_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{1} x_{ij} \quad \min Z_2 = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{2} x_{ij}$$

$$\ldots\ldots\ldots$$

$$\min Z_r = \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{c}_{ij}^{r} x_{ij} \quad \min Z_{r+1} = \max_{i,j} \{\, t_{i,j} |\, x_{i,j} > 0\} \tag{1}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad \forall i = \overline{1,m}, \quad \sum_{i=1}^{m} x_{ij} = b_j, \quad \forall j = \overline{1,n},$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \quad x_{ij} \geq 0 \; for\ all\ \ i\ and\ j,$$

where : $\tilde{c}_{ij}^{k}$, $k = 1, 2 \ldots r$, $i = 1, 2, \ldots m$, $j = 1, 2, \ldots n$ are costs or other amounts of fuzzy type, t_{ij} – necessary unit transportation time from source i to destination j, a_i – disposal at source i, b_j – requirement of destination j, x_{ij} – amount transported from source i to destination j.

In the model the criteria of maximum also may exist, which however does not complicate it.

3 Theoretical analysis of fuzzy cost multi-criteria transportation model

Since the parameters and coefficients of transportation multi-criteria models have real practical significances such as unit prices, unit costs and many other, all of them are interconnected with the same parameter of variation, which can be calculated by applying various statistical methods. We propose to calculate it using the following formula:

$$p_{ij}^{k} = \frac{c_{ij}^{k} - \underline{c}_{ij}^{k}}{\bar{c}_{ij}^{k} - \underline{c}_{ij}^{k}}, \tag{2}$$

where: $\underline{c}_{ij}^{k}$, $\bar{c}_{ij}^{k}$ – are the limit values of variation interval for each cost coefficient c_{ij}^{k}, where: $i = \overline{1,m},\ j = \overline{1,n},\ k = \overline{1,r}$.

Agreeing to the formula (2), the parameters $\{p_{ij}^{k}\}$ can be considered as the probabilistic parameters of belonging for every value of coefficients $\{c_{ij}^{k}\}$ from their corresponding variation intervals.

The main idea of the method that follows, is the simultaneously and interconnected variation of objective functions coefficients. This makes possible to reduce the model (1) to a set of deterministic models that can be solved by applying the fuzzy techniques [1].

4 Some reasoning and algorithms

Seeing that the model (1) is of multi-criteria type, for its solving usually it builds a set of efficient solutions, known also as Pareto-optimal solutions. Since solving model (1) involves its iterative reducing to some deterministic we should propose firstly the following definitions.

Let us suppose that: $\left(\bar{\mathrm{X}}, \bar{\mathrm{T}}\right)$ is one basic solution for the model (1), where: $\bar{\mathrm{T}} = \max\limits_{i,j} \{\bar{t}_{ij}/\bar{x}_{ij} > 0\}$ *and* $\overline{X} = \{\overline{x_{ij}}\}$, $i = \overline{1,m}$, $j = \overline{1,n}$ is one basic solution for the first r-criteria model (1).

Definition 1. *The basic solution $\left(\bar{\mathrm{X}}, \bar{\mathrm{T}}\right)$ of the model (1) is a basic efficient one if and only if for any other basic solution $(X,T) \neq \left(\bar{X}, \bar{T}\right)$ for which exists at least one index $j_1 \in (1,...r)$ for which the relation $Z_{j_1}(X) \leq Z_{j_1}\left(\bar{X}\right)$ is true, there immediately exists another, at least, one index $\exists j_2 \in (1,....r)$, where $j_2 \neq j_1$, for which at least, one of the both relations $Z_{j_2}\left(\bar{X}\right) < Z_{j_2}(X)$ or $\bar{T} < T$ is true. If all of these three inequalities are verified simultaneously with the equal sign, it means that the solution is not unique.*

Definition 2. *The basic solution $\left(\bar{\mathrm{X}}, \bar{\mathrm{T}}\right)$ of the model (1) is one of the optimal(best) compromise solution for a certain time $\bar{T}$, if the solution $\bar{\mathrm{X}}$ is located closest to the optimal solutions of each criterion.*

In order to solve deterministic model (1) we can use the ***fuzzy technique*** [1] and iteratively solve the deterministic model (3) for the best – L_k and the worst U_k values of k-criterion.

Max λ *in the same availability conditions as in* (1) *and:*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}^{k} x_{ij} + \lambda \cdot (U_k - L_k) \leq U_k, \quad k = \overline{1, r}, \tag{3}$$

Applying iteratively the fuzzy technique for each increasing time level, we could get the set of all its optimal compromise solutions.

5 Conclusions

By applying the hypothesis about the interconnection and similar variation of the model's objective functions coefficients, we reduce the model (1) to several models of deterministic type, each of which may be solved using fuzzy technique.

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Optimal and Stackelberg controls of linear discrete processes influenced by echoes

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Abstract

We develop mathematical models of Stackelberg control for linear discrete-time processes influenced by echoes as the development of Stackelberg control models and principles — particular models and principles of the more general mathematical models of Pareto-Nash-Stackelberg control introduced by V. Ungureanu in [10]. Application of a straightforward method to solve investigated problems along with the Pontryagin's principle produces important theoretical and practical results. Software benchmarks confirm and illustrate value of the results.

Keywords: linear discrete control problem, echo, Stackelberg control, Stackelberg equilibrium.

1 Introduction

Mathematical models of Stackelberg control are special models of control [10], which may be considered a particular case of more general models of Pareto-Nash-Stackeberg control processes, based on mixture of control and games of simultaneous and sequential types [9, 10, 11, 6, 1, 8, 4, 5, 2]. A new kind of mathematical models appears if the influence of the echoes of precedent stages phenomena occurs on the system state and control. Such models need a definition of solution concepts and the applying of straightforward principle or the maximum principle of Pontryagin [6, 10] to obtain the solutions.

2 Optimal control of linear discrete-time processes with periodic echoes

Let us consider a dynamic system with a state evolution and control described (governed) by the following mathematical model:

$$\begin{array}{rcl} f(x,u) & = & \sum\limits_{t=1}^{T}(c^t x^t + b^t u^t) \rightarrow \max, \\ x^t & = & \begin{cases} A^{t-1}x^{t-1} + B^t u^t, & t \notin [k\tau, k\tau+\omega), \\ A^{t-1}x^{t-1} + B^t u^t + C^{t-\gamma}u^{t-\gamma}, & t \in [k\tau, k\tau+\omega), \end{cases} \\ D^t u^t & \leq & d^t, \\ & & t = 1, ..., T, \\ & & k = 1, 2, ..., \left\lfloor \frac{T}{\tau} \right\rfloor, \end{array} \qquad (1)$$

where $x^0, x^t \in R^n$, $u^t \in R^m$, $c^t \in R^n$, $b^t \in R^m$, $A^{t-1} \in R^{n\times n}$, $B^t \in R^{n\times m}$, $d^t \in R^k$, $D^t \in R^{k\times n}$, $t = 1, ..., T$. Evolution parameters are fixed and have obvious interpretation: $\tau \in \mathbb{N}$ is a time period with which echo start to be active and to influence on system, $\omega \in \mathbb{N}$ is a length of time interval for which echo is active, $\gamma \in \mathbb{N}$ is the length of time between control applying and its echo producing. Some controls may not have echoes. Without loss of generality, we can assume that $\gamma, \omega \leq \tau$. Evidently, x^0 is the initial state, $x = (x^0, ..., x^T)$ is the system's trajectory, $u = (u^1, ..., u^T)$ forms the system's trajectory control.

Theorem 2.1. *Let (1) be solvable. The control $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, is optimal if and only if $\bar{u}^t$ is the solution of linear programming problem*

$$\begin{array}{rcl} \psi_t(u^t) & \rightarrow & \max, \\ D^t u^t & \leq & d^t, \end{array}$$

for $t = 1, ..., T$, where $\psi_t(u^t)$ is a linear function on u^t obtained by direct/forward substitution in (1).

3 Stackelberg control of linear discrete-time processes influenced by periodic echoes

Let us modify (1) by considering a dynamic system with a state evolution and Stackelberg control described (governed) by the following mathematical model [10]:

$$
\begin{array}{rcl}
f_t(x, u_t) & = & \displaystyle\sum_{t=1}^{T}(c^t x^t + b^t u^t) \rightarrow \max, \\
x^t & = & \left\{ \begin{array}{ll} A^{t-1}x^{t-1} + B^t u^t, & t \notin [k\tau, k\tau + \omega)\,, \\ A^{t-1}x^{t-1} + B^t u^t + C^{t-\gamma}u^{t-\gamma}, & t \in [k\tau, k\tau + \omega)\,, \end{array} \right. \\
D^t u^t & \leq & d^t, \\
& & t = 1, ..., T, \\
& & k = 1, 2, ..., \left\lfloor \frac{T}{\tau} \right\rfloor,
\end{array}
\tag{2}
$$

where $f_t(x, u_t)$ is the gain function of the t^{th} player. At every stage t the player t mooves.

Theorem 3.1. *Let (2) be solvable. The control $\bar{u}^1, \bar{u}^2, ..., \bar{u}^T$, is optimal if and only if $\bar{u}^t$ is the solution of linear programming problem*

$$
\begin{array}{rcl}
f_t(u^t) & \rightarrow & \max, \\
D^t u^t & \leq & d^t,
\end{array}
$$

for $t = 1, ..., T$, where $f_t(u^t)$ is a linear function on u^t obtained by direct/forward substitution in (2).

4 Conclusion

We present seminal results from a more large work dedicated to PNS control processes with echoes which will follow soon.

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A Hierarchical Approach to Multicriteria Problems

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Abstract

It is shown, that any multicriteria problem can be represented by a hierarchical system of criteria. Individual properties of the object (alternative) are evaluated at the bottom level of the system, using a criteria vector; and a composition mechanism is used to evaluate the object as a whole at the top level. The problem is solved by the method of nested scalar convolutions of vector-valued criteria. The methodology of the problem solving is based on the complementarity principle by N. Bohr and the theorem of incompleteness by K. Gödel.

Keywords: hierarchical structure; nested scalar convolutions; multicriteria approach; decomposition; composition.

1 Problem Description

The problem of decision making in general view can be represented by the scheme

$$\{\{x\}, Y\} \to x^*,$$

where $\{x\}$ is a set of objects (alternatives); Y is the function of choice (rule establishing a prefer ability on a set of alternatives); x^* is the chosen alternatives (one or more).

The function Y is used to solve the problem of analysis and evaluation of alternatives. On results of estimation the choice of one or a few best alternatives from the given set follows. In decision theory, there are two different approaches to evaluating objects (alternatives) subject to choice. One of them is to evaluate an object *as a whole* and to choose an alternative by comparing objects as gestalts (holistic images of objects without detailing their properties). The second approach is detailed elaboration and assessment of various object vectors of properties and making decisions after comparing these properties. If a holistic approach implies choosing x^* directly using choice function Y, the *vector approach* requires a mechanism to carry out decomposition of Y into a set (vector) of the choice functions y. By decomposition of the choice function Y is understood its equivalent representation by a certain set of other functions y which composition is the initial choice function Y.

Separation of properties of alternatives on the basis of the analysis is the decomposition leading to the hierarchical structure of properties.

Properties, for which there exist objective numerical characteristics, are called *criteria.* The approach of comparison on separate properties, at all its attraction, derivates a serious problem of return transition to required comparison of alternatives as a whole.

2 Statement of the problem

Quality of an alternative is determined by hierarchical system of vectors

$$y^{(j-1)} = \{y_i^{(j-1)}\}_{i=1}^{n^{(j-1)}}, \quad j \in [2, m],$$

where $y^{(j-1)}$ is the vector of criteria on the $(j-1)$-th level of the hierarchy, by the components of which the quality of properties of alternatives for the j-th level is assessed; m is the amount of levels of the hierarchy; $n^{(j-1)}$ is the amount of estimated properties on $(j-1)$-th level of the hierarchy. The numerical values of n criteria $y^{(1)} = y$ of the first level of the hierarchy for the alternative are given.

The same criterion on $(j-1)$-th level can participate in the evaluation of several properties of the j-th level, i.e. possible cross-links in the hierarchy. It is clear that $n^{(1)} = \sum\limits_{i=1}^{n_1} r_i = n$ and $n^{(m)} = 1$.

Importance (significance) of each of the components of the criterion of $(j-1)$-th level in the evaluation of properties of k-th level is characterized by a property coefficient of the priority, their set forming the priority vectors system

$$p_{ik}^{(j-1)} = \{p_{ik}^{(j-1)}\}_{k=1}^{n(j)}, \quad j \in [2, m].$$

It is required to find an analytical evaluation $y*$ and qualitative evaluation of the effectiveness of this given alternative, and from the alternatives available to choose the best.

3 The method of solution

At the study, the approach is used consisting in the creation and simultaneous co-existence of not one but many theoretical models of the same phenomenon, and some of them conceptually contradict each other. However, no one can be neglected, as each describes a property of the phenomenon and none can be taken as a single because it does not express the full range of its properties. Compare the said with the principle of complementarity, introduced into science by Niles Bohr: "... To reproduce the integrity of the phenomenon should be used mutually exclusive 'complementary' classes of concepts, each of which can be used in its own, special conditions, but only when taken together, exhaust the definable information." It is the principle of complementarity that allows for separating and then linking these criteria in multicriteria evaluation. Only a full set of individual criteria (vector criterion) enables an adequate assessment of the functioning of a complex system as a manifestation of the contradictory unity of all its properties.

However, this possibility represents only necessary but not sufficient condition for the vector evaluation of the entire alternative as a whole.

For a complete evaluation it is necessary to go out from the lower level of the hierarchy and to rise on the following tier, i.e. to carry out an act of criteria composition. Let's compare this with the incompleteness theorem of Kurt Gödel "... In every complex enough not contradictory theory of the first order there is a statement, which by means of the theory is impossible neither to prove, nor to deny. But the self-consistency of a particular theory can be established by means of another, more powerful formal theory of the second order. But then the question of the self-consistency of this second theory arises, and so forth." We can say that Gödel's theorem is a methodological basis for the study of hierarchical structures.

With reference to our problem it means that for an adequate estimation of an alternative as a whole we should solve a task of the criteria composition on levels of hierarchy, consecutively passing from the bottom level up to top.

A scalar convolution of criteria can serve as a tool for the act of composition. The scalar convolution – it is a mathematical technique for data compressing and quantifying its integral properties by a single number.

A scalar convolution on **nonlinear compromise scheme** for the criteria subject to be minimized is proposed

$$Y[y(x)] = \sum_{k=1}^{s} \alpha_k A_k \left[A_k - y_k\left(x\right)\right]^{-1},$$

applied in cases where the decision-maker considers as the preferred those solutions in which the values of individual criteria $y_k(x)$ are farthest from their limit values, A_k. This convolution has a number of essential advantages, which include flexibility, universality and analyticity.

The choice of a compromises scheme is made by the DM and appears as explicitly conceptual.

4 Nested scalar convolutions

It is proposed for analytical evaluation of hierarchical structures to apply a method of nested scalar convolutions. The composition is performed on the "matryoshka principle": the scalar convolutions of the weighted components of vector criteria of lower level serve as the components of the vectors of higher level criteria. Scalar convolution of criteria obtained at the uppermost level is automatically considered as the expression for the analytical evaluation of effectiveness of the entire hierarchical system.

The algorithm for nested scalar convolutions is represented by an iterative sequence of operations of the weighed scalar convolutions of criteria for each level of the hierarchy from the bottom up, taking into account the priority vectors, based on the selected compromise scheme

$$\left\{\left(y^{(j-1)}, p^{(j-1)}\right) \rightarrow y^{(j)}\right\}_{j\in[2,m]} \tag{1}$$

and the searching and evaluating of effectiveness of the entire hierarchical system (alternative) as a whole is expressed by the problem of determining the scalar convolution of criteria on the top level of the hierarchy:

$$y^* = y^{(m)}.$$

When using the recurrent formula (1) important is the rational choice of the compromise scheme. For the method of nested scalar convolutions the adequate is a nonlinear compromise scheme. It is established that, without loss of generality, a premise for its use is that all the partial criteria were non-negative, were subject to minimization and were limited:

$$0 \leq y_i \leq A_i, \quad A = \{A_i\}_{i=1}^{n},$$

where A is the vector of restrictions on the criteria of the current level of the hierarchy; n is the amount of them.

Proceeding from (1) the expression to evaluate k-th property of an alternative for the j-th level of the hierarchy by using the nonlinear compromise scheme looks like

$$y_k^{(j)} = \sum_{i=1}^{n_k^{(j-1)}} p_{ik}^{(j-1)} [1 - y_{0ik}^{(j-1)}]^{-1}, \quad k \in [1, n^{(j)}], \tag{2}$$

where criteria of the $(j-1)$-th level are normalized (reduced to unity). Thus, $y_{0ik}^{(j-1)}$ are the normalized vector's $y_0^{(j-1)}$ components involved in the evaluation of properties of the k-th alternative on the j-th level of the hierarchy; $n_k^{(j-1)}$ is their amount; $n^{(j)}$ is the amount of evaluated properties of the j-th level.

The structure of the nonlinear compromise scheme enables normalizing the convolution (2) not to the maximum (which in this case is difficult), but to the minimum value of criteria convolution. Indeed, the ideal values for the criteria that are subject to be minimized are their zero points. Putting in (2)

$$y_{0ik}^{(j-1)} = 0, \quad \forall i \in [1, n_k^{(j-1)}]$$

and taking into account the normalization $\sum_{i=1}^{n} p_i = 1$, we obtain $y^{(j)}_{k\,\min} = 1$. After calculations, the final expression for the recurrent formula for calculating analytical assessments of the alternatives properties at all levels of the hierarchy becomes

$$y^{(j)}_{0k} = 1 - \{ \sum_{i=1}^{n_k^{(j-1)}} p^{(j-1)}_{ik} [1 - y^{(j-1)}_{0ik}]^{-1} \}^{-1}, k \in [1, n^{(j)}], j \in [2, m]. \quad (3)$$

5 Conclusion

The foregoing leads to the conclusion that any problem of vector assessment of an alternative can be represented by a hierarchical system of criteria, resulting from the decomposition of an alternative properties. The lower level of the hierarchy is an object (alternative) assessment on selected properties, using initial criteria vector, and the upper level is obtained through the mechanism of the composition as a whole object evaluation. Central here is the problem of the composition of criteria for levels of the hierarchy to be solved by the method of nested scalar convolutions.

The methodological basis of an alternative properties decomposition to obtain the initial criteria vector is the Bohr's principle of complementarity. This is a *necessary* condition for vector estimation of alternatives.

The methodology of a criteria composition for levels of the hierarchy is based on the Gödel's theorem of incompleteness. This is a *sufficient* condition for vector estimation of alternatives.

We dare to say that above inferences about notions of criteria decomposition and composition can be extended on the more general notions of analysis and synthesis.

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Optimization problems on graphs and transport networks

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Abstract

This paper examines the problems of determining the center, the main center, the absolute main center on undirected graphs and proposes the efficient algorithms to solve them.

Keywords: main center, absolute main center, median

1 Introduction

The necessity to determinate the location of service points in practical problems, on transport networks, appears in order to minimize the travel time to the farthest point (consumer), taking into account the frequency displacements at each point, too.

The problems of determining the location points on the transmission network of the producers (deposits), so that transport costs related to satisfying consumers to be minimal, are another type of problems.

Effective algorithms have been developed to solve such problems in different metric spaces and transport networks.

2 The problems of absolute center and median on numerical axis

Consider m points on numeric axis with coordinates $a_1 < a_2 < \ldots < a_m$. Let us associate with every point i a number (frequency) $p_i = p\left(a_i\right) > 0,\ \ i = \overline{1\,;\ m}$. Consider functions $F_1\left(x\right) = \max\limits_{1 \leq i \leq m} \left\{p_i \cdot\ \left|x - a_i\right|\right\}$ and $F_2\ \left(x\right) = \sum\limits_{i=1}^{m} p\ \left(a_i\right) \cdot\ \left|x - a_i\right|$. The absolute center for this system of points is the point x_1^*, for which: $F_1\left(x_1^*\right) = \min\limits_{x}\ \max\limits_{1 \leq i \leq m}\ \left\{p_i \cdot\ \left|x - a_i\right|\right\}$.

The convexity of functions $f_i(x) = p_i \cdot \ |x - a_i|\,, \quad i = \overline{1;\, m}$, implies the convexity of $F\ (x) = \max\limits_{1 \le i \le m} f_i\ (x)$. So, the local minimum of this function is global minimum, too. Different methods, based on this property, are proposed for determining the absolute center of this system of points on numeric axis.

The median of a system of points of numeric axis is the point x_2^*, for which: $F_2\ (x_2^*) = \min\limits_x \sum\limits_{i=1}^{m} p\ (a_i) \cdot\ |x - a_i|\,.$

The function $F_2\ (x)\,$, for $p\ (a_i) > 0\,,\ \ i = \overline{1;\, m}$, as the finite sum of convex functions, is a convex function and its local minimums are global, too.

The function $F_2\ (x)$ has a minimum in $x_2^* = a_q$, if the inequality $\sum\limits_{i=1}^{q-1} p\ (a_i) \le \frac{1}{2} Q \le \sum\limits_{i=1}^{q} p\ (a_i)$ holds, where $Q = \sum\limits_{i=1}^{m} p\ (a_i)$.

This inequality constitutes the basis of the algorithm for determining x_2^*. The point x_2^* is named median, too, because this problem is related to median problem of statistic series.

3 The problems of absolute center and median in R_1^2

Consider m points $M_i\,(a_i\,,\ b_i)\,,\ \ i = \overline{1;\ m}$ in space R_1^2. Let us associate with every point $M_i\,(a_i\,,\ b_i)$ a number (frequency) $p_i = p\,(M_i) > 0$. The distance between two points $M\,(x,\ y)$ and $M_i\,(a_i,\ b_i)$ in R_1^2 is defined: $d\,(M\,;\ M_i) = |x - a_i| + |y - b_i|\,,\ \ i = \overline{1;\ m}.$

Consider the functions: $\varphi_1\,(x, y) = \max\limits_{1 \le i \le m} f_i\,(x, y)$, where $f_i\,(x, y) = p_i \cdot (|x - a_i| + |y - b_i|)$ and $\varphi_2\,(x, y) = \sum\limits_{i=1}^{m} p_i \cdot (|x - a_i| + |y - b_i|)\,.$

The problem of absolute center needs to determine a point $P_1^*(x^*, y^*)$ for which: $\varphi_1\,(x^*, y^*) = \min\limits_{(x,y)} \max\limits_{1 \le i \le m} \{p_i \cdot (|x - a_i| + |y - b_i|)\}\,.$

The function $\varphi_1\ (x\,,\ y)$ is convex and its local minimums are global, too. An efficient algorithm is proposed for determining the absolute center in R_1^2.

The median problem needs to determine a point $P_2^*\,(x^*,\ y^*)$ for which: $\varphi_2\,(x^*, y^*) = \min\limits_{(x,\ y)} \sum\limits_{i=1}^{m} p_i \cdot (|x - a_i| + |y - b_i|)\,.$

The median location problem in R_1^2 may be solved by reducing it to two problems on respective axis.

The median problem in metric space R_1^n with norm $\|X\| = \sum\limits_{i=1}^{n} \left| x^{(i)} \right|$ is examined, too. It's reduced to n problems on respective axis, too.

4 The problems of absolute center and median R_∞^2

The distance between two points $M\,(x, y)$ and $M_i\,(a_i, b_i)$ in metric space R_∞^2 is defined by: $d\,(M; M_i) = \max\left\{|x - a_i|\,; |y - b_i|\right\}, \quad i = \overline{1; m}$.

We prove that the problem of absolute center and median determination in R_∞^2 with coordinates axes xOy are reduced to problems of absolute center and median determination in R_1^2 with coordinates axes $\xi_1 O\,\xi_2$, obtained by xOy rotation on 45°.

5 The problems of absolute center, main center, main absolute center and median on undirected graphs

Consider undirected graph $G = (V, U)$, where $V = (v_1, v_2, \ldots, v_n)$ is the set of vertices, $U = (u_1, u_2, \ldots, u_m)$ is the set of edges. Let us associate with every vertex v_i a number (frequency) $p_i = p\,(v_i) > 0$, and with every edge u_k – a length $l\,(u_k)$.

The problem of absolute center on graphs needs to determine the point x^* on edges of G, for which: $\psi_1\,(x^*) = \min\limits_{x} \max\limits_{1 \le i \le n}\ \{p_i \cdot d\ (x\ ,\ v_i\,)\}$, where $d\ (x\ ,\ v_i\,)$ is the minimal distance from the point x on edge to vertex v_i.

The absolute center on undirected graphs is determined by different methods, such as Hachimi method and iterative method.

The problem of main center on undirected graphs needs to determine a vertex v_{i^*}, for which: $\psi_2\ (v_{i^*}) = \min\limits_{x} \max\limits_{1 \le i \le n}\ \{p\,(u_k) \cdot d\ (v_i\ ,\ u_k)\}$ where $p\,(u_k)$ is the frequency on edge $u_k = (v_r\ ,\ v_s)$, and

$$d\ (v_i\ ,\ u_k) = \frac{d\ (v_i\ ,\ v_r) + d\ (v_i\ ,\ v_s) + l\ (v_r\ ,\ v_s)}{2}.$$

The main absolute center method is related to the absolute center method with a unique distinction as in first problem the distance point-vertex is applied, and in a second problem – point-edge distance is applied.

The median problem on graphs consists in determination of points x^* on graph G edges, for which: $\psi_3(x^*) = \min\limits_{x} \sum\limits_{i=1}^{n} p_i \cdot d(x, v_i)$.

An efficient algorithm is proposed for solving the problem on tree (undirected connected graph without cycles).

6 The Weber problem on spaces R_1^2, R_∞^2, R_1^n and on tree

The optimization practical problems solving on plane needs the minimization of the sum of weighted distances between points, where the manufacturers X_i, $i = \overline{1, M}$ will be located and between these points and A_j, $j = \overline{1, N}$, where the consumers are located, already.

A model of this type is the Weber problem, which needs to minimize the function

$$F(X_1, X_2, \ldots, X_M) = \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_{ij} d(X_i, A_j) + \sum_{i=1}^{M-1} \sum_{k=i+1}^{M} \beta_{ik} d(X_i, X_k),$$

where α_{ij} and β_{ik} are positive numbers, $d(X_i, A_j)$ and $d(X_i, X_k)$ are distances between respective points.

It is proved that the Weber problem solving in R_1^2 is reduced to two respective problems on each coordinate axes. Every such problem on axes needs to solve no more than $N-1$ problems of maximal flow determination (a minimal section) in a network with $M+2$ vertices. The Weber problem in R_∞^2 with coordinate axes xOy is reduced to Weber problem in R_1^2 with coordinate system $\xi_1 O \xi_2$, which are obtained from xOy by 45° rotation.

The Weber problem is considered in the metric space R_1^n, too. It's proved that the solution of such problem in the metric space R_1^n needs the solution of n problems on the axis with respective coordinates.

The Weber problem on graphs has its interesting features. The consumers A_j, $j = \overline{1, N}$ and manufactures X_i, $i = \overline{1, M}$ are located on $G = (V, U)$ vertices. An efficient algorithm for Weber problem solving on tree is constructed. The complexity of such algorithm is about $O(n \cdot M^3)$.

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Section 4

Computer Science

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An Approach to Implementation of Hybrid Computational Paradigm *

Artiom Alhazov, Lyudmila Burtseva, Svetlana Cojocaru, Alexandru Colesnicov, Ludmila Malahov

Abstract

We present recent advances on hybrid computations within the P systems framework with quantum functionalities. This model benefits from both biomolecular and quantum paradigms and is supposed to overcome some of their inherent limitations.
Keywords: Computational models, Parallelism and concurrency, Quantum computing, Biomolecular computing, P systems

Introduction In computability theory, a model defines feasible computational operations with their execution time/space. This research concerns the capability of membrane computing, see [5], to provide the hybrid computing framework, enhanced with quantum functionality.

Among unconventional computing paradigms quantum and biological ones were developed mostly in parallel; both are considered as tools for solving hard tasks. However, some hard tasks could not be efficiently solved by pure quantum or bioinspired methods, so efficient solutions are searched in the hybrid model. We propose hybrid computations, where classical P system formalism serves as the framework providing computations by elements of other models and communications between them. Only classical scheme of quantum device, see [10], is used for representation of quantum functionality incorporated in the proposed hybrid. At the current stage, each problem solving employs particular computation methods in the quantum part of hybrid model. The main components of hybrid (P system with quantum functionality)

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computing model (HpqCM) are P system and quantum computation parts and their intercommunication, implemented by P system objects mapped to the base states of each initial and resulting qubit of the quantum device. The appearance of such objects starts the quantum computation. Based on implemented solutions of several problems, we extract the common feature of the hybrid computation: solution has to be based on generating of candidates lists, which are used for comparison to pattern, e.g., applying Grover search, in the quantum device.

Membrane Level of the Hybrid Model Three problems were solved in the hybrid framework, using, respectively, 1) transitional P systems with inhibitors, 2) tissue P systems with symport/antiport, and 3) P systems with active membranes. Lacking space, we only illustrate the kinds of rules allowed by these models. 1) Rewrite a in region i to u if inhibitor b is absent. 2) Move multiset u from region i to region j and multiset v from region j to region i. 3) Evolution (a), send-in (b), send-out (c), dissolution (d), division (e).

$$
\begin{array}{lll}
1) & a \to u|_{\neg b} \in R_i, & a, b \in O, u \in (O \times Tar)^*, \\
2) & (i, u/v, j), & i, j \in H \cup \{0\},\ u, v \in O^*, \\
3(a) & [\ a \to u\ ]_h^e, & a \in O,\ u \in O^*,\ h \in H,\ e \in E, \\
3(b) & a[\ \ ]_h^e \to [\ b\ ]_h^{e'}, & a, b \in O,\ h \in H,\ e, e' \in E, \\
3(c) & [\ a\ ]_h^e \to [\ \ ]_h^{e'} b, & a, b \in O,\ h \in H,\ e, e' \in E, \\
3(d) & [\ a\ ]_h^e \to b, & a, b \in O,\ h \in H, e \in E, \\
3(e) & [\ a\ ]_h^e \to [\ b\ ]_h^{e'} [\ c\ ]_h^{e''}, & a, b, c \in O,\ h \in H,\ e, e', e'' \in E.
\end{array}
$$

Membrane $\leftrightarrow$ Quantum Communication We discuss the interaction between the biological sub-system and the quantum one(s). The tuple $\beta = (\Pi, T, T', H_Q, Q_N, Q_M, Inp, Outp, t, q_{h_1}, \cdots, q_{h_m})$ defines the hybrid system, where Π is a P system, $H_Q = \{h_1, \cdots, h_m\}$ is a subset of membrane labels in Π used for quantum calculations, T is a trigger and T' is the signal on obtaining the quantum result. For labels in Q_H, the quantum sub-systems are $q_{h_1}, \cdots, q_{h_m}$. The rest of the tuple components of β specify the interaction between Π and q_{h_j}, $1 \le j \le m$.

The input size for quantum systems is $Q_N : H_Q \to \mathbb{N}$ qubits. The output size is $Q_M : H_Q \to \mathbb{N}$ bits. Function t defines the timing: $t(h_j)$ is the number of membrane steps for the quantum calculation in device q_{h_j}, currently proposed to be 3 (initialize, transform, measure).

For the behavior of β to be well-defined (even for the situations not used in our constructions), we introduce the trigger-object $T \in O$, where O is the alphabet of Π. The work of a quantum sub-system of type q_{h_j} starts when T appears in that membrane. The quantum state is initialized by objects from $Inp(h_j) = \{O_{k,h_j,b} \mid 1 \le k \le Q_N(h_j),\ b \in \{0,1\}\} \cup \{T\}$, so $Inp : H_Q \to 2^O$ is a function describing the input sub-alphabet for each quantum sub-system type, object $O_{k,h_j,b}$ initializing input qubit k by value b. Value $|0\rangle$ is assumed if qubit k is not initialized. For multiple objects, say, $(O_{k,h_j,0})^s(O_{k,h_j,1})^t$, qubit k is set to state $\frac{s|0\rangle+t|1\rangle}{\sqrt{s^2+t^2}}$. The technique is restricted to non-entangled states. We assume the input objects wait until the trigger appears.

The output of quantum sub-systems is returned to the P system as objects from $Outp(h_j) = \{R_{k,h_j,b} \mid 1 \le k \le Q_M(h_j),\ b \in \{0,1\}\} \cup \{T'\}$, object $R_{k,h_j,b}$ meaning the output bit k is b. One-bit output is often denoted by `yes` and `no`.

Problems Solved A well-known **satisfiablily problem**, SAT, is solved in [8] by a transitional P system with inhibitors, and a quantum sub-system doing Grover search. P system is given the input objects coding occurrences of variables in clauses; it takes 3 steps to find variables missing in clauses, codes all in bits and calls the quantum system.

Image retrieval problem is solved in [8] by tissue P systems with symport/antiport.

Finding the **longest common subsequence** (NP-complete) is solved in [7] by P systems with active membranes and quantum subsystems. The P system performs division to consider possible subsequences of the first string, counts their lengths, builds the actual subsequences, and calls the quantum sub-systems to check whether such sub-sequences are found in other input strings; the search is done in longer-first order.

Conclusions We summarize advances of hybrid model of membrane computing framework with other (now, quantum) approaches.

Hybrid character of computation is provided by mutual accepting of input/output by computation models of different nature and even macro/micro levels. P system formalism as the framework provides procedure of converting multisets to quantum registers contents and back. The returned registers content is prepared by measurement that is final step of quantum computation.

Since the main reason of hybrid model was practical needs of several domains delivering hard tasks, we tested our approach on problems of different range: from theoretical computing to everyday practical application. Besides being a testbed the sample problems solutions enrich hybrid computational model with new features and methods.

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Just In Time Platform for weather forecast and agrimarketing information and insurance for Moldavian farmers

Ion Amarfii, Anatolie Fala, Sergiu Gafton

Abstract

The work deals with a subject characteristic for a rapid information dissemination process for specific segments of beneficiaries. Because of the need of rapid informing of farmers about agricultural sector threats, predicting hydro-weather phenomena and agricultural market situation in Moldova, there was developed a software solution able to strengthen data collection using automatic strategies, automated and manual, in order to rapidly inform farmers.

Keywords: GSM, ATMOS-AMIS, SMS, IVR, Just In Time Platform, Performing Agriculture, integration

1 Introduction

Recently appeared solutions, based on monitoring the evolution of weather systems via satellite, radar and weather stations integrated with specialized programs, allow one to forecast, anticipate and mitigate potential risks from natural hazards.

At the same time rapid development of communication technologies, particularly global computer networks and cell phones, boosted creation of effective channels for rapid dissemination of information. It is obvious that agriculture is a sector sensitive to weather developments and it would be imperative to develop solutions capable of providing timely information to farmers on specific measures to prevent, mitigate and overcome the negative effects of natural disasters.

For this purpose ATMOS-AMIS platform provides an operative tool for rapid dissemination of critical weather information and agro-related information for a large number of farmers and rural communities in the Republic of Moldova.

2 Problem formulation

Severe damage of the agricultural sector by natural hazards in recent decades imposed the necessity of development and implementation of an information system in disaster management and climate risks, enabling timely information for producers and rural entrepreneurs in order to reduce the effects of phenomena generating risk in agriculture.

In response to these challenges, at the initiative of the Ministry of Agriculture and Food Industry of Moldova within the World Bank Management Disaster and Climate Risk Project implemented under the agreement signed between the Government of Moldova and the International Development Association of World Bank, there was created the ATMOS Platform, management being ensured by the National Agency for Rural Development (ACSA).

3 ATMOS-AMIS Architecture

As it is shown in Figure 1 ATMOS-AMIS IT solution involves the interaction of a number of subsystems for the implementation of the objectives.

In particular it is the interaction of the following components:

- **ATMOS** – represents the Alert Through Mobile Service platform that is ensuring issuance and the sending of SMS;
- **AMIS** – Agricultural Marketing Information System that provides all marketing information for SMS and Newsletters;
- **IVR** – Interactive Voice Response, that processes ad hoc requests for information seekers;

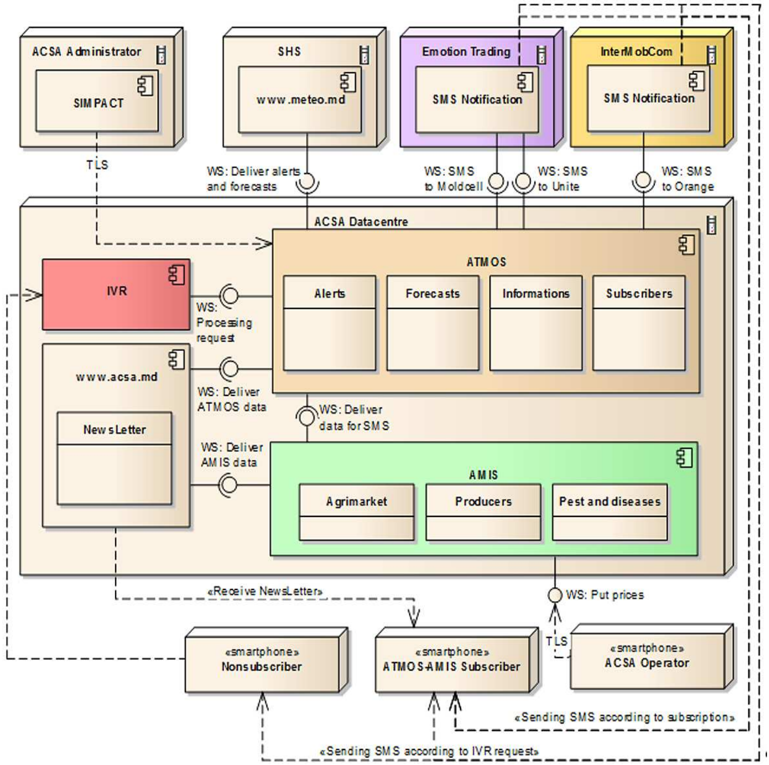

Figure 1. ATMOS-AMIS Architecture

- **www.acsa.md** – ACSA web site that generates and sends Newsletter's;

- **SHS** – computer system of the State Hydrometeorological Service providing weather forecasts and alerts.

- **Emotion Trading** – computerized system of the integrator which ensures sending of SMS to Moldcell Unite subscribers;

- **InterMobCom** – computerized system of the integrator which ensures sending of SMS to Orange subscribers;

- **Smartphone Application** – information system designed for ACSA operators for instant collecting of agricultural prices from Moldovan markets.

At the basis of the information solution the SOA architecture stands so that the interaction of these components is done through Web services presented in Figure 1.

Therefore, although each computer subsystem operates and is administered relatively independently, through these services, all the described subsystems interact in order to implement the objectives of the ATMOS-AMIS operation.

4 The basic functionalities of the information system

To ensure functional objectives of ATMOS-AMIS platform 9 categories of functionality available for different categories of information system users have been implemented:

- **Issuing and sending SMS messages automatically**. Represents all functionalities for perfecting and automatic sending of SMS messages with alerts, weather and marketing information as requested.
- **Processing untyped requests or erroneous SMS requests**. Represents all functionalities for manual processing of untyped requests (which is not provided for automatic processing) or of the erroneous SMS (which cannot be processed automatically) received from customers or internet visitors.
- **Managing subscribers**. Represents all the functionalities for creating customer profiles to ensure their access to the system.
- **Interaction via IVR messages**. Is an interactive mechanism of interaction with information seekers regarding their delivery of information required by an interactive audio menu application.
- **Sending NewsLetter**. Represents all functionalities designed for content generation and automatic dispatch of newsletters to ATMOS-AMIS subscribers.

- **Strengthen Knowledge Database**. Represents flow used to complement the contents of a Knowledge Database designed for agricultural production processes.

- **Payment processing**. Represents all the functionalities for processing payments made by customers to determine the period of validity or automatic blocking access when subscription has expired.

- **Data Aggregation**. Represents all the functions designed for extracting aggregate reports for operating ATMOS-AMIS used to analyze market potential, agriculture information system audit and supervision activities of subordinates.

- **Exploring ATMOS-AMIS public resources**. Represents all functionality designed for Internet users regarding access to public information related to ATMOS-AMIS available on the site http://www.acsa.md.

5 ATMOS-AMIS development perspectives

Architecture based on which the ATMOS-AMIS platform is carried, permits adding new information subsystems for widening its spectrum of action.

In particular it is considered appropriate the integration with systems of Service of Civil Protection and Emergency Situations of Moldova (for automatic data taking over for natural disasters and techno genic alerts) and National Agency for Food Safety of Moldova (for automatic data taking over for alerts on plant pests and diseases).

Also service packages offered to customers must be diversified in order to increase the number of subscribers, which will allow the accumulation of funds for the operation and further development of the platform.

6 Conclusion

ATMOS-AMIS currently has 1150 subscribers for SMS / Newsletter, over 38 800 customers that access Internet services and more than 1000 unique daily visits of users from Republic of Moldova, Romania, Ukraine, Russia and the European Union.

From the development, launching and operational development period ATMOS-AMIS has:

- dispatched 101,5 thousand SMS with weather forecast, 35,9 thousand SMS with prices, 66,2 thousand SMS with hydrological alerts, 95,9 thousand SMS with weather alerts and 11,3 thousand SMS with environmental quality alerts;
- dispatched 92,6 thousand News Letters, for 775 ATMOS-AMIS subscribers having the placement of 1560 pieces of information and news, 580 offers and requests of commercialization/purchase of agricultural products and production means;

Since December 2013, ATMOS-AMIS in partnership with the companies ТОВ "Фруктовий Проект" in Ukraine and DLV Plant from the Netherlands has started a weekly survey of the fresh apple market from Ukraine, Moldova, Poland, Germany, Italy, Russia and the Netherlands markets, for ATMOS-AMIS subscribers: National Extension Service of Moldova, Ministry of Agriculture and Food Industry an Association of Producers and Exporters of Fruits from Moldova. Apples is the most exported horticultural product, annual exports from Moldova being over 140 000 tones.

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Approaches to the ontology alignement and identity resolution problems

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Abstract

This paper describes approaches to the vocabulary normalization and identity resolution problems arising during the use of the LOD datasets to enrich the content of scientific knowledge bases. The dataset of the Open Archive of the Russian Academy of Sciences, and several bibliographic datasets are used as test examples.

Keywords: Linked Open Data, ontology, identity resolution

1 Introduction

One of the projects carried out at the A.P. Ershov Institute of Informatics Systems SBRAS is aimed at enriching the SBRAS Open Archive structured by the BONE ontology [1] with the data of the Open Linked Data cloud [2]. A four-step strategy for the integration of Linked Data into an application consists of access to linked data(1), vocabularies (schema, ontology) normalization(2), identity resolution(3), and data filtering(4). There exist specialized tools for solving separate problems [3, 4]. We present in this paper a new pattern making possible automatic generation of SPARQL queries and establishing correspondence between the entities of two ontologies. As for the identity resolution problem, we show that equivalence of strings that identify two persons cannot guarantee the identity of these persons and propose additional methods to solve this problem.

2 Vocabulary normalization

One specific feature of the BONE ontology is using the "qualified relation" modeling pattern: entities usually described by means of relationships in other ontologies are described as instances of classes in the BONE ontology. This feature compensates the absence of attributes of the RDF predicates. For example, using the "*from-date*" and "*to-date*" properties of the *bone:participation* class, we are able to specify facts such as "Academician A.P. Ershov was the head of a department at the IM SB AS from 1959 to 1964 and the head of a department at the CC SB USSR AS from 1964 to 1988".

This additional capability makes necessary to establish correspondence between one or several groups of the form "Class1–relation1–Class2" of the first ontology and one or several groups of the form "Class3–relation2–Class4–relation3–Class5" of the second ontology. In particular, a new instance of the Class4 for every triple <Class1:instance1, relation 1, Class2:instance2> should be created. This kind of translation can be carried out by an appropriate SPARQL-query. Since the needed SPARQL-queries are rather tedious, we have created a program that can generate this kind of queries using the visualization of two ontologies.

An example generating instances of the *bone:participation* class with respect to the akt:has-affiliation relation is shown in Fig. 1.

3 The identity resolution problem

Let us consider an instance of the *bone:person* class describing Academician A.P. Ershov. We can find 18 persons with distinct identifiers whose *akt:full-name* is "Andrei P. Ershov" at the bibliographic portal http://dblp.rkbexplorer.com. A person identified by the http://dblp.rkbexplorer.com/id/people-e1ac8593dbc7db6ec5766ea313914be4-1211d4d9974a0a977bd166da859d928f identifier is the author of the "Mixed computation in the class of recursive program schemata". Another person identified as

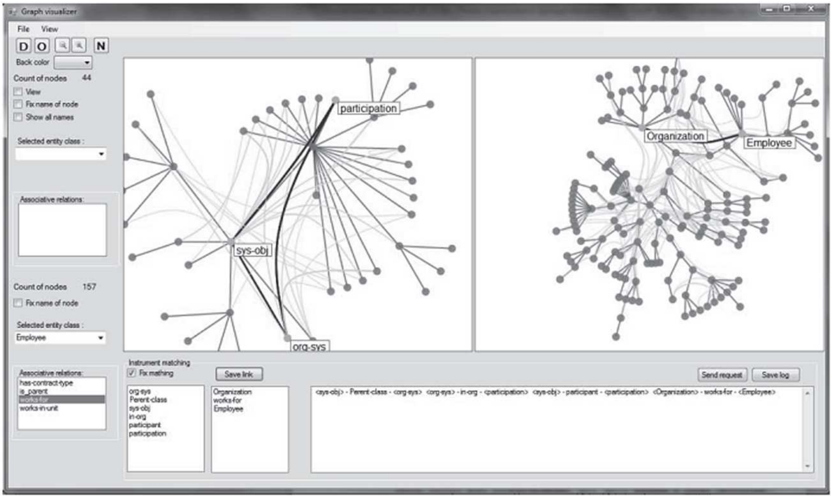

Figure 1. Interactive matching between two groups of classes and relations 2

http://dblp.rkbexplorer.com/id/people-e1ac8593dbc7db6ec5766ea313914be4-2fd1e3b39206345ab05fd9be97bc0d00 has a publication entitled "Time sharing: the need for re-orientation." In addition, there exist other persons with *akt:full-name* attributes such as "A. P. Yershóv", "A. Yershov" and even "Andrew Ershov". All of them have their own lists of publications. Which of these identifiers correspond to the same physical object, and, therefore, can be connected by the relation *owl: sameAs* and which of them describe different physical objects? Whether all publications attributed to a person do belong to this person?

Obviously, the answers to these questions have a significant impact on the calculation of various characteristics, such as citation index. We are trying to answer these questions by checking full-text versions of the publications.

1. **Check the workplace**. Publication date and authors' affiliation are extracted from the textual version of the publication and compared with the person's list of jobs of the Open Archive.

2. **Check the reference list**. The name of the author of each publication is compared with the names of the authors of cited

in them publications. If coincidence of names is found, the cited publication of the same author is combined into one set with the current publication. Then, the same procedure is applied to the added publications.

Conclusion

In this paper we have considered an unusual pattern arising during the ontology normalization step and created a program generating the appropriate SPARQL- queries based on the visualization of ontologies. We have also demonstrated that the conventional tools used for identity resolution are not suitable for clear identifying the authors of publications. Alternative approaches to solving this problem have been proposed.

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Automation Development in e-Learning of Personalized Tasks of "Problem Solving" Type

Maria Beldiga, Tudor Bragaru, Gheorghe Căpăţână

Abstract

Were developed new methodologies and technologies to assist professors in developing necessary numbers of individual tasks of "*problem solving*" type for such activities as (self) training and / or evaluation and uploading the elaborated tasks to an e-Learning/e-testing platform with aim to perform multiple sessions of e-training/e-testing.

Keywords: family problems, generic model, individual tasks, e-Learning/e-testing platforms, distance learning, assessment.

1 Introduction

Investigated problem. The following challenges of the Modern Society regarding:
– integration of the Information and Communication Technologies (ICT) in education;
– continuous lifetime training, especially in open form, remotely accessible to anyone, anytime and anywhere;
– directing modern training not only to obtain knowledge, but mainly to get skills;
– placing trainees in the center of the educational system;
and many other challenges, require composing and checking a large number of individual tasks with open response of "*problem solving*" type. Manual checking under limited time available to professors is almost impossible.

Individual tasks make use of more objective evaluation results in achieving educational outcomes, to exclude fraud of cribbing or Internet retrieval of answers prepared beforehand, keep learners' motivation, competitive spirit etc. On the other hand, automat generation and checking of tasks enable more efficient (self) training opened remotely to anyone, anywhere and anytime, these being the main requirements and the most relevant trend outlined by the current educational system reform. A set of research was performed regarding mathematical ***modeling of generic and specific families of problems, generation*** (*according to models*), ***solving*** (*as formalized knowledge*), ***automatic verification*** of individual tasks with open response type of problem solving in educational activities laboratory and / or (self-) evaluation.

Individual tasks with open response of "*problem solving*" type have enormous potential in the educational process, but cannot be used because of the large time frame needed to compile / solving which professors do not have.

Necessary time for professors for developing a number of tasks K, can be deduced *from*:

$$T = K * (A + R + N + C), \tag{1}$$

$$T_{SSI} = S, \tag{2}$$

where:

- T – the summary time, necessary for performing all operations to solve and check the tasks for K respondents,
- K – the required number of parallel tasks (non-faceted), same statements, goals etc., each with its response, calculated from individual values of problem's parameters,
- A – the necessary time for elaboration of the task,
- R – the necessary time to solve the task,
- N – the necessary time to check/evaluate the task,
- C – the dependent time on the complexity of the task,
- T_{SSI} – the required time for specifying a set of tasks in the process of automation,

- S – the required time for individual specification of a set of individual parallel tasks.

2 Results

It was developed an ***information support system*** (*ISS*) for professors intended for processes of individual specification, automatic generation, solving and checking of individual tasks that have the aim to ***reduce essentially*** the professor's time and effort required for developing individual tasks for formative evaluation, formative and / or final summarized.

The developed system can be regarded as a *set of computational tools of tasks specification according to some generic and specific models, problem solving methods and techniques*, which ensures automatic generation, solving and checking.

In order to illustrate basic components and functions of the new technology it was chosen the discipline "*Decision Support Systems* (*DSS*)". The result can be used directly by: (1) professors in the processes of development and exploitation of digital educational content, (2) students in traditional training and/or open distance and (3) enterprise managers as decision support system.

At the stage I it was performed the analysis of types of decision problems from curriculum and grouping them into families of decision problems that cover curriculum objectives discipline DSS. The decision problems (DP) treated by DSS curriculum were grouped into two families, which have been developed to solve specific models with different methods and techniques:

1. mono-criteria DP family;
2. multi-criteria DP family.

At the stage II there were designed and built the following components of the system:

1. ***Knowledge Base.*** The SSI knowledge representation is performed through *knowledge frames* (for simplicity *frames*). A frame is a hierarchical structure that describes procedural concept and objective of the scope of authority of the discipline taught.

2. ***Composer of custom tasks*** illustrated in ISS allows automatic generation of a set of custom mono-criteria and multi-criteria tasks FPD from generic model.

3. ***Solver of the personalized tasks***. The role of the solver is to find solutions for sets of individual tasks generated by *Composer*. The solution maker employs for this aim the procedural knowledge base of the ISS, which make a link between personified specification of the user and the procedural knowledge, necessary to solve these problems.

4. ***Database*** of SSI keeps the personalized tasks generated by composer. Also it keeps the DB solutions, customized tasks generated by composer and DP formulated by the decision maker – solution found by Solver.

5. ***User Interface*** of ISS is a mechanism of interaction between the user and SSI. It allows the user to make more operations.

3 Conclusion

The main original performances for the SSI are: *automated generations* of *individual tasks* according to professors specification, *automatic solving of individual tasks*, *uploading in the EC environment the SSD personalized tasks*, *checking* and *automatic evaluation of respondents' answers* during the activities of practice/training and/or (self) evaluation.

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On Using Mersenne Primes in the Public-key Encryption

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Abstract

The paper considers design of the public-key encryption algorithm providing security against known decryption text. The algorithm is based on computational difficulty of discrete logarithm problem in multiplicative group of the binary finite field $GF(2^s)$, where s is a Mersenn exponent ($s = 1279$, $s = 2203$ and $s = 4253$).

Keywords: Mersenne primes, discrete logarithm problem, binary polynomials, finite fields, public-key encryption

1 Introduction

The discrete logarithm problem (DLP) in the finite fields is used as the difficult computational problem put into the base of many public-key cryptoschemes [1]. The binary fields $GF(2^n)$ have significant advantage for designing encryption algorithms, since computational difficulty of the exponentiation operation in $GF(2^n)$ is comparatively low. The multiplication operation in such fields is performed as multiplying binary polynomials modulo an irreducible binary polynomial of the degree n. This operation is especially fast in the case of using irreducible trinomials $x^n + x^k + 1$, where $k < n/2$. In this paper it is justified using DLP, that in the binary fields $GF(2^s)$, where s is a Mersenn exponent, the multiplicative group of $GF(2^s)$ has prime order 2^{s-1}. This fact is used by designing public-key encryption algorithms, like the ElGamal algorithm.

2 The Public-key encryption by ElGamal

The ElGamal public-key encryption algorithm [2] uses the difficulty of the DLP in the fields $GF(p)$ and can be used for sending a secret message via a public channel to the owner of the public key $y = a^k \bmod p$, where k is private key; p is a large prime; a is a primitive element mod p. The algorithm performs as follows:

1. The sender generates the single-use private key u and computes the single-use public key $R = a^u \bmod p$. Then he computes the single-use secret key $Z = y^u \bmod p$ and encrypts the message M: $C = MZ \bmod p$, where C is the produced ciphertext.

2. Then the values C and R are sent to the owner of public key y.

The decryption procedure is performed by receiver, using his private key k, as follows:

1. Using the single-use public key R the receiver computes the single-use secret key $Z = R^k \bmod p$.

2. Then he decrypts the ciphertext C and obtains the message $M = CZ^{-1} \bmod p$.

Suppose that except large prime q, some small primes r_i (i=1, 2,..., g) divide the number $p-1$. Then an adversary can implement some potentially known decrypted text attack on the ElGamal algorithm that relates to the following scenario.

The attacker selects a value $R' < p$ having composite order $\omega' = \prod_{i=1}^{g} r_i$ modulo p, generates a random value $C < p$, and then sends the values R' and C to the owner of the public key y.

The receiver computes the value $M' = (CZ'^{-1} \bmod p) =$ $= (CR'^{-k} \bmod p)$ that becomes some way known to the attacker. The last one computes the value $Z' = (CM'^{-1} \bmod p) = (R'^k \bmod p)$. Then, using the baby-step-giant-step algorithm [5], the attacker obtains the value $k \equiv k \bmod \omega'$. If $\omega' > q > k$, then $k' = k$. If $\omega' < k$, then $k = k' + \eta\omega'$, where η is a natural number such that $\eta < k$. Evidently, finding η is easier than finding the private key k, therefore one can claim that the known decrypted text attack provides computing of at least part of the private key. The highest security of the ElGamal algorithm is provided in the case of using the prime values p such that $p = 2q + 1$, where q is also a prime. In the last case the considered attack outputs only one bit of the information about the private key k. Next section proposes a modification of the ElGamal algorithm against which the known decrypted text attack outputs no information about k.

3 Implementation over binary finite fields

Full security against the known decrypted text attack can be provided with using binary finite fields $GF(2^s)$, the multiplicative group of which has prime order $2^s - 1$, to construct a public-key encryption algorithm like the ElGamal algorithm. This case corresponds to the Mersenn exponents s and Mersenn primes $2^s - 1$ as well as to interpreting the message M as a binary polynomial $M(x)$ of the degree $m < s$ and using an irreducible binary polynomial $\pi(x)$ having the degree s. Instead of an integer a of the ElGamal algorithm one can use any non-zero polynomial $\alpha(x)$ in the public-key encryption algorithm defined over $GF(2^s)$. Indeed, each non-zero element in $GF(2^s)$, where s is a Mersenn exponent, has order $2^s - 1$. Thus, the public key is computed as the polynomial $\chi(x) = (\alpha(x))^k \bmod \pi(x)$, where k is the private key.

The public-key encryption is performed as follows:

1. The sender generates at random the single-use private key $u < p - 1$ and computes the single-use public key $\rho(x) = (\alpha(x))^u \bmod \pi(x)$. Then he computes the single-use secret key as the polynomial $\lambda(x) = (\chi(x))^u \bmod \pi(x)$ and encrypts the message $M(x) : C(x) = M(x)\lambda(x) \bmod \pi(x)$, where $C(x)$ is the ciphertext.

2. Then the values $C(x)$ and $\rho(x)$ are send to the owner of public key y, i.e. to the receiver of the message $M(x)$.

The decryption procedure is performed as follows:

1. Using the single-use public key $\rho(x)$, the receiver computes the single-use secret key $\lambda(x) = (\rho(x))^k \bmod \pi(x)$.

2. Then he decrypts the ciphertext $C(x)$ and obtains the message $M(x) = C(x)(\lambda(x))^{-1} \bmod \pi(x)$.

There are known the following values s to which correspond Mersenne primes: s=1279; 2203; 2281; 3217; 4253; 4423; 9689; 9941; 11213 [http://oeis.org/A000043] which are appropriate for application in the proposed modification of the ElGamal public-key encryption method. Besides, to provide higher performance one can use the irreducible binary trinomials as polynomial $\pi(x)$, for example $x^{1279} + x^{216} + 1$; $x^{2281} + x^{715} + 1$; and $x^{3217} + x^{67} + 1$ [3]. Earlier the application of the Mersenn primes was proposed for designing the commutative ciphers and zero-knowledge protocols [4].

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The research of order tables as algebraic structures

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Abstract

The purpose of the paper is research the order tables of algebraic structures. The order tables with operations union, intersection and difference will be researched. Algebraic properties of tables and operations on them were investigated. It was proposed in C ++ a generic parameterized class with abstract class records tabular representation which allows modeling work with order table in terms of algebraic structures [1,2].

Keywords: key, algebraic structure, generic class, union, intersection.

1 Introduction

In what follows we will try to study the order tables with different operations in terms of algebraic structures. There will be considered the most important operations on the order tables, such as adding an item in the table tidy, orderly linking two tables difference.

We investigate which of these operations form a lot of order tables monoid.

Table is a set of elements, each of which has a special indicator called key property that elements are added to the table and looking for the key. Besides key element, table (also called a registration table) typically contains data, carrying some information.

We say that the table is sorted by the key element that facilitates the retrieval of information it contains.

2 Algebraic structure of the ordered tables

Let $E = \{e_1, e_2, ..., e_n\}$ be the set of all tabulated records by the same key structure highlighted. For serial keys, a relationship that allows us to order the tabulated records after key is established. Key is determined by the relationship of the order allowing us ordering tabular entries after key.

Definition 1. Through ordered tables the crowd E is denoted $T(e_{i_1}, e_{i_2}, ..., e_{i_k})$, we understand the crowd orderly of $\{e_{i_1}, e_{i_2}, ..., e_{i_k}\}$, where $e_{i_j} \in E, j = 1, 2, ..., \kappa$, as k – is a number of elements in tables $T(e_{i_1}, e_{i_2}, ..., e_{i_k})$, and element e_{i_1}– it's called the first element in the table T, e_{i_k}– it's called the last element of the table T.

Notice 1. The table can also be ordered blank, if it has no element. We denote the empty ordered table by "$\emptyset$", so $T = \emptyset$. Ordered tables $T(e)$ are composed of a single element $e \in E$.

We denote by $\Re = \{\mathrm{T}_1, \mathrm{T}_2, ..., \mathrm{T}_m, ...\}$ – the crowd of all possible ordered tables on crowd E. Add a set of operations for ordered tables similar to those described in [2, 3].

2.1 Difference ordered tables

We will introduce the operation of difference that will give us the opportunity to extract records from the ordered tables and we will note the sign "-".

Definition 1. The operation "-" that extracts of the ordered tables $T_1 = T(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}})$, ordered tables $T_2 = T(e_{j_1}, e_{j_2}, ..., e_{j_{k_i}})$, is operation $T_1(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}}) - T_2(e_{j_1}, e_{j_2}, ..., e_{j_{k_i}})$, that results in table $T(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}} - e_{j_1}, e_{j_2}, ..., e_{j_{k_i}})$ with elements that are in T_1, and are not T_2 for other $T_i, T_J \in \Re$. Thus the set of all tables ordered the lot E with operation "-" is a couple denoted $(\Re, -)$.

2.2 Intersection ordered tables

The intersection of two tables $T = T_1 \bigcap T_2$ means the lot that contains the common elements of T_1 and T_2 with equal keys.

Definition 1. The operation "$\bigcap$" that intersects order tables $T_1 = T(e_{r_1}, e_{r_2}, ..., e_{r_{k_i}})$, with order tables $T_2 = T(e_{l_1}, e_{l_2}, ..., e_{l_{k_i}})$, is operation $(T_1(e_{r_1}, e_{r_2}, ..., e_{r_{k_i}}) \bigcap T_2(e_{l_1}, e_{l_2}, ..., e_{l_{k_i}}))$, that results in order tables $T(e_{j_1}, e_{j_2}, ..., e_{j_{k_i}})$, so every element belongs to the order tables $T_1 = T(e_{r_1}, e_{r_2}, ..., e_{r_{k_i}})$, and order tables $T_2 = T(e_{l_1}, e_{l_2}, ..., e_{l_{k_i}})$, for other $T_r, T_l \in \Re$. Thus the set of all tables ordered the lot E with operation "$\bigcap$" is a couple denoted $(\Re, \bigcap)$.

Notice 1. If the key in both tables is different (does not intersect), then you get blank crowd $T_1 \bigcap T_2 = \emptyset$.

Teorema 1. Couple $(\Re, \bigcap)$ has identity record.

Demonstration: As a neutral element will get an empty table $T()$. We take any arbitrary table $T_i \in \Re$. Let $T_r = T(e_{r_1}, e_{r_2}, ..., e_{r_{k_i}})$, then $T_i \bigcap \emptyset = T(e_{i_1}, e_{i_2}, ..., e_{i_k}) \bigcap \emptyset = \emptyset \bigcap T(e_{i_1}, e_{i_2}, ..., e_{i_k}) = T(e_{i_1}, e_{i_2}, ..., e_{i_k}) = T_i$.

The theorem is proved.

Teorem 2. Couple $(\Re, \bigcap)$ forms a semigroup.

Demonstration: Let T_i, T_j, T_r be three arbitrary tables in $\Re$. We prove that $(T_i \bigcap T_j) \bigcap T_r = T_i \bigcap (T_i \bigcap T_r)$.

Let $T_i = T(e_{i_1}, e_{i_2}, ..., e_{i_k})$, , $T_j = T(e_{j_1}, e_{j_2}, ..., e_{j_k})$, $T_r = T(e_{r_1}, e_{r_2}, ..., e_{r_k})$. Then

$$(T_i \bigcap T_j) \bigcap T_r = (T(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}}) \bigcap T(e_{j_1}, e_{j_2}, ..., e_{j_{k_i}})) \bigcap$$

$$\bigcap T(e_{r_1}, e_{r_2}, ..., e_{r_{k_i}}) = T(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}}) \bigcap (T(e_{j_1}, e_{j_2}, ..., e_{j_{k_i}}) \bigcap$$

$$\bigcap T(e_{r_1}, e_{r_2}, ..., e_{r_{k_i}})) = T_i \bigcap (T_j \bigcap T_r)$$

for other $T_i, T_j, T_r \in \Re$.

The theorem is proved.

Theorem 3. Couple $(\Re, \bigcap)$ forms a monoid.

The Theorem 1 shows that couple $(\Re, \bigcap)$ possesses identity element, the Theorem 2 shows that couple $(\Re, \bigcap)$ is a semigroup. So, couple $(\Re, \bigcap)$ is a monoid.

Theorem 4. Couple $(\Re, \bigcap)$ forms a commutative monoid.

Demonstration: Let T_i, T_j be two arbitrary tables in $\Re$. We prove that $T_i \bigcap T_j = T_j \bigcap T_i$.

Let $T_i = T(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}})$, $T_j = T(e_{j_1}, e_{j_2}, ..., e_{j_{k_j}})$. Then

$$T_i \bigcap T_j = T(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}}) \bigcap T(e_{j_1}, e_{j_2}, ..., e_{j_{k_j}}) =$$

$$= T(e_{j_1}, e_{j_2}, ..., e_{j_{k_j}}) \bigcap T(e_{i_1}, e_{i_2}, ..., e_{i_{k_i}}) = T_j \bigcap T_i.$$

for other $T_i, T_j \in \Re$.

The theorem is proved.

3 Conclusions

The purpose of the article was to demonstrate that the order tables can be algebraic structure. It has been shown that order tables T with concatenation operation is commutative monoid. It is proved that the set of common and uncommon records from two tables where keys are unique can be reunited. Difference is the set of all records belonging to the first table and the second table does not belong, that are equal key records.

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Implementation of Scientific Cloud Testing Infrastructure in Moldova

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Abstract

Development of scientific computing infrastructure significantly influences on various fields of science that require solving complex computational problems. In Moldova over last years modern computing infrastructures (Grid, HPC) were deployed, but new technologies and new applications domains have led to new approached of computational resources and services offering. Service oriented infrastructures based on Cloud paradigm simplify use of distributed computing infrastructure, wider users community and applications areas. In the paper the approach of federated cloud infrastructure deployment in Moldova is briefly described.

Keywords: Federated Cloud infrastructure, on-demand services, integrated Grid and Cloud infrastructure.

1 Introduction

Cloud computing – data processing technology in which distributed computer resources and capacity are provided to the user as a service available over the Internet. Although there are many commercial realizations of cloud infrastructures that are widely used for providing various services of common use, scientific cloud infrastructures for supporting of research activities are only at the initial stage of development. It is explained by specific requirements of complex applications used for research purposes, relatively large amount of testing and debugging procedures, necessity permanently adopt computing resources to needs of complex applications and in many cases a short lifetime of scientific applications that cannot be used as reusable services.

The mentioned above pushed away new initiatives in this area that are mainly base on open source software platforms.

2 Regional federated cloud infrastructure deployment

One of the regional initiatives implemented in Moldova was the project Experimental Deployment of an Integrated Grid and Cloud Enabled Environment in BSEC Countries on the Base of g-Eclipse (BSEC gEclipseGrid). The g-Eclipse framework provides a middleware independent distributed computing model implemented for different Grid and Cloud middleware. The main purpose of the Project is to deploy a regional integrated Grid and Cloud enabled environment based on g-Eclipse for the South-East Europe region including Armenia, Georgia, Moldova and Romania.

In order to find out which cloud middleware better suits the project needs, two main open source platforms OpenStack and OpenNebula were implemented and analyzed. Although OpenStack seemed to have more built in features, we choose OpenNebula as a cloud middleware, because it had oZones module, which is especially designed to create federated cloud infrastructure. Unlike other open source alternatives, OpenNebula does not embrace a particular hypervisor. It also does not have any specific infrastructure requirements, fitting well into any pre-existing environment, storage, network, or user-management policies. The deployed suggested platform is based on OpenNebula Cloud Management Platform. The first level of the platform consists of resources provided by each partner of Cloud infrastructures. The national cloud resources have been joined together using the OpenNebula Zones (oZones) approach, which allows centralized management of multiple instances of OpenNebula (zones), managing in turn potentially different administrative domains.

3 Approaches of cloud infrastructure implementation

For realization of basic elements of scientific cloud infrastructure in Moldova, we examined different software products, its specifications and infrastructure realization approaches. After examination and analysis of available resources, we elaborated implementation plan, which specifies ways of computational resources allocation for the experimental cloud segment. Now this experimental Cloud infrastructure consists of one virtual machine as a master node that holds OpenNebula Sunstone GUI for easily infrastructure management through graphical web interface and one worker node based on a physical server with four computational cores, 8 Gb RAM and 500 Gb of hard disk space. This infrastructure interconnected via 1Gbit is dedicated to private management network that ensures better performance of deploying virtual machines, snapshotting and live migration features. In addition, it has a public network connection on a separate physical interface, which provides connectivity to virtual machines over Internet. Resources of this infrastructure are enough for various testing purposes and can be easily scaled up in the future by adding if required more worker nodes, networks, data storage devices, etc. to the existing infrastructure.

Due to good security options and our previous experience, CentOS was selected as a basic operating system for our experimental Cloud segment and OpenNebula 4.4.1 was installed with using KVM as hypervisor. We also selected SharedFS file system for data sharing between worker nodes and master node, and 802.1Q network driver, which allows flexible network management, dividing virtual machines to separate virtual networks (VLANs) for security purposes or unite them together to satisfy any users community specific communication capacity needs.

The main approach prosed for implementation in the BSEC gEclipseGrid project is ability of realization in perspective joint computational environment that will combine Grid and Cloud resources to offer the united enhanced computational power that can adaptively, on demand allocate computational resources depending on workflow requirements. As an example, if the user requires parallel computational resources, he will submit a job on the Grid, but if the user needs any specific software or environment to solve some special problem, he can use a dedicated Cloud service or virtual image for that purpose.

4 Conclusion

The created regional cloud testing infrastructure, although it has limited computational resources, is the successful example of adaptation of new technologies and open source software platforms for providing computational resources to scientific community.

There are perspectives to continue development of the scientific cloud infrastructure and technologies at national and regional levels. We intend together with BSEC gEclipseGrid project partners to exploit BSEC Programme funding in future and submit new proposal focused on adaptation and implementation new open source tools for creation federated infrastructures. Other perspective direction is cooperation with partners within new projects initiating by European Grid Initiative and comprises integration of the national Grid and cloud computing infrastructures.

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Complex applications porting to HPC infrastructure

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Abstract

In this article the development of programming environment for integrating individual cluster systems of the IMCS of ASM and the cluster of the SUM in a single, integrate parallel HPC systems is described. The elaborated applications can be ported to the resources of the integrated HPC system.

Keywords: computer science, HPC system, mathematical model of semiconductor diode, informational extended games.

1. Adapting the local HPC systems for execution parallel applications

For ensuring various parallel applications development and execution there were adopted the local HPC systems of the Faculty of Mathematics and Computer Science of USM and IMI ASM for use of open source software that is needed to implement execution of the parallel applications. The works included:

1. Analysis requirements of the elaborated algorithms and determine comprehensive set of basic open source software for support of the elaborating parallel programs execution.

2. Adapting the local HPC systems and installing basic software and open source software packages that needed to form effective run time environment for elaboration and execution of wide set of parallel applications.

To achieve formulated objectives we realized the following approaches:

a) The existing technical basis for providing necessary resources for development and execution of the parallel applications were improved;
b) There were analyzed requirements and provided ability and conditions of wide usage of the existing HPC parallel clusters in different fields of science, technology and economic activities;
c) Software environment was developed for extending the possibility of multiprocessor cluster's systems usage for solving wide range of complex problems;

Creation of integrated system at national level as a part of regional HPC and distributed computing infrastructures and use it for various complex applications porting and execution is an important outcome of these activities.

2. Interactive Training Course "Parallel programming models on clusters"

To acquire the necessary theoretical and practical skills on using regional and European HPC clusters, it was proposed an online course available to learners through Sakai CLE and BigBlueButton open source software. To support the video conferencing, in Sakai was integrated the BigBlueButton plugin that supports sharing of slides, webcams, whiteboard, chat, voice over IP, and presenter's desktop.

The content of the online course is focused on 2 main chapters. The first chapter provides learners with information on parallel computing systems and models, Rocks type clusters and basic commands, Grid computing elements, the models of shared memory (OpenMP) and distributed memory (MPI) parallel programming. In the second chapter learners will study the parallel software packages like ScaLAPACK and PETSc, and use these libraries in developing and launching their application on the cluster.

3. Finite volume method for semiconductor device problem.

Let's consider the mathematical model of semiconductor diode. Its mathematical formulation is based on Drift-Diffusion model. We denote by Γ_D the part of the boundary of domain where the Dirichlet boundary conditions are specified. The remaining part of the boundary we denote by Γ_N and we require the flow normal to the boundary to be equal to zero, so we get the Neumann boundary conditions on Γ_N. Unknown functions $\varphi,\ \varphi_n,\ \varphi_p$ are to satisfy the following system of nonlinear differential equations:

$$-\nabla\cdot(\varepsilon\nabla\varphi)=q\,(p-n+N)\,; \tag{1}$$

$$-\nabla\cdot(J_n)=-q\,(R_{SRH}+R_{AUG})\,;\ J_n=-qn\mu_n\nabla\varphi_n; \tag{2}$$

$$n=n_i exp\left(\frac{\varphi-\varphi_n}{\varphi_T}\right);\ p=n_i exp\left(\frac{\varphi_p-\varphi}{\varphi_T}\right). \tag{3}$$

As the equations are strongly nonlinear, then in order to obtain the convergent solution we apply the iterative procedure of gradually increasing of the input voltage V_a with small step. The obtaining solutions are used for equation linearization. To solve numerically the system of equation we apply the generalization of Gauss-Seidel iterative procedure (in the specialized literature method is called the *Gummel's* algorithm). According to this algorithm we set the initial iteration $\varphi^{(0)},\ \varphi_n^{(0)},\ \varphi_p^{(0)}$, then solve quasi-linear Poisson equation $-\nabla\cdot(\varepsilon\nabla\varphi)=q\left(p^{(0)}-n^{(0)}+N\right);\ n^{(0)}=n_i exp\left(\frac{\varphi-\varphi_n^{(0)}}{\varphi_T}\right);$ $p^{(0)}=n_i exp\left(\frac{\varphi_p^{(0)}-\varphi}{\varphi_T}\right)$ and obtain the function $\varphi^{(1)}$. Then excluding J_n in (2) we obtain the equation $\nabla\cdot\left(qn^{(01)}\mu_n\nabla\varphi_n\right)=-q\left(R_{SRH}\left(n^{(01)},p^{(01)}\right)+R_{AUG}\left(n^{(01)},p^{(01)}\right)\right);n^{(01)}=n_i exp\left(\frac{\varphi^{(1)}-\varphi_n}{\varphi_T}\right);$ $p^{(01)}=n_i exp\left(\frac{\varphi_p^{(0)}-\varphi^{(1)}}{\varphi_T}\right)$ for determining the function $\varphi_n^{(1)}$. Next we exclude J_p in (3) in order to obtain the equation $-\nabla\cdot\left(qp^{(10)}\mu_p\nabla\varphi_p\right)=-q\left(R_{SRH}\left(n^{(10)},p^{(10)}\right)+\ R_{AUG}\left(n^{(10)},p^{(10)}\right)\right)$ for determining the function $\varphi_p^{(1)}$.

4. Parallel algorithm to find the set of all equilibrium profiles in the lot of bimatrix games

Let be given a lot of matrices $\{(A_p\,;\,B_p)\}_{p=\overline{1,k}}$,were p denotes the node of the parallel cluster, and $A=||a_{ij}||_{i\in I(p)}^{j\in J(p)}$, $B=||b_{ij}||_{i\in I(p)}^{j\in J(p)}$. Consider the bimatrix game $\Gamma_p=\langle I(p),J(p),A_p,B_p\rangle$, where $I(p),\ J(p)$ is the set of pure strategies of the player 1, player 2 and A_p, B_p are the pay-off matrices of the player 1 and 2 respectively. Denote by $NE(A_p, B_p)$ the all Nash equilibrium profiles in the bimatrix game $(A_p\,;\,B_p)$. The parallel algorithm to find the set of all equilibrium profiles $(i^*(p), j^*(p))$ consists of the following steps.

1. Using the MPI programming model and open source library ScaLAPACK-BLACS, the matrices $(A_p; B_p)$ are scattered through the MPI process $p = \overline{1,k}$;

2. MPI process, $p = \overline{1,k}$, using the OpenMP and ScaLAPACK, eliminates from matrix A_p and B_p the lines that are strictly dominated in matrix A_p and columns that are strictly dominated in matrix B_p. Finally we obtain the matrices $\left(A'_p\,;\,B'_p\right)$, where $A'_p = ||a'_{ij}||_{i\in I'(p)}^{j\in J'(p)}$, $B = ||b_{ij}||_{i\in I'(p)}^{j\in J'(p)}$ and cardinals $|I(p)| \geq |I'(p)|$, $|J(p)| \geq |J'(p)|$.

3. MPI process, $p = \overline{1,k}$, using the OpenMP and ScaLAPACK, determines all strategy profiles from $NE(A'_p, B'_p)$ and constructs all strategy profiles from $NE(A_p, B_p)$.

4. Using ScaLAPACK-BLACS, the root MPI process is gathering the strategy profiles $(i^*(p), j^*(p))$ from MPI process, $p = \overline{1,k}$.

For this algorithm a C++ program has been developed using MPI functions, OpenMP directives and ScaLAPACK routines. Program has been tested on the control examples. The test results were consistent with theoretical results.

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Disproportionality of Multi-optional PR Voting Systems

Ion Bolun

Abstract

Results on qualitative and quantitative comparison of Hamilton, d'Hondt, Sainte-Laguë, Huntington-Hill and Mixed "votes-decisions" methods for multi-optional PR voting systems are systemized. By simulation it is shown that Mixed method is better in majority of cases than the d'Hondt one.

Keywords: disproportion, index, votes-decision method, optimization, proportional representation, voting system.

1 Introduction

Multi-optional voting systems with proportional representation (PR) are widely used in practice. An example would be the election of Members of Parliament on party lists. A major requirement to such systems is the equal representation, as far as possible, in the decision of each of the voters' will. However, because of nature in integers of the problem, such representation usually fails. Various "votes-decision" (VD) methods of minimizing the disparity in question are proposed and used, including [1]: Hamilton (Hare), Jefferson, Webster, d'Hondt, Sainte-Laguë and Huntington-Hill.

In [2], the opportunity of using the Average relative deviation (ARD) index (I_d) to estimate the disproportionality of PR voting systems is argued. Index I_d conveys the average relative deviation of the representation in the decision of electors will from the mean value of these wills. Studies [1-5] show that in different situations VD methods behave differently. The known theoretical results do not show unequivocally which VD method to use in a specific situation. Therefore it is appropriate the comparative analyses of these methods by simulation, partly carried out and described in this paper.

2 Qualitative comparisons of some VD methods

Let: M – number of seats in the elective body; n – number of parties that have reached or exceeded the representation threshold; V – total valid votes cast for the n parties; $d = M/V$ – rights of each elector; V_i, v_i – number and percentage of valid votes cast for party i, respectively; x_i, m_i – number and percentage of seats to be allocated to party i, respectively; I_d – value of ARD index.

It has been proved [1, 2], that the minimum value of ARD index is obtained using Hamilton method. However, its use can lead, in some cases, to paradoxes "Alabama", "of Population" or "of the New state" [1]. The d'Hondt, Sainte-Laguë, Huntington-Hill [1] and Mixed methods [3] are immune to these paradoxes. These methods are compared using such indices as: disproportionality by I_d; quota rule; immunity to paradoxes; non-favoring large parties and non-favoring small parties.

The "quota rule" is satisfied for party i, if $\lfloor dV_i \rfloor \leq x_i \leq \lceil dV_i \rceil$, $i = 1, 2, \ldots, n$ [3]. A method is called non-favoring parties, when it doesn't allocate seats to one party, exceeding its upper quota ($\lceil dV_i \rceil$), at the expense of small ones, non-assuring their lower quota ($\lfloor dV_j \rfloor$), and vice versa.

Comparative characteristics of the five VD methods by the nominated above criteria, basing on results from [1, 3-5], are summarized in Table 1.

According to Table 1, none of the five compared methods does not prevail completely to the other, excepting the Mixed one to Sainte-Laguë. Although, the d'Hondt method fails only three parameters to Hamilton method, one can not, however, say that this is preferable to Sainte-Laguë one, which fails to Hamilton method by four parameters

Table 1. Comparative characteristics of the five VD methods

VD method	Minimal dispro-portion	Satisfying		Immune to para-doxes	Non-favoring	
		lower quota	upper quota		small parties	large parties
Hamilton	yes [1,4]	yes [1]	yes [1]	no [1]	yes [1]	yes [1]
Huntington-Hill	no [1,4]	no [5]	no [5]	yes [1]	no [5]	no [5]
D'Hondt	no [1,4]	yes [1,4]	no [5]	yes [1]	yes [5]	no [1,5]
Sainte-Laguë	no [1,4]	yes [1,4]	no [5]	yes [1]	no [5]	no [5]
Mixed (at $c = 2$)	no [3]	yes [3]	yes [3]	yes [3]	no [3]	no [3]

and even the Huntington-Hill method that fails to Hamilton one by five parameters.

3 Quantitative comparisons of some VD methods

Quantitative comparisons of some VD methods for particular cases were done in [1, 3, etc.]. A specific case is shown in Table 2.

From Table 2 one can see that Huntington-Hill method prefers small parties at the expense of the largest party – party 1 ($\Delta x_1 = -2$), when d'Hondt method, on contrary, prefers the party 1 ($\Delta x_1 = 3$). Here $\Delta x_i = x_i - \lfloor dV_i \rfloor$.

Note also that the average disproportionate representation of electors' will on five ballots investigated in [3] are: the Hamilton method – 6.84%, the Mixed method – 7.09%, the Huntington-Hill method – 7.90%, the Sainte-Laguë method – 8.42% and the d'Hondt method – 10.89%. So, from four monotone methods, the Mixed one shows, for examined cases, the best result.

Table 2. Distribution of seats for the ballot: $M = 98$; $n = 10$; $V_1 = 11600$; $V_i = 1000 - i + 2$, $i = \overline{2, 10}$

VD method	$\Delta x_1 > 1$ $\Delta x_1 < 0$	Favoring parties	x_i, $i = \overline{1, 10}$	I_d, %
Hamilton	-	-	$x_1 = 55$; $x_i = 5$, $i = \overline{2, 8}$; $x_i = 4$, $i = \overline{9, 10}$	3.55
Huntington-Hill	-2	small	$x_1 = 53$; $x_i = 5$, $i = \overline{2, 10}$	4.66
d'Hondt	3	large	$x_1 = 58$; $x_i = 5$, $i = \overline{2, 5}$; $x_i = 4$, $i = \overline{6, 10}$	7.52
Sainte-Laguë	-	-	$x_1 = 55$; $x_i = 5$, $i = \overline{2, 8}$; $x_i = 4$, $i = \overline{9, 10}$	3.55
Mixed (at $c = 2$)	-	-	$x_1 = 55$; $x_i = 5$, $i = \overline{2, 8}$; $x_i = 4$, $i = \overline{9, 10}$	3.55

4 Comparison of some VD methods by simulation

The methodology and the software application SIMRP for multi optional PR voting systems simulation are described in [6]. Subject to simulation are only quantities V_i, $i = \overline{1, n}$. Basic criterion for comparison of VD methods is index I_d. By I_d criterion the best is Hamilton method; also [3] $I_d(\text{Mixed}) \leq I_d(\text{Sainte-Laguë})$.

Some results of calculations using SIMRP, at normal distribution of quantities V_i, $i = \overline{1, n}$ and initial data: $M = 20, 100$; $n = 4, 6, 8, 10$; $V = 10^8$; sample size of 20000 ballots for each pair M, n, are presented in Table 3. Here P is the percentage of ballots, for which

I_d(Mixed) = I_d(Hamilton), and R is the ratio of the percentage of ballots, for which I_d(d'Hondt) $>$ I_d(Mixed), to the percentage of ballots, for which I_d(d'Hondt) $<$ I_d(Mixed).

From Table 3 one can see that for more than 80% of ballots Mixed method gives the same distribution of seats as Hamilton one does. Also, Mixed method gives a better distribution of seats for a number of polls at least 10-30 times higher than the d'Hondt one does. Elsewhere, P index is decreasing and R index is increasing with the increasing of the number n of parties. So, the more parties, the less efficient is the Mixed method in comparison with the Hamilton one and d'Hondt method in comparison with the Mixed one.

Table 3. Some results of calculations using SIMRP

	$M = 20$				$M = 100$			
	$n = 4$	$n = 6$	$n = 8$	$n = 10$	$n = 4$	$n = 6$	$n = 8$	$n = 10$
P, %	91,56	87,62	84,54	82,68	91,34	87,66	83,97	81,52
R, times	12,86	17,19	24,01	33,44	14,50	20,53	27,19	37,02

5 Conclusions

From the three monotone methods, d'Hondt, Sainte-Laguë and Mixed ones, the Mixed method is better, by the minimum of ARD index, than Sainte-Laguë one and, in majority of cases, is better than the d'Hondt one.

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Fuzzy logic in automatic analysis of Google Analytics data

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Abstract

This article presents a method for automation of the process of data analysis generated from Google Analytics, using fuzzy logic. After the method implementation, the user will get the most useful and suitable content depending on the age he/she has. Since Google Analytics doesn't provide any automatic analysis tools, implementing this model would bring an appreciable number of advantages, like time saving.

Keywords: Fuzzy logic, web analytics, automatization, self-tuning, fuzzyfication.

1 Introduction

One of the most popular search engines – Google, provides specialized tools – Google Analytics(GA). It allows the storage of the data about the website users, such as: geographic location, age, preferences and interests, etc. [1]. This information is used to optimize the website, but at the moment there are no available methods or tools for automating the decision-making process, so this responsibility is taken by the SEO (Search Engine Optimization) administrator. There are frequent instances when the SEO performs the same routine of operations repeatedly, leading to a decrease in yield and productiveness and also to the adoption of poor quality decisions. In this article, the possibility of implementing fuzzy logic to automate the decision making process in order to save time and free the administrator of such routine activities, will be investigated. One of the GA compartments is the users age, which will be used in this research and based on it, a method of automatic data analysis and optimal decision making will be proposed, using the maximization function in the defuzzyfication process.

2 The fuzzyfication process

In the section bellow, Google Analytics gathered the collected data into 6 groups, G_1–G_6 (input variables):

- G_1 – from 18 to 24 years old;
- G_2 – from 25 to 34 years old;
- G_3 – from 35 to 44 years old;
- G_4 – from 45 to 54 years old;
- G_5 – from 55 to 64 years old;
- G_6 – above 65 years old.

But, the information on the site is divided into 3 groups [2]:

- FG_Y – for young people;
- FG_M – for middle-age people;
- FG_O – for old people.

Since there are no clear criterias of division between these age categories, the dividing into 6 groups (G_1 – G_6) which Google Analytics offers, can't be used directly, being necessary to implement an algorithm that could determine to which group of those 3 (FG_Y, FG_M, FG_O) does the user belong (Table 1).

Table 1. The representation of fuzzy age distribution

	18	24	34	44	54	64	65+
Young	1	0.75	0.25	0	0	0	0
Midle	0	0.25	1	1	0.25	0	0
Old	0	0	0	0.25	0.75	1	1

Here we can apply the Fuzzy logic [3] to present a specific content depending on the group of users. This will define the linguistic variable age, that can receive one of the following values: young, middle-aged, elderly (Table 1). Basing on the data from the Table 1, the diagram of fuzzy age distribution is proposed (Fig. 1).

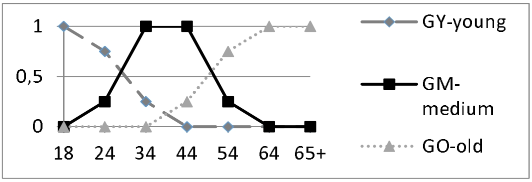

Figure 1. Fuzzy age distribution diagram.

3 Rule base definition

As it can be seen from Fig. 1, a user is “very” young if he is 18 years old (maximum value 1), “still” young at the age of 24 (value 0.8), “quite” young (value 0.2) at age 34 years and at the age of 44 years not young anymore (value 0). The same approach will be applied to the other two groups: middle-aged and elderly. Hereby, a rule can be created [3], which will transform the input data into output data as follows:
IF age = young THEN content = for young
IF age = medium THEN content = for medium
IF age = old THEN content = for old

In this way, the system receives at the input – numerical age of the user and using the rule base created earlier will determine the category (group) of the users and which content should be submitted.

4 The defuzzyfication process

At this stage, the linguistic variable content will indicate to system from which group should be the content presented. If the content of the website is intended for middle age people, it will be very useful (maximum usefulness will be 1) for the 34-44 aged users. For users which age is 24-34 and 44-54 it will have a medium usefulness (0.2 and

0.8) and for the users younger than 24 years and older than 54 years, this content will have a lower usefulness, so will present no interest. For this reason the maximization function will be used [4]
optimalContent = Max(content)
because the aim is to present the most relevant content for the corresponding user age. So, if the value of the affiliation function would be 1, the best suitable content would be presented.

5 Conclusion

The proposed method allows the automatic selection of the most optimal and suitable content to be presented to the website users. The main advantage is that if the user is 25-34 years old (the affiliation function of G_M will have the maximal value – 1), and if the content for this group of user is available, it will be displayed. Otherwise the best suitable content will be displayed.

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Regeneration of printed cultural heritage: challenges and technologies

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Abstract

The paper is concerned to techniques for creation and use linguistic resources for historical Romanian, departing from existing documents of the corresponding periods. Problems related to Moldavian Cyrillic alphabet for 1951–1991 in Bessarabia (Moldavian SSR) are discussed in details. Descyrillization of the literature from this period would permit to inject the most valuable parts of this heritage in the cultural life. A textbook by Victor Ufnarovski «Аквариу математик» ("The Mathematical Aquarium") is used as an illustration.

Keywords: OCR, recognition of historical printed text, transliteration, electronic lexicography, language development, preparation of historical texts to re-edition.

Restoration of Romanian Cyrillic texts. We present a methodology to restore Romanian texts printed in Cyrillic scripts. Historically, three types of Romanian Cyrillic scripts were used [1]. Romanian Cyrillic in 47 letters existed in Romania (including Bassarabia) till the middle of 19$^{\text{th}}$ century. Bassarabians used Cyrillic script based on Russian civil script till the 1$^{\text{st}}$ quarter of the 20$^{\text{th}}$ century. Then Moldavian Cyrillic script was used in Bassarabia (Moldavian ASSR, Moldavian SSR) till 1989. The latest variant is used till present in Transnistria.

We take 1951–1989, and restrict ourselves by the scientific and technical texts. We are interested in preparation of manuals, textbooks, and monographs for their re-edition. Electronic linguistic resources created for the said task can be used at research on development of the Romanian language.

We selected for our experiment a popular science textbook [2]. Its Russian variant [3] was in great demand, and was re-edited in Russia three times in 2010, 2011, and 2014. The edition of 2014 was the extended one[1]. The Romanian edition in the Latin script may therefore be of interest for specialized publishing houses.

Regeneration of Romanian Latin text of the book consists of the following stages:

- scan and image preprocessing;
- recognition (OCR);
- manual correction of recognition errors in the Cyrillic text;
- transliteration of Romanian text from Cyrillic to Latin script;
- editing of transliterated text;
- preparation of camera ready manuscript in LaTeX.

Scan and image preprocessing. The print quality was quite satisfactory. There were no special difficulties during scan. Before the OCR stage, the images were preprocessed by the *ScanTailor* program[2].

OCR. We used the *FineReader* OCR engine to recognize Romanian Cyrillic text of the said book. *FineReader* recognizes all Unicode characters in many typefaces. We defined the corresponding subset of Unicode as a new "user language". We took the Russian alphabet as the base, deleted letters **ё**, **щ**, **ъ**, and added letter **ӂ**. The last letter was introduced in 1967 for sound [dʒ].

FineReader may be provided with a lexicon (word list). Lexicon is used at recognition of poorly printed words, and at the hyphenation processing. We had not any list of Romanian words in the Cyrillic script during our experiment.

Manual correction of recognition errors. The print quality was good so the hyphens elimination was mainly necessary on this stage. The following statistics of the text in Moldavian Cyrillic

[1] `http://www.ozon.ru/context/detail/id/5021600/`

[2] `http://scantailor.org/`

script was obtained: 244 pages; approx. 364,000 characters (incl. blank spaces); approx. 61,300 words; approx. 8,050 different words.

Transliteration of text. Transliteration between Cyrillic and Latin scripts for Romanian is a complicated process due to irregularities of the Moldavian Cyrillic writing. For example, the letter **я** can be mapped to Latin **ia**, **ea**, and **a** (*функция* = *funcţia*). A special program is necessary for the process.

Editing of transliterated text. It was noted that the text does not fully correspond to the modern standard norms of the Romanian language. Therefore the additional lexical and stylistic editing is necessary after transliteration.

Preparation of camera ready manuscript. This includes insertion of formulas and diagrams, adaptation and insertion of artistic graphics, and LaTeX processing.

The book contains a lot of equations and drawings. They should be retyped manually in LaTeX. The problem of equations recognition is not solved till now.

We should convert graphical items to formats suitable for LaTeX processing (EPS of PDF). Several pictures include inscriptions that should be transliterated by manual correction of the image. The corresponding LaTeX packages should be used to integrate pictures into the text.

LaTeX processing, with the necessary editing of the LaTeX source, should be performed repeatedly till obtaining the camera ready result.

Other problems and approaches. To obtain Moldavian Cyrillic lexicon transliterating the modern Romanian word list from the Latin script is possible but equally difficult. For example, there are approx. 20 rules for letter **i** that can be mapped to **и**, **й**, **ь**, **ю**, **я**, **ы**, or deleted. We are developing the corresponding software.

The paper [4] describes an experiment with creation of a parallel Romanian corpus at the automated alignment when both Cyrillic and Latin variants of a text are available. Some additional peculiarities of the descyrillization process were unveiled. The pure transliteration "letter to letter", "letter to letter combination", and "letter combination to letter" for the text used in [4] covers 98.2% of words. The remaining 1.8% of words were edited as "word to word", "word to word combination" and "word combination to word". The following differences in the orthographic rules were noted: use of hyphen instead of apostrophe (*ынтр'о* = *într-o*); elimination of hyphen (*вре-ун* = *vreun*).

Conclusion. We see that the restoration of printed book from Moldavian Cyrillic script is complicated even for a relatively near period. The first task is to develop a transliteration program and to obtain the corresponding lexicon. It would permit to inject into the cultural context the texts. The developed linguistic resources and obtained de-scyrillized texts could be used to study development of the Romanian language in the corresponding period and place.

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Managing for knowledge through the quality management of information systems

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Abstract

In this article quality management (QM) is presented as an efficient link between information management (IM) and knowledge management (KM), and the main condition of the alignment of IM to the enterprise's business objectives. IM is discussed as a foundation for KM. It is argued that significant benefits can be achieved from the integration of IM, KM and QM in the form of improved profitability and customer satisfaction.

Keywords: integrated information system, information management, knowledge management, quality management.

1 Introduction

The experience accumulated in the field of informatization until present leads to a conclusion, that despite the importance of information technologies (IT) for the organizations success, they are still not sufficient to guarantee the success of the main organization's activity. The question is how to manage and use the information and IT to help organizations take best advantage of it.

In the past the main focus was concentrated on the development of IT infrastructure (networks, information systems, etc.). Now we reached the level of IT, when we can process huge amounts of data and obtain enormous volumes of information. But sometimes the surplus of information (impossible to be assimilated) causes more problems than the lack of it. At present we need instruments to obtain only the information, that corresponds to special requirements of users: to be relevant to specific decision situations, to be understood by users according to their experience, and being taken as a base for decisions they have a positive impact on the main activities efficiency. In other words, users need only the information, corresponding to the definition of knowledge. That's why it is very important to focus on the development of mechanisms for improving quantitative and qualitative aspects of information infrastructure and services.

2 Main directions of research and practice

In the context of the development of Information society, among the most frequently used concepts and, respectively, the main directions of research and practice, oriented to ensure effectiveness of informatization, the following ones are considered: Information Management (IM), Knowledge Management (KM), Quality Management (QM) of Information Systems (IS). In the increasing amount of publications in these fields we see that they are developed mostly as independent domains of science and practice. Nevertheless, we can observe that during the latest period of time more and more different interdependencies among these directions of science are discussed.

A clear relationship between KM and IS was mentioned in [1], where it was argued that IS, which has traditionally primarily operated in technical domain and has been pulled "kicking and screaming" into developing complementary people skills and organizational skills, must move yet another major step in the same direction if it is to play an important, integrative role in KM. The idea that Information Resource Management should be a Foundation for Knowledge Management has been discussed in [2]. The necessity of implementation of principles of Total Quality Management (TQM) in IS management has been examined in [3] as useful techniques for effective management of the IS function and, we would add, to the IM in general. An extensive literature exists on this domain but it is based on fragmented research.

In [4, 5] we made an attempt to do a systemic analysis of literature specialized in this area, which allowed to identify mutual interdependencies (relationships) among all these types of management (IM, KM, QM). They appeared to be highly correlated, and being realized in parallel in the process of development of IS, provide an essential synergistic effect of all information and knowledge activities.

3 Quality management of IS and IM

The relationships between IM, KM and QM should be taken into consideration by IS designers and IM practitioners. Analyzing the mentioned relationships we can identify the role of each type of management in order to integrate them into the system, helping organizations take best advantage of it. For example, IM is an essential factor supporting the implementation of such an information-intensive management system as QM. On the other hand, QM in the field of IS and IM could be a mechanism that, improving the quality of data, soft, information, etc., according to the requirements of users of the given enterprise, can become an effective mechanism of achieving a good level of KM. Thus, assessing the quality of the received information and ensuring a permanent feedback in the framework of QM ensure the relevance of accumulated data, information and knowledge in accordance with real needs of users and with their experience. In this case knowledge management (at least in the framework of explicit knowledge) will be based on the QM, implemented in IM.

The feedback mentioned above supposes an intensive user involvement that improves IS, IM and KM common success in complex development projects. It is not enough to involve users at the beginning of IS project, at the stage of elaboration of new IS concept. It is impossible to define final requirements to the system quality (software and hardware quality) and information quality in a very dynamic environment. Thus, it is necessary to build a mechanism which ensures a permanent monitoring of user's satisfaction by information results provided by IS. This information should be used for relevant and permanent improvement of the system. That mechanism should be based on implementation and utilization of QM techniques to IM and KM.

Integration of KM (at least in the framework of explicit knowledge) into the strategy of IS development is an efficient way of the IS development. KM for explicit knowledge (documented and organized on supporters) has many common features with IM. Based on permanent feedback between users of the information and information managers, QM in IS provides data on the quality of the information in relation to the area of interest of the users (influenced by their experience, etc.), necessary for permanent improvement of IM.

The QM in IS links the IM with the general management and, in addition, with KM. Thus, information system, specialized databases, document management, other components of IT must be designed and developed under a general concept of the integrated IS, in order to ensure informational reflection of all real processes, facts in their dynamic in time.

4 Conclusion

While the three types of management (IM, KM,QM) are developed as the independent ones, the results of the study found them highly correlated, and being realized in parallel in the process of development of IS, provide an essential synergistic effect.

Significant benefits can be achieved from the integration of IM, KM and QM in the form of improved profitability and improved customer satisfaction. Quality management of IS, when permanently should be collected the information about the user's satisfaction on the quality (usefulness, relevance, timelines, etc.), can be an efficient subsystem, transforming IM in a good foundation for KM.

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Recognition and Prediction for Implicit Contrastive Focus in Romanian

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Abstract

We investigate the issue of Information Structure (IS) and attempt to provide solutions to the prosody prediction problem of the Implicit Contrastive Focus (ICF) concept. Explicit Contrastive Focus (ECF) is the intonationally F marked entity introduced by overt lexical contrastive markers; ICF occurs without the lexical presence of the contrastive Focus markers. The ICF problem means to obtain reliable algorithms and procedures on the Discourse-Prosody interface in order to accurately predict the contrastive Focus distribution within the Romanian ICF-type affirmative finite clause. We describe algorithms for solving the ICF problem for Romanian, using dislocated constituents in the finite clause to predict Prosodic Prominence (PP).

Keywords: prosody prediction for the Romanian clause, Communicative Dynamism degrees; Systemic Ordering, Implicit Contrastive Focus in Romanian.

1 Systemic Ordering and Communicative Dynamism for Prosody Prediction

Information Structure (IS) and Intonation are seen as two autonomous and independent components of grammar, closely related to each other: intonational phrasing and patterns express informational structure, while a great part of IS is linguistically conveyed by prosody. The interaction between IS and intonation is studied on the following grammar interfaces: "intonation and phonology, Focus and phonological phrasing, intonation (focus) and syntax, IS and discourse analysis"[5].

The Systemic Ordering (SO) refers to a pre-established linear order of the clause constituents (syntactic-semantic roles) in a finite clause. SO is the statistical result of the most frequent linear ordering of the semantic roles in a finite clause, for all the predicates in the active diathesis of a certain language. For a specific predicate (or predication) p, the SO for p is denoted with $\mathrm{SO_p}$. Thus SO can be understood as the corresponding p-free, statistical ordering, of all the $\mathrm{SO_p}$-orders. SO and $\mathrm{SO_p}$ notions are language-dependent, possibly with strong similarities for sibling natural languages. For the Romanian language there are not reliable results on the SO and $\mathrm{SO_p}$ orderings, supported by computational linguistic consistent studies.

When referring to $\mathrm{SO_p}$ (statistically p-depending) or SO (statistically p free) order of the semantic roles in a specific clause, within a certain language, we can have the $\mathrm{SO_p}$-disorder, respectively SO-disorder that may occur on a particular clause [4], [2]. The SO for Czech and English is given in [3].

2 Explicit and Implicit Contrastive Foci

Explicit Contrastive Focus (ECF) and *Implicit Contrastive Focus* (ICF) phrases should be used with the following meanings [1]: ECF describes those categories of contrastive Focus introduced by specific lexical markers, while ICF designates the situations where contrastive intonational focusation is covered by dislocation / disorder of the semantic roles within the finite clause, but without the lexical presence of the contrastive Focus markers. The only device to introduce the contrastive focusation on certain constituents is the syntactic dislocation from their standard position in the Systemic Ordering (SO) of syntactic-semantic roles for the Romanian finite clause. The ICF problem consists in obtaining reliable algorithms and procedures on the Discourse-Prosody interface in order to predict realistically the contrastive Focus distribution in the Romanian ICF-type affirmative finite clause.

3 Algorithms for ICF Prosody Prediction

Our purpose is to estimate the Focus categories in a finite-clause of ICF affirmative type. Two positions of a syntactic-semantic role within a finite clause are important: (1) The position of the corresponding constituent in the Systemic Ordering (SO). The constituent first position is naturally denoted as its place into the SO-order. (2) The position of a constituent as syntactic-semantic role within a finite affirmative clause to be analysed. The measure of SO-disorder is proposed to be the distance between the CD-order and SO-order positions, and computed as the number of permutations necessary to remove the dislocated constituent from its CD-order and to reposition to its location in the SO order. Conventionally, the predicate does not change its position relative to the SO-order in the clause.

For the textual SO-order of an affirmative finite clause, we established experimentally an ordering for the PP (Prosodic Prominence) values of the SO-ordered constituents. This order is represented as Focus weights taking values in the interval [0, 1], assigned to each syntactic-semantic role in the clause, for the SO-ordered constituents. The SO-coefficients are used for the estimation of PP values on the basis of the CD-degrees computed for the constituents in CD order (thus SO-disorder) in an affirmative ICF-type finite clause using an algorithm we have designed.

4 Conclusion

In this paper we have presented a method for determining Prosodic Prominence of constituents in Romanian affirmative clauses on the basis of Systemic Ordering and Communicative Dynamism.

The current approach we proposed is to compute the distance of a CD-dislocated constituent compared to the SO-position of the constituent, for the SO-order of the constituents in the ICF-type affirmative finite clause at-hand.

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Particular Aspects of the Cyrillization Problem

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Abstract

We study the problem of converting Romanian words from Latin to Cyrillic alphabet. A case study of letter *i*, which could be translated to either one of *и*, *й*, *ь*, *ё*, *ю*, *я*, or nothing, depending on the context and some non-trivial factors is presented.

Introduction Why is cyrillization problem actual? Currently there is a huge heritage of texts written in Romanian in Cyrillic script. For making them usable they should be accessible to all. A good solution is digitizing texts of this heritage. Digitization process consists of two stages:

1. Scanning texts written in Cyrillic script.
2. OCR scanned text.

For implementation of the second stage, OCR program, one needs to train a lexicon of words written in Cyrillic, spelled according to the historical period. Moreover, the obtained text should be spell checked. So, the need for a corrector of the Romanian language words written in Cyrillic script arises. Such a corrector does not exist yet, it would have to rely on the linguistic resources of Romanian words in Cyrillics. A database of about 1 million of annotated Romanian words in Latin alphabet was developed [2, 3]. The cyrillization problem becomes very important for obtaining a corresponding database in Cyrillics.

We continued developing and describing the algorithm [1] of transliterating words of Romanian language written in Latin script to its equivalent written in Cyrillic script, according to grammatical rules stated by MSSR in the period 1967–1989. Below we give the elementary rules.

The following letters are translated independently of others:

 *Acklowledging Artiom Alhazov for assistance in article preparation.

Latin	ă	â	b	d	f	î	j	k	l	m	n	o	p	r	s	ş	t	ţ	v	z
Cyrillic	э	ы	б	д	ф	ы	ж	к	л	м	н	о	п	р	с	ш	т	ц	в	з

The rest of conversion depends on the context, e.g.,

Latin	a	c	g	h	u
Cyrillic	а	к	г	х	у
If not in	ia, ea	c(e), c(i)	g(e), g(i)	(c,g)h(e, i)	iu
Cyrillic	я	ч	дж	-	ю

Some cases were presented in [1], it remains to discuss the behavior of letters *e*, *i*, *q*, *w*, *x*, *y*, as well as the subsequences in the last row of the table above.

In a natural and efficient solution of the problem, the translation should be performed left-to-right. The rules in any group are sorted lexicographically. We look at the first untranslated symbol. The words with letters *q*, *w*, *y* are borrowed from other languages, sometimes reducing the problem to the language of the origin; a dictionary of exceptions is needed where a systematic approach fails. The letter *x* is translated as either *кс* or *кз*. All other cases contain at least one of letters *e*, *h*, *i*.

A special case presents the transliteration of words containing the letter *i*. Further processing is a study of letter *i* in different cases, the algorithm was implemented.

Main contextual rules In transliteration algorithm we used the following rules. Symbol "-" means no letters. Word boundaries are denoted by \$. Capital symbols C/V indicate consonant/vowel, respectively. Sign ¬ negates a condition. Symbol ' represents palatalization.

Left context	Latin	Right context	Cyrillic	Condition
(c)	i	(o,u)	-	
(c)	i	(a)	и	
(c)	i	\$	-	Singular noun
(c,g)	i	\$	ь	Unstressed (e.g., plural noun or imperative verb)

C	*i*	C	*и*	
V	*i*	\$	*и*	if it forms a syllable
V	*i*	\$	*й*	if it does not form a syllable
C	*i*	\$	*ь*	to indicate palatalization
\$|V	*ia*		*я*	
i	*a*		*я*	e.g., *România-Ромыния*, *teoria-теория*
i	*a*		*а*	in borrowed/internat. words
	ia		*ья*	in roots of domestic words
\$|V	*i*	*e*	-	
	i	*e*	*и*	e.g., *теорие*
	iii	\$	*ии*	
¬(*i*)	*ii*	\$	*ий*	
C	*ii*	C	*ии*	
\$	*i*	*o*	*й*	rarely *и* instead of *й*
V	*i*	*o*	*й*	
C	*i*	*o*	*и*	
	i	*u*	*и*	e.g., in suffix *-циуне*
\$|V	*iu*		*ю*	
C	*iu*		*ю*	only if read as '*u*
				e.g., *ochiul-окюл*

More complicated rules When developing our algorithm the rules described above were maximally taken into consideration. However, the specific character of Romanian language does not permit completely formalize them. It should be noted that we process words in Romanian language basis of language resources [2, 3]. This allows us to use additional information about the word, such as the part of speech (noun, verb, adjective), singular/plural, etc.

Letter *i* in the end of a word Nouns ending in consonant+*i*. If the word ends with *i*, preceded by one of groups (*bl*, *dr*, *cp*, *pl*, *fl*, *ştr*, *tr*) in plural, rule *i*→*и* applies, e.g., *codri*⇒*кодри*. The same rule applies for specific words of other parts of speech:

noştri	*ноштри*	*voştri*	*воштри*	*muţunachi*	*муцунаки*
simpli	*симпли*	*umpli*	*ымпли*	*umfli*	*ынфли*
intri	*ынтри*	*umbli*	*ымбли*	*iluştri*	*илуштри*
umpli	*ымпли*	*afli*	*афли*	*scaraoţchi*	*скараоцки*
culi	*кули*	*maori*	*маори*	*efendi*	*ефенди*
henri	*хенри*	*hicori*	*хикори*	*bengali*	*бенгали*
lori	*лори*	*peni*	*пени*		

In the nouns ending with *ci*, *chi*, *gi*, *ghi*, if the plural ends with *uri*, rules *ci*→*ч*, *gi*→*ж*, *chi*→*кь*, *ghi*→*гь* are applied, e.g., *ochi*⇒*окь*. The same rule also applies for other parts of speech:

aici	*аич*	*atunci*	*атунч*	*deci*	*деч*
nici	*нич*	*cinci*	*чинч*		

Rule *i*→*й* is applied to the nouns ending in *ei*, *ai*, *ui*, *îi*, *oi*, *ăi*, e.g, *pui*⇒*пуй*.

In the end of the following words after *ş*, rule *i*→- is applied:

acuşi	*акуш*	*acelaşi*	*ачелаш*	*aceluiaşi*	*ачелуяш*
aceiaşi	*ачеяш*	*cîtuşi*	*кытуш*	*sieşi*	*сиеш*
totuşi	*тотуш*	*însuşi*	*ынсуш*	*iarăşi*	*ярэш*
However,		*îşi*	*ышь*		

For 2nd person verbs, singular and plural, in imperative and in present conjunctive, rule *i*→*ь* is applied, e.g., *cînţi*⇒*кынць*.

Exceptions, for which rule *i*→*и* applies:

intri	*ынтри*	*umbli*	*ымбли*	*umpli*	*ымпли*	*afli*	*афли*

For the verbs whose 1st person singular ends with *n*, rule *i*→*й* is applied, e.g., (*spun*) *spui*⇒*спуй*.

Letter *i* in the beginning of a word Rule *i*→*ы* applies for the verbs starting with *i* and their derivatives. Examples:

introduce	*ынтродуче*	*intitula*	*ынтитула*	*insera*	*ынсера*
incontinuu	*ынконтинуу*	*investi*	*ынвести*	*intona*	*ынтона*
invedera	*ынведера*	*intrare*	*ынтраре*	*implica*	*ымплика*
investiţie	*ынвестициe*	*inserat*	*ынсерат*	*intra*	*ынтра*
introductiv	*ытродуктив*				

For the following words, the starting diphtong *ie* changes to *ue*:

ie	*ue*	*iezuit*	*uезуum*	*iezuitic*	*uезуuтuк*

Yet, for the following words, the starting diphtong *ie* changes to *e*:

ied	*ed*	*ieduţ*	*eдуц*	*iei*	*eй*
iele	*eле*	*ieri*	*eрь*	*ierna*	*eрна*
ierta	*eрта*	*iete*	*eте*		

The starting diphtong *ia* changes to *я*. Examples:

iarăşi	*ярэш*	*iad*	*яд*	*iarbă*	*ярбэ*	*ia*	*я*

Letter *i* inside a word Diphtong *ie* not in the beginning of a word is replaced by letter *e*. Example: *taie*⇒тае.

Exception (*ie*→*uе* applies instead): *ijienă*⇒*ижиенэ*

In the following words, diphtong *io* is replaced by *йо*:

iobag	*йобаг*	*iod*	*йод*	*iordan*	*йорган*
iotaciza	*йотачиза*	*bălăior*	*бэлэйор*	*voios*	*войос*
maior	*майор*	*ploios*	*плойос*	*raion*	*район*

In words allowing root alternations, rules *ia*→*я* and *ie*→*e* are applied after consonants *b*, *r*, *m*, *p*, *t*. Examples:

amiază	*амязэ*	*amiezi*	*амезь*	*biată*	*бятэ*
biete	*беme*	*viaţă*	*вяцэ*	*vieţi*	*вець*
dezmiardă	*дезмярдэ*	*dezmierd*	*дезмерд*	*piardă*	*пярдэ*
pierd	*перд*	*piatră*	*пятрэ*	*pietre*	*петре*
fiarbă	*фярбэ*	*fierbe*	*фербе*		

In the words without above mentioned alternations the rule *ie*→*e* is also applied. Examples:

viespe	*веспе*	*miel*	*мел*	*miercuri*	*меркурь*
mierlă	*мерлэ*	*piedică*	*педикэ*	*piele*	*пеле*
piept	*пепт*	*piersic*	*персик*	*fiere*	*фере*
vier	*вер*				

Rule *io*→*ьо* is applied in the middle of a syllable. Examples:

ghiol	*гьол*	*chiomb*	*кьомб*	*chioşc*	*кьошк*
mintios	*минтьос*	*miorlăi*	*мьорлэи*	*ştiolnă*	*штьолнэ*

Rule *iei*→*ей* is applied in old words after a vowel:

apăraiei	*апэраей*	*ghionaei*	*гионаей*
droaiei	*дроаей*	*zgripţuroaiei*	*згрипцуроаей*

Exceptions, using *iei*→*ией*:

bogăţiei	*богэцией*	*bărbăţiei*	*бэрбэцией*
împărăţiei	*ымпэрэцией*	*tăriei*	*тэрией*

Conclusions Thus, the algoritm of converting Romanian words from Latin to Cyrillic alphabets is obtained. Certainly, we have not pretensions to the completeness. The algorithm will be developed further. We should especially note that the Reusable Resources for Romanian [3] were very useful for us because they contain the morphological information about words.

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Preliminary analysis of the reporting support implementation and testing for IAS IMCS

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Abstract

This article describes results of preliminary analysis of implementation of reporting support and testing for Information Analytical System for Research Institution – IAS IMCS.

Keywords: reporting indicators, Information Analytical System, research institution.

1 Introduction

In recent decades, the number of different kinds of reports, documentation on projects preparation, and other materials for presentation of research institution (RI), its departments and individual researchers has significantly increased. Preparation of such documentation takes considerable time, which a researcher initially prefers to spend directly on scientific activities. As a result, representatives of RI, in particular those related to information systems development, have found themselves in the role of a shoemaker without shoes: it turned out that there does not exist an information system to support the activities of RI from the standpoint of a researcher and RI' administration.

Experience in the development of such systems, when representatives of institutions who administer research activities from outside, act as a customer [1], or experience in scientific activity and its administration are not united, showed lack of effectiveness of these systems. There are quite powerful and useful systems of information character which do not provide analytic functions [2]. Also there are various types of bibliographic and other systems aimed at one side of the RI' activity. But they do not support the process of RI' reporting. Specificity of RIs requires that demands for information analytical system (IAS) were advanced by group of persons including researchers, administrators of the RI itself, and, which is especially desirable, by persons, possessing both, scientific and administrative experience.

This article describes results of preliminary analysis of implementation of reporting support and testing of IAS for RI – IAS IMCS [3]. The system is still under development; in parallel its database (DB) is filled with data of the Institute of Mathematics and Computer Science (IMCS) of Academy of Sciences of Moldova and is tested on the input data for obtaining values of certain reporting indicators.

2 Reporting support

What is "reporting indicators"? As a rule, it is a list and/or a number of objects with certain properties at definite moment/period of time (Figure 1). E.g., one needs to get in a report the following:

- number of monographs, published by RI researchers in 2005-2010;
- number of researchers younger than 35 years at January 1, 2010.

For reporting, these indicators usually are calculated manually, which takes a lot of time and brings a lot of inconvenience. It is significant that all the data required for calculation are time-related, e.g.:

- publication – with a year of publishing;
- age of the employee – with the birth date and the date on which the information is requested;
- number of researchers – with the date on which the data is requested and the period of time when an employee holds a post.

Factor of time representation in IAS IMCS. 1) In most DB tables of IAS IMCS each record contains moment or period of time to which the stored data are attached. 2) On forms for sampling data

Human resources (without combining jobs)*

Total persons	
including:	
scientific researchers	
PHDs	
doctors habilitation	
scientific researchers younger than 35 years	
PHD students	
postdoctoral scholars	
master students	**

* Organizations in science and innovation indicate also persons employed without combining jobs internally

** Ratio of master students who study on the base of *budget/contract* financing

Figure 1. One of the forms (with the list of indicators) in the annual report both for each subdivision and the institution as a whole.

for reporting indicators calculation, field(s) for indicating respective moment/period of time, is/are present.

Sampling data in accordance with the right balance of time ranges in the items 1) and 2) gives the desired result (list) for the indicator. The difficulty is that most of the indicators are often based on several tables, each containing its own time slot. E.g., when counting the number of articles published by researchers of some laboratory in a specified year, you need to consider publication date and periods of researchers work in this laboratory. This information is in several tables.

DB of the system "IAS IMCS" filling. Since this is not just the system for reporting, but for reports creation support, the issue of the amount of relevant objects, accompanied by a list of the objects helps to take control of data reliability. For example, one of the staff, looking the report materials, notes that in the list of publications there is no monograph, about which he knows exactly that it has been already appeared. So information is not input into the system DB or it is input incorrectly, or the report is incorrectly made.

In the “proper” scientific institutions there is a motivation of researchers in publications and other evidences of their scientific achievements. Having IAS RI, information about these evidences have to be selected from the IAS DB. Consequently, the researchers are interested in relevant data input into the IAS RI database correctly, as early as possible, and also in their validation by the respective officials [3]. Naturally, that better and earlier than the researcher himself, this information no one will input. In addition, they can also use this data, such as information on publications, in preparation of individual, institutional, laboratory reports, project reports, and in preparation of new projects, new publications, CV. Thereby, taking into account the specificity of scientific institutions, the problem of the database filling is being resolved.

Representation form for reports being generated in the IAS IMCS. The IAS IMCS generates reports in format of Excel table. It permits not overloading the system by details in the data representation mode, especially of graphical nature, and to use varied widely known and widely applicable arsenal of Excel. So, having the report generated in the Excel format, one can build demonstrative charts.

Another handy feature of Excel – the use of formulas for cell contents construction – is being used in IAS IMCS for concatenation of separate cells of the generated report with information about publications. This allows one to create lists of publications and adapt them to different standards of publications lists representation. For this purpose the generated report contains an additional column “Summary”. In this column the information about publication, which is contained in other columns, is “stuck” together as it looks in usual publications lists. The reports generation tools constructs formula for column “Summary”. Changing this formula by means of Excel allows approximating to the desired standard much closer.

The system is being developed and transferred into operation step by step. At first, the part of the system responsible for the most labour intensive and often used indicators, is implemented and became available for use: publications, participation in forums and staff of the

institution (age, gender, scientific title etc.).
The system IAS IMCS testing from the very beginning is carried out on real data. At the step of designing, the analysis of the authors' experience when elaborating the system "Scientific Potential of Moldova" (elaborated by the order of Supreme Council for Science and Technological Development) [1] was conducted. The real data from this system were extracted and input into IAS IMCS. This, to some extent, guaranteed the accuracy of the data. Since the concept and objectives of IAS IMCS are substantially richer than of the system "Scientific Potential of Moldova", the extracted data were adapted and new parts are started to be input.

At the first step, the system was tested by the developers directly and updated with new valid data by the necessity, and not always through the user interface.

At the step, when the interface began to function for data input and editing, for the purpose of testing the system by the potential users, a group of researchers-nondevelopers was involved in the system DB replenishment. The system developers periodically analyze and "clean" the input data. Revealing "typical" errors of data input and problems of this group of researchers permit improving the system interface.

To date, the system DB contains information about 660 publications of the IMCS researchers, while table for authors contains 1820 records, table for collections of publications – 260 records, table for publishers – 164 records, table for persons – 138 records, participation in the forums – about 252 records, table for projects – 62 records, table for researchers participation in projects – 119 records.

3 Conclusion

– At the beginning, input of inquiry information (dictionaries) takes a lot of time: titles of journals, publishers, scientific specialities etc. But such information is input just once when used for the first time, but is used repeatedly. The positive effect of the system usage increases as its database is replenished.

– The dictionaries content, classification of scientific papers, conferences, etc. depend on the internal needs of the institution, on the reporting that is required.
– Usage of the system, and in particular, of DB dictionaries, facilitates self-discipline in the RI and enhancement of documentation, as well as self-discipline of major users – researchers.
– Advantage of the system is DB completion by the institution researchers, who are the authors of publications, executors of projects, participants of scientific forums etc. As direct players, they know better than anyone the titles of publishers, forums, projects ciphers and other components of the complete information. In addition, they are more interested in the presence of correct information about their research activities in the DB, and more than others will actively use it for periodic reporting, list of publications and CV writing.

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Methods of exploratory data analysis of the ultrasound investigations

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Abstract

This paper is an introduction to the tools of exploratory data analysis of the ultrasound investigations and methods of correlation analysis and regression analysis.

Keywords: Exploratory Data Analysis, Correlation Analysis, multiple linear regression analysis, univariate data analysis, multivariate data analysis.

1 Exploratory Data Analysis

EDA (Exploratory Data Analysis) was created and named by the American statistician John Tukey.

EDA is approach analysis that is focused on identifying patterns in the data, and identifying features of the data. EDA is the first step in data analysis [1].

Methods of analysis generally fall under the following categories:

1. Univariate data analysis. There exist several ways to summarize a univariate distribution. Often, simple descriptive statistics is computed and plotted on a corresponding histogram; however, such univariate statistics is very sensitive to extreme values and, which is possibly more important, does not provide any spatial information [2].

2. Multivariate data analysis. Working with multiple variables requires the use of multivariate data statistics. If we want to express

the relationships between two variables, we do so through the regression study and correlation analysis. In a regression study, we estimate the relationship between two variables by expressing one as a linear (or nonlinear) function of the other. In correlation analysis we estimate how strongly two variables vary together [3].

3. Normal-score transform. Many statistical techniques suppose that the data have an underlying Gaussian distribution. The transformed data are used for some geostatistical analyses and can be reconfigured to their original state in a back transform, when correctly executed. Thus, it is a temporary state and is used for the convenience of satisfying the Gaussian assumption, when necessary [2].

2 Correlation Analysis

Correlation analysis represents a method for measuring the covariance of two random variables in a matched data set. Covariance is usually expressed as the correlation coefficient of two variables X and Y. The correlation coefficient presents a unitless number that changes from -1 to $+1$. Value of this coefficient is the standardized degree of association between X and Y. The sign is the direction of the association, which can be positive or negative [4].

3 Multiple linear regression analysis

A multiple linear regression analysis is carried out to predict the values of a dependent variable, Y, given a set of p explanatory variables $(x_1,x_2,\ldots,x_{\mathrm{p}})$.

As in the case with simple linear regression and correlation, this analysis does not allow us to make causal inferences, but it does allow us to investigate how a set of explanatory variables is associated with a dependent variable of interest [5].

4 Data analysis methods used in tools for exploratory data analysis in ultrasound domain

Ultrasound investigation is one of the most widespread diagnostic methods of medical imaging.

Ultrasound is easy-to-use paraclinical investigation method, non-invasive and highly effective. It is very accurate in its application area and is easy to be realized by a well-trained specialist [6].

On the other hand, the operator dependency problem is known, which implies a degree of subjectivism in description of organs and formulation of conclusions. To reduce this dependency and obtain qualitative standardized conclusions the SonaRes decision support system was developed. It is aimed at using it for diagnostics of abdominal region organs.

The SonaRes system helps doctor to obtain quickly correct information about a specific pathology. It is not the case that the system replaces the doctor, it just proposes conclusions, which do not contradict the data introduced by the doctor, and, more importantly, gives an explanation about the process of their obtaining.

As a result of ultrasound investigation the SonaRes system accumulates a considerable data stock, which presents an inestimable value for research and improvement of diagnostics methods and treatment.

The knowledge base serves for storing the data accumulated in the process of knowledge acquisition and formalization.

The user database serves for storing the data, accumulated in the process of collecting information about patient and his diagnostics: personal data, investigation sessions history and results, generated reports, etc.

Diagnostics module serves for carrying out the investigation process. The doctor has possibility to interact directly with the system through the interface block, which ensures dialogue between the user and the system. Diagnostics module realizes diagnostics deduction based on data obtained from the user, the knowledge base and the inference engine (Fig. 1).

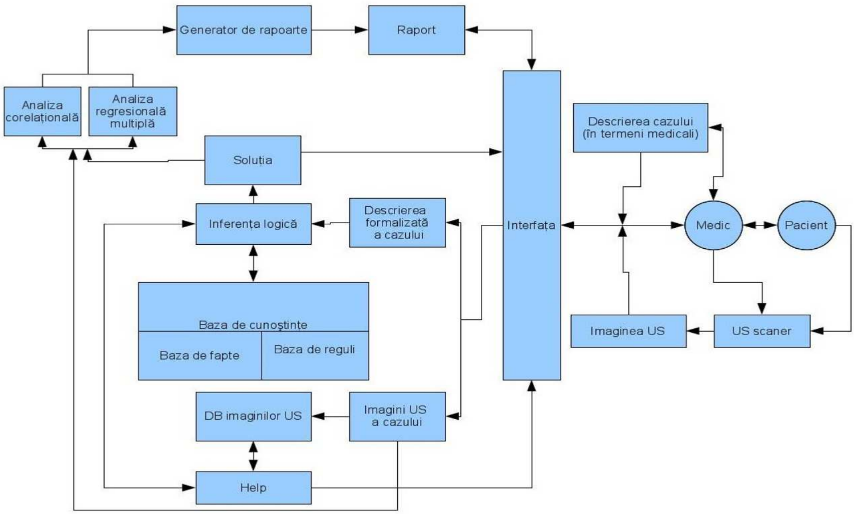

Figure 1. Components of the SonaRes system

To create tools for exploratory data analysis (EDA) we use the database of the SonaRes system, namely: patient personal data (first name, last name, age, contacts, rural or urban environment) and investigations data. In addition, for EDA purposes we will request accumulation of the data about living and working conditions (degree of environmental pollution in a certain locality, nutrition, harmful factors in working conditions, etc.).

To realize data analysis two additional modules were included in the functional scheme of the SonaRes system, namely, "Correlation analysis" and "Multiple regression analysis" (Fig. 1). These two methods will be used to analyze the relationships between some pathologies and action of the living environment or of some harmful factors, which can influence the evolution of pathologies.

From Fig. 1 we can observe that the patient data and investigation results are stored in the knowledge base, and the conclusion of process of ultrasound investigation is obtained through the logical inference.

All data from the component "Solution" go to the components "Correlation analysis" and "Multiple regression analysis", where the analysis of relationships between pathologies and factors that lead to their evolution is performed.

To perform the analysis of pathologies evolution two types of multivariate analysis correlation and multiple regression were selected. The choice is motivated by the fact that these techniques focus more on intensity of dependencies between the variables, less on their distribution levels.

In our case we investigate a series of correlations, which establish both, the dependence between some pathologies and factors, which may determine their evolution, and dependence between some pathologies.

For example, it is known that an excessive use of pesticides leads to the development of toxic hepatitis and cirrhosis. It is necessary to establish the correlation between degree of pollution with pesticides in certain areas and existence of people with mentioned pathologies, to estimate the severity of diseases, dependence on workplace, age, etc.

5 Conclusion

Analysis of relationships between pathologies and harmful factors, which lead to their development, allow reducing risks of pathologies progression.

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The emotional orientation estimation

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Abstract

The paper presents two sentiment analysis classifiers in order to configure a tool for automatic sentiment analysis to respond optimally in this important issue of the natural language processing. A huge amount of texts on topics of public interest, especially on forums, was annotated morpho-syntactically and semantically, manually and automatically. The statistics are relevant for developing the EAT tool, i.e. with more features than opinions extraction (as age and gender of writer, topics, tonality, etc.). This study is intending to help direct beneficiaries (public consumer, marketing managers, PR firms, politicians, journalists), but, also, specialists and researchers in the NLP, linguists, sociologists, etc.

Keywords: sentiment analysis, classifier, semantics, statistics.

1 Introduction

The option for this topic of sentiment analysis (SA), known as opinion mining, encountered in texts exposed in *Forums,* relies on the need to clarify the descriptive behavior of commentators, affected by the amount of different approaches of articles from online print press, especially in the crisis communication, regardless of their nature and purpose. Part of this research was reported already (Gîfu, 2012, 2014) and it concerned this issue, aiming to improve and validate the Romanian tools.

The question is: *How much a computational tool can capture the nature and the intensity of sentiment in a text?*

The paper is structured in five sections. The section 2 mentions some important works focused on sentiment analysis. The section 3 describes two sentiment analysis classifiers used in this research, the section 4 presents statistics, and finally, the section 5 depicts some conclusions and discussions.

2 Background

The extracting emotional orientation from a text is assimilated as subjectivity analysis [Dave et al., 2003], evaluating affection [Gîfu & Cioca, 2014] or objectivity analysis [Mihalcea et al., 2007]. SA defines the processing search results from a text, generating a list of qualities and aggregating opinions for each of them. Moreover, SA has been interpreted as including various types of analysis and evaluation [Liu, 2010], [Pang and Lee, 2008]. The approaches take in consideration two classes (positive and negative), sometimes three (e.g. the neutral class with value 0), assigning each word with one value of a scale of values (shorter or longer) [Gîfu, 2012].

3 Text classifiers

SAT (*Sentiment Analysis Tool*) was created using an API[1]. This classifier determines if a text is positive or negative, and was based on 40000 Amazon product reviews and mood training data. Moreover, the tool can measure the state of mind of the user (*Mood classifier*), the gender of the user (*Gender Analyzer Classifier*), which age group the user belongs (*Age Analyzer Classifier*), the tonality of a message (*Tonality Classifier*).

EAT (*Emotional Analysis Tool*) is a tool able to detect and to explain the appreciations about some entities (persons, products, organizations, etc.). EAT is based on information like labeling of parts

[1] http://www.uclassify.com/

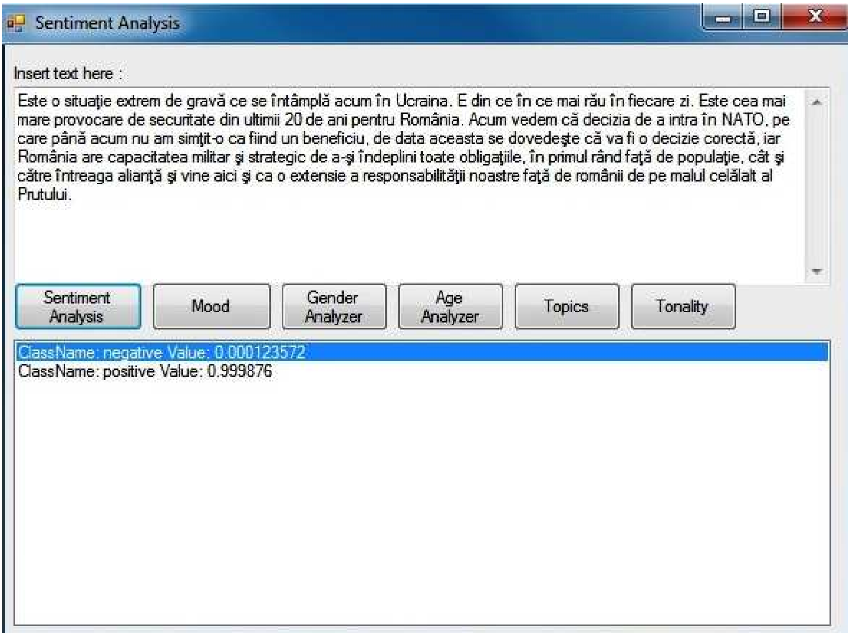

Figure 1. The SAT interface

of speech (e.g. Example 1), extraction of interest nominal groups, automatic extraction of entities and anaphoric connections. Moreover it was developed an important ontology of entities, categories and values.

Example 1

```
<?xml version="1.0" encoding="UTF-8" standalone="no"?>
<DOCUMENT>
<P ID="1">
<S ID="1">
<W EXTRA="intranzitiv" ID="11.1" LEMMA="fi"
MSD="Vmip3s" Mood="indicative" Number="singular"
POS="VERB" Person="third" Tense="present"
Type="predicative" offset="0">Este</W>
<NP HEADID="11.3" ID="0" ref="0">
```

```
<W Case="direct" Gender="feminine" ID="11.2"
LEMMA="un" MSD="Tifsr" Number="singular"
POS="ARTICLE" Type="indefinite" offset="5">o</W>
<W Case="direct" Definiteness="no"
Gender="feminine" ID="11.3" LEMMA="situa?ie"
MSD="Ncfsrn" Number="singular" POS="NOUN"
Type="common" offset="7">situa?ie</W>
<W ID="11.4" LEMMA="extrem~de" MSD="Rg"
POS="ADVERB" offset="16">extrem de</W>
<W Case="direct" Definiteness="no"
Gender="feminine" ID="11.5" LEMMA="grav"
MSD="Afpfsrn" Number="singular" POS="ADJECTIVE"
offset="26">grav?</W>

</DOCUMENT>
```

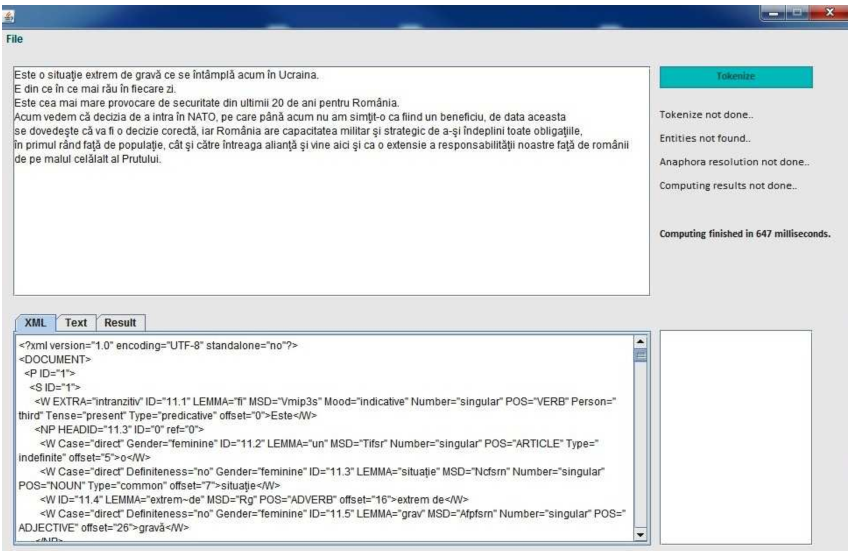

Figure 2. The EAT interface

4 Statistics

For the elaboration of preliminary conclusions over the efficiency of the two instruments, a manual and automatic analysis was performed on a corpus of 121 texts (7448 words), starting from the comments on the Romanian forums in first week of May. The topic was *the crisis from Ukraine.*

Table 1. Statistical results for the emotion detection

	Precision	Recall	F-measure
SAT	61%	72 %	66%
EAT	87%	81%	84%

5 Conclusions and discussions

The statistical results show that a classifier based on Romanian language, does not preserve the semantics of API standards. In addition, the previously analyzed texts reflected the users' opinions on products and companies and not on political crises. The communication context requires the diversification of the corpus and the improvement of the rules set for values. Furthermore, the EAT algorithm will be improved by calculating the distances between categories and entities found in the same sentence.

Acknowledgments. I express my appreciation to all colleagues from NLP-Group@UAIC-FII who developed the tools for natural language processing used in this research.

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A syntactic-semantic analysis system

Veronica Gîsca

Abstract

This paper presents a formal representation of natural language, taking account of the syntactic and semantic information simultaneously. The grammatical formalism used is that of attribute grammars, and representation knowledge is provided by formal logic.

Keywords: natural language processing, parser, syntax, semantics, attribute grammars.

1 Introduction

A natural language processing (NLP) system consists of at least several components including syntactic and semantic analyzers. A common assumption in the design of an NLP system is that these components are separate and independent. This allows researchers an abstraction necessary to promote substantial steps forward in each task, plus such a separation permits for more convenient, modular software development. However constraints from higher level processes are frequently needed to disambiguate lower level processes. Thus, an approach that integrates syntactic and semantic processing is essential for proper analysis of sentences.

In this paper we present a system for natural language parsing that targets a tight integration of syntax and semantics using attribute grammars.

2 System description

The system consists of the following parts: a lexical analyzer, a syntactic parser, a semantic interpreter.

A lexical entry consists of a word and a set of codes that are necessary for subsequent analysis. In fact, it is in the lexical entry that basic morphological, syntactic, and semantic information is stored. From the point of view of syntax, the lexicon helps the parser assign a token to a particular word. The semantic information contained in the entry is the basic synthetized attribute underlying the final process of analysis.

To implement the described process within attribute grammars approach, we first manually specify a dictionary that contains semantic information sufficient for annotating leaf nodes stemming from the words in an input sentence. We then use logic rules to define annotations for non-leaf nodes of parse trees and restrict the produced parse trees only to those that are semantically coherent.

The input to the syntactic parser is a string of tokens and terminals to be processed into a sentence with some syntactic structure. The syntactic rules represent a context-free grammar description. As form the type of look-ahead, the syntax is basically of the top-down type.

Components of a top-down process can be easily added or edited without affecting the rest of the system. In this way, the top-down parser can be constructed piecewise [1]. In our top-down system, a new component consisting of syntactic and the related semantic rules can be developed and tested separately and then added as a new alternative [3].

Our system is ideally suited to incorporate compositional semantics that have correspondence between the rules defining syntactic constructs and the rules stating how the meaning of phrases are constructed from the meanings of their constituents.

Alongside with piecewise syntactic extension, we can glue together semantics rules for corresponding syntax rules to compute meanings of a larger set of languages.

3 Analysing a sentence

The process of analysis from sentence to semantic representation can be separated into three sub-processes. After the sentence has been segmented, we obtain the lexical items in XML format.

We used POS Tagger [2] – a web service developed by the Faculty of Computer Science from Iasi, Romania, and the WordNet for Romanian to obtain lexical items.

After parsing, we obtain the syntactic structures of the sentence. For example, consider the syntactically ambiguous sentence: Omul priveşte o pasăre cu telescopul.

Its verb phrase allows for two syntactic structures under the figure (see Fig.1).

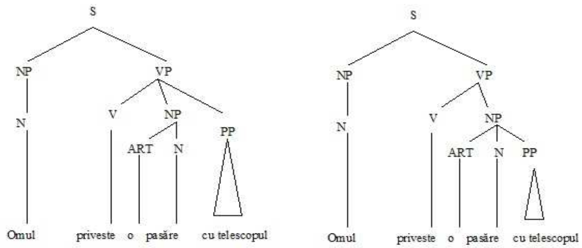

Figure 1. Syntactic structures.

In a prepositional phrase PP, a preposition demonstrates an adjectival or adverbial property by quantifying either the noun phrase NP or the verb phrase VP. According to the second parse, the NP's head noun pasarea is quantified by the preposition "cu", which can be misinterpreted as "o pasăre cu telescopul". But in the second parse, the preposition "cu" quantifies the verb "priveste" from the VP, which is semantically more meaningful. Based on this argument, we can make sure that the second parse is discarded while generating only the first parse by using the attribute grammar.

In order to assign the proper syntactic structure to each of these sentences one has to take into account selectional restrictions, i.e., the semantic restrictions that a word imposes on the environment in which it occurs.

4 Conclusion

In this paper we presented a syntax-semantic analysis system for natural language processing. For integration of syntax and semantics analyses we used the notation of attribute grammars. In contrast to purely syntactic grammars, attribute grammars are capable of describing features in natural language. Attributes are associated with each symbol occuring in the production rule.

In the future we plan to investigate further incorporating dependencies involving inherited attributes along with synthesized attributes used for the natural language disambiguation.

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Diversification in an image retrieval system

Adrian Iftene, Lenuta Alboaie

Abstract

In this paper we present an image retrieval system created within the research project MUCKE (Multimedia and User Credibility Knowledge Extraction). Our discussion in this work will focus mainly on components that are part of our image retrieval system proposed in MUCKE. One of the problems addressed by our system is related to search diversification. In order to solve this problem, we used text processing on user query and image processing to create clusters with similar images.

Keywords: image retrieval, search diversification.

1 Introduction

In last years, the web has become a support for social media where users of social networks are pushing multimedia data directly from cameras, phones, etc. MUCKE project addresses this stream of multimedia social data with new and reliable knowledge extraction models designed for multilingual and multimodal data shared on social networks. One of the aims of this project is related to give a high importance to the quality of the processed data, in order to protect the user from an avalanche of equally topically relevant data. In this context we built a novel image retrieval framework which perform a semantic interpretation of user queries and return diversified and accurate results.

Over time, various theories involving search results diversification have been developed, theories that have been taken into consideration [1]: (i) *content* [2], i.e. how different are the results to each other, (ii) *novelty* [3], [4], i.e. what does the new result offer in addition to the previous ones, and (iii) *semantic coverage* [5], i.e. how well covered are the different interpretations of the user query. In the MUCKE project, we create a collection with over 80 millions images with their associated metadata, mainly extracted from Flickr[1]. Over this collection we perform processing at text level (on associated metadata) and at image level. These processing modules help the information retrieval system to retrieve images and to offer results in a diversification way.

2 Text processing module

The text processing module is used to process the associated metadata and to process the user query. For text processing standard tools are used for POS-tagging, lemma identification and named entity identification. After text processing of associated metadata, the collection is indexed with Lucene. Then, the diversification is made on user query with: Yago ontology and a special module for query expansion.

Yago ontology comprises well known knowledge about the world [6]. It contains information extracted from Wikipedia and other sources like WorldNet and GeoNames and it is structured in elements called entities (persons, cities, etc.) and facts about these entities (which person worked in which domain, etc.). For example, with Yago we are able to replace in a query like "tennis player on court", the entity "tennis player" with instances like "Roger Federer", "Rafael Nadal", etc. Thus, instead to perform a single search with initial query, we perform more searches with new queries, and in the end we combine the obtained partial results in a final result.

Query expansion module uses a technique of processing a given query in order to obtain new ones that are both more efficient and more relevant in the context of information retrieval operations. In this case, we faced with two major issues that occur when the end user enters the query: it is *not precise enough*, meaning that there are too many results returned, most of them being irrelevant or it is *not abstract enough*,

[1]Flickr: http://www.flickr.com

meaning that the search does not return any results at all. Here, we applied two approaches: (1) *a global technique*, which analyses the body of the query in order to discover word relationships (synonyms, homonyms or other morphological forms from WordNet[2]), to remove stop words (the, a, such, at) and to correct any spelling errors; (2) *local feedback* which implies the analysis of the results returned by the initial query, leading to re-weighting the terms of the query and relating it with entities and relationships originating from the target ontology.

3 Image processing module

The information retrieval system searches in its collection with the new obtained queries and returns a collection of relevant metadata with their associated images. The image processing module performs diversification on these returned results. The main aim of this module is to create clusters with similar images and instead offer to user all obtained results, the system will offer only one representative image from every cluster. In this way the similar images are hidden, the user will not be overwhelmed and will be able to see all the pictures on request.

For that, with the help of Mathlab and its predefined functions we extracted visual characteristics such as shape, color, texture, etc. Also, a naive algorithm that calculates an euclidean distance between the average color of the two images was implemented. Then, the clustering module organizes in clusters the images, according to certain metrics conducted from their features and based on the distances between them. For the clustering process the DBSCAN algorithm is used [7] and the resulting clusters represent the application's output.

[2]WordNet: http://wordnet.princeton.edu/

4 Conclusions

In this paper we present our current work in MUCKE project. The paper addresses the diversification aspects that can be useful for an image retrieval system. For that we perform text processing on user query with Yago and WordNet resources and image processing in order to create clusters with similar images.

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A Universal Generalized Register Machine with Seven States

Sergiu Ivanov, Sergey Verlan

Abstract

It is known that there exists a universal register machine with 32 instructions (states) allowed to either increment a register, decrement a register, or check a register for zero. In this paper, we consider more powerful instructions capable of checking multiple registers for zero, as well modifying multiple registers, and we construct a universal generalized register machine with only seven states.

Keywords: generalized register machine, universality, encoding.

1 Introduction

The concept of universality dates back to the paper [4] by A. Turing, in which he constructs a universal Turing machine capable of simulating any other Turing machine. In general, the universality problem for a class of computing devices $\mathcal{C}$ consists in finding a *fixed* element $\mathcal{M} \in \mathcal{C}$ capable of simulating the computation of any other element $\mathcal{M}' \in C$.

Register machines represent a class of computing devices having a finite number of registers storing integer numbers and a finite number of states, with which instructions are associated [2]. In the most basic versions of register machines, the allowed instructions are incrementing a register, decrementing a register, and checking if a register is zero. It is known (e.g. [3]) that such register machines can compute any partial recursive function. In [2], I. Korec constructed a universal register machine with 8 registers and 32 instructions of this kind.

In this paper we consider more powerful instructions capable of checking multiple registers for zero, as well modifying multiple registers, and we construct a universal *generalized* register machine with only seven states.

2 Preliminaries

A deterministic *register machine* is defined as a 5-tuple (Q, R, q_0, q_f, P), where Q is a set of states, $R = \{R_1, \ldots, R_k\}$ is the set of registers, $q_0 \in Q$ is the initial state, $q_f \in Q$ is the final state and P is a set of instructions of the following form: $(p, RiPq)$ – in state p, increment register R_i and go to state q; (p, RiM, q) – in state p, decrement register R_i and go to state q; and (p, Ri, q, s) – in state p, go to q if register R_i is not zero or to s otherwise. We remark that for each state p there is only one instruction of one of these types above. It is habitual to consider register machines in which instructions "decrement and zero check" are used instead of the latter two types [1].

3 A Universal Generalized Register Machine

We will rely on a different representation of register machines as graphs. We label the nodes of the graph with states, while the operations and conditions are attached to the arcs. In our notation, the symbols RiP and RiM stand for the operations of incrementing and decrementing R_i. Such a construct is more general than a register machine, because more than one operation or condition may be attached to an arc. A transition along an arc may happen only when all conditions associated with the arc are satisfied. It is therefore possible to "compress" the states of the universal register machine with 32 instructions from [2], for example, to obtain a generalized universal register machine with fewer states. Note that an arc cannot repeatedly check and modify the same register in one step, which essentially means that some states of the cited universal machine cannot be merged with other states.

Table 1. The universal generalized register machine with seven states

q_i	q_j	Conditions	Operations
1	1	$R_1 \neq 0$	$R1M$, $R7P$
1	2	$R_1 = 0$	$R6P$, $R7P$
2	2	$R_5 = 0$, $R_6 = 0$	
2	2	$R_5 \neq 0$	$R5M$, $R6P$
2	3	$R_5 = 0$, $R_6 \neq 0$	$R5P$, $R6M$
3	1	$R_1 \neq 0$, $R_6 = 0$, $R_7 = 0$	$R1M$
3	1	$R_1 \neq 0$, $R_4 \neq 0$, $R_6 \neq 0$, $R_7 = 0$	$R1M$, $R4M$
3	2	$R_6 = 0$, $R_7 \neq 0$	$R1P$, $R7M$
3	2	$R_1 = 0$, $R_4 \neq 0$, $R_6 \neq 0$, $R_7 = 0$	$R4M$, $R6P$
3	2	$R_1 = 0$, $R_6 = 0$, $R_7 = 0$	$R6P$
3	3	$R_6 \neq 0$, $R_7 \neq 0$	$R1P$, $R5P$, $R6M$, $R7M$
3	4	$R_4 = 0$, $R_6 \neq 0$, $R_7 = 0$	
4	1	$R_0 \neq 0$, $R_1 \neq 0$, $R_2 = 0$, $R_5 = 0$	$R0M$, $R1M$
4	1	$R_1 \neq 0$, $R_2 \neq 0$, $R_4 \neq 0$, $R_5 = 0$	$R1M$, $R2M$, $R4M$
4	1	$R_0 = 0$, $R_1 \neq 0$, $R_2 = 0$, $R_4 \neq 0$, $R_5 = 0$	$R1M$, $R4M$
4	2	$R_0 \neq 0$, $R_1 = 0$, $R_2 = 0$, $R_5 = 0$	$R0M$, $R6P$
4	2	$R_1 = 0$, $R_2 \neq 0$, $R_4 \neq 0$, $R_5 = 0$	$R2M$, $R4M$, $R6P$
4	2	$R_0 = 0$, $R_1 = 0$, $R_2 = 0$, $R_4 \neq 0$, $R_5 = 0$	$R4M$, $R6P$
4	5	$R_5 \neq 0$	$R5M$
4	7	$R_0 = 0$, $R_2 = 0$, $R_4 = 0$, $R_5 = 0$	
4	7	$R_2 \neq 0$, $R_4 = 0$, $R_5 = 0$	$R2M$
5	1	$R_1 \neq 0$, $R_3 = 0$, $R_5 = 0$	$R0P$, $R1M$
5	1	$R_1 \neq 0$, $R_3 \neq 0$, $R_4 \neq 0$, $R_5 = 0$	$R1M$, $R3M$, $R4M$
5	2	$R_1 = 0$, $R_3 = 0$, $R_5 = 0$	$R0P$, $R6P$
5	2	$R_1 = 0$, $R_3 \neq 0$, $R_4 \neq 0$, $R_5 = 0$	$R3M$, $R4M$, $R6P$
5	6	$R_5 \neq 0$	$R5M$
5	7	$R_3 \neq 0$, $R_4 = 0$, $R_5 = 0$	$R3M$
6	1	$R_1 \neq 0$, $R_4 \neq 0$, $R_5 = 0$	$R1M$, $R2P$, $R3P$, $R4M$
6	2	$R_1 = 0$, $R_4 \neq 0$, $R_5 = 0$	$R2P$, $R3P$, $R4M$, $R6P$
6	4	$R_5 \neq 0$	$R4P$, $R5M$
6	7	$R_4 = 0$, $R_5 = 0$	$R2P$, $R3P$

We will now briefly describe a universal generalized register machine with seven states by listing its arcs in Table 1. Each line in this table corresponds to a transition from the state q_i to state q_j, checking the conditions listed in the "Conditions" column, and performing those operations on registers which are given in the "Operations" column.

4 Conclusion

In this paper, we considered generalized register machines, which are oriented graphs of states, in which the arcs have (multiple) conditions and (multiple) operations assigned. We constructed a universal generalized register machine with seven states. Such a construction may be useful in finding the universal elements of certain classes of computing devices (for the case of Petri nets with inhibitor arcs, see [1]).

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Stroke Volume Equation Validation: Impedance Cardiography vs. Ultra-sonography in Humans Exposed to Earth, Moon, Mars and Zero-gravity Conditions

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Abstract

We hypothesized that stroke volume (SV) values obtained from thoracic impedance (Sramek-Bernstein equation) would be similar to those obtained from ultra-sonography and that indexes of human segmental fluid volume shifts as well as regulatory responses would be similar at Earth and under simulated Mars and Moon gravities produced by Lower Body Positive Pressure and Head Up-Tilt models. SV comparisons indicated that impedance cardiography provided a legitimate estimate of SV. Even though the impedance formula generated higher absolute values for SV than did the ultrasound technique, percentage changes in response to matched levels of orthostatic stress were similar for both techniques.

Keywords: Impedance Cardiography, Ultra-Sonography, Stroke Volume, Sramek-Bernstein Equation, Gravity.

1 Introduction

In today's era of space exploration, models of activity in reduced and increased gravity are needed to predict human cardiovascular responses to short and long duration space flight, from liftoff to landing on other surfaces, like Earth's Moon, Mars, or an asteroid and to safe return of crew to Earth. Currently, there are a variety of human models used in simulating reduced or increased gravity conditions, like water immersion, centrifugation [1], parabolic flight, body suspension, head up tilt, head down tilt and bed rest. In all of these models, a common problem is accurate, noninvasive measurement of stroke volume (SV). This variable is also the one most likely to provide insights into the effectiveness of reproducing the effects of reduced gravity fields [2]. Use of lower body positive pressure (LBPP) to control stroke volume in upright subjects shows great potential as an approach for reliably simulating cardiovascular changes similar to those that occur in reduced gravity. In this study we compare head-up tilt (HUT) vs. body un-weighting using LBPP for simulating responses to standing on the Moon (20% Body Weight (BW) vs. 10° HUT), on Mars (40% BW vs. 20° HUT) and on Earth (100% BW vs. 80° HUT). We hypothesized that stroke volume values for each subject obtained from the thoracic impedance equation (Sramek-Bernstein) would be similar to the one obtained from ultrasonography and that the indexes of healthy human segmental fluid volume shifts (thorax (THX), abdomen (ABD), upper (UL) and lower (LL) leg), and cardiac output, as well as regulatory [MAP, HR, total peripheral resistance (TPR)] responses to standing would be similarly affected by LBPP and HUT models.

2 Methods

2.1 Subjects / Protocol

Cardiovascular responses of healthy, young men (n = 12, Mean ± S.E.M., Age = 26.1±1.5 yr, Ht = 178.7±3.1cm, Wt = 79.9±5.2kg) and women (n = 9, Mean ± S.E.M., Age = 26.2±1.5 yr, Ht = 164.5±0.9cm, Wt = 64.1 ± 4.6kg) to lying supine followed by standing on Earth and in simulated reduced gravity were measured using an Alter-G device (Mfg. G Trainer, Alter-G, Inc) at 20%BW, 40%BW and 100%BW. The Alter-G device is a computerized treadmill enclosed in the Alter-G LBPP chamber. The use of this treadmill requires the subject to wear special neoprene shorts which zip into the opening at the top of the pressure chamber in a special collar. The pressure inside the Alter-G chamber can be adjusted and can reduce the subject's body weight from 0 to 80% of original weight by pushing the body upwards. This allows for simulation of various gravity environments while the subject stands passively or exercises (the treadmill can be set for walking, running or climbing). The same data were also acquired during HUT (Bailey Mfg. 9505 Tilt Table) at tilt angles modeling Moon (10°) Mars (20°) Earth (80°) and space flight (0°) body weights. Subjects were non-smokers and not suffering from hypo- or hypertension. Data collection lasted a total of 4 hrs for each subject. Subjects were randomly assigned to an evenly distributed matrix by treatment type (Alter-G or HUT) and treatment level (Alter-G ctrl, Alter-G 20%BW, Alter-G 40%BW, Alter-G 100%BW, HUT0°, HUT10°, HUT20°, HUT80°). Subjects were fitted with Alter-G shorts, worn for all activities, during both AG and HUT conditions. They were then instrumented for measurements of ECG (3-lead "UFI -2121/1"), beat-to-beat blood pressure of the finger ("Portapres", Finapres Medical Systems), oscillometric brachial blood pressure every 3 min (AND – UA – 767) and segmental impedance/fluid shifts (10-lead "Thrim Tetrapolar –UFI 2994"). The ECG and segmental impedance leads used monitoring electrodes (3M "Red Dot" 2560). The tilt table was equipped with an accelerometer, while the Alter-G device had a force platform and an LBPP pressure sensor ("CyberSense Cy Q 301")

Prior to testing, subjects were weighed ("Taylor Professional" Scale), their height measured ("American Std") and the distance between impedance leads recorded. Following this, 10 min of supine (HUT0°) data were collected. The HUT table activities consisted of tilting the subject to one of the required angles (HUT0° – for Space / weightlessness, HUT10° for Moon gravity, HUT20° for Mars gravity and HUT80° for passive response to Earth gravity). Data collection consisted of 10-15 min of echocardiography, followed by three minutes of cardiovascular data collection (SBP, DBP, MAP, HR, SV, CO, TPR, segmental impedance [THX RO, ABD RO, UL RO, LL RO]) at each stress level. Alter-G activities consisted of adjusting chamber pressure to obtain the ground reaction force required to yield the desired body-weight of the subject (Alt-G ctrl was taken in a second supine session conducted to simulate Space/ weightlessness, Alt-G 20%BW for Moon gravity, Alt-G 40%BW for Mars gravity, Alt-G 100% BW for Earth gravity). Data collection consisted of 10-15 min of echocardiography, followed by three minutes of cardiovascular data gathering for each of the body weight levels in Alter-G.

Two-dimensional echocardiography (ultra-sonography) for determining stroke volume, heart rate and cardiac output was administered. Aortic blood velocity time integral was acquired by sonography during the last minute of each HUT or LBPP level from continuous wave Doppler measurements made at the apical window using a 2 to 4 MHz phase array probe with a standard ultrasound machine (CX-50, Philips Healthcare, Andover, MA). Images were stored digitally for offline analysis (ProSolv w 3.0, Problem Solving Concepts, Inc., Indianapolis, IN). Doppler images from at least three cardiac cycles for each minute were independently analyzed by two experienced, registered sonographers. Stroke volume (annulus cross sectional area x velocity time integral), cardiac output (stroke volume x heart rate), and total peripheral resistance (mean arterial pressure/cardiac output) were calculated. In addition to the ultra-sonography method, above, SV was determined using thoracic impedance (Sramek-Bernstein impedance formula, [3]):

$$SV = \delta \cdot \frac{(0.17 \cdot H)^3}{4.25} \cdot \frac{dZ/dt_{max}}{Z_0} \cdot LVET \tag{1}$$

Eq. (1). Sramek-Bernstein Stroke Volume Equation, where δ = weight correction factor, H = subject height (cm), dZ/dt_{max} = maximal impedance change (Ω/s), Z_0 = basic Thoracic impedance (Ω), $LVET$ = Left Ventricular Ejection Time (s)

Sramek-Bernstein developed this equation based on original Kubicek's equation, Eq. (2), who assumed that the estimated volume of the electrically participating tissue in the thorax is modeled as a cylinder instead of a truncated cone as in the Sramek-Bernstein version.

$$SV = p \cdot \frac{L^2}{Z_0^2} \cdot dZ/dt_{max} \cdot LVET \tag{2}$$

Eq. (2). Kubicek's Equation, where SV is stroke volume (ml), p the resistivity of blood (Q cm), L the distance between the voltage measuring electrodes (cm), Z_0 the basic thoracic impedance (Q), dZ/dt_{max} the maximal impedance change (Q/s) and $LVET$ the left ventricular ejection time (s).

A typical impedance signal (dZ), its first derivative (dZ/dt) with marks on the important points of the wave form, and the ECG are shown in Fig. 1. Since Kubicek *et al.* introduced their methodology for stroke volume calculation, the first derivative of the impedance signal has extensively been studied by many investigators to discover its physiological correlates and origin. Karnegis *et al.* first showed that the A-wave follows the P wave of the ECG, and the C-wave is associated with ventricular contraction. During diastole, they noticed another upward deflection of the dZ/dt signal: the O-wave. Lababidi *et al.* compared the dZ/dt signal with simultaneously performed phonocardiography in 91 subjects. They found that the B-point coincides with the aortic valve opening and the X-point with aortic valve closure. These observations have been confirmed by several investigators using echocardiography and aortic pressure recordings. Using today's technology, the dZ/dt signal has been determined to be highly sensitive for systolic time intervals. The origin of the impedance cardiographic signal appears to be complex [3] and the exact physiological and anatomical basis still needs further explanation. Many investigators have dealt with this subject in the past. In general, evidence to support the origin of the impedance cardiographic signal has been derived from studies, which tried to correlate physiological parameters to the dZ/dt signal, modelling studies and studies performed in animals.

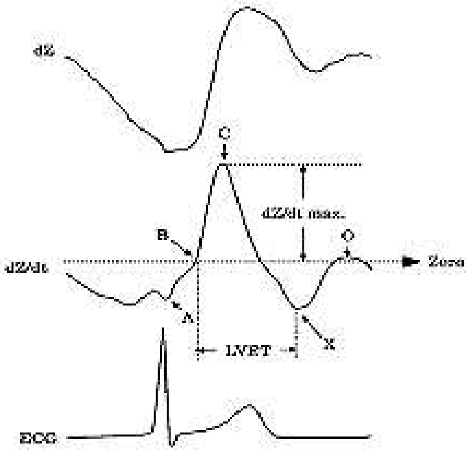

Figure 1. Characteristic dZ, dZ/dt and ECG signal

2.2 Validation

In the past 30 years many validation studies have been performed [3] comparing the impedance cardiographic method according to Kubicek *et al* and according to Sramek and Bernstein, with other methods to assess stroke volume. Sramek and Bernstein's method is by far the most frequently used impedance cardiographic method since 1986. This is the result of the implication of this method in a practical, commercially available set-up (the NCCOM, BoMed Medical Manufacturing Ltd., Irvine, CA, U.S.A.). Most investigators found a significant correlation between stroke volume measured with impedance cardiography and stroke volume measured with other methods. However, various investigators also reported a wide dispersion of the impedance stroke volume data. This has especially been reported by investigators using Sramek-Bernstein's method. It also appears from these studies that the impedance method is not equally valid under all physiological conditions. Aortic valvular pathology, the first 12 h after coronary artery surgery and sepsis appear to be less favourable conditions for impedance cardiography. Studies using Kubicek's method show better correlations with the reference method. Pickett *et al.* and Woltjer *et al.* seem to confirm this observation. Nevertheless, Kubicek's method is far less standardized. A critical element of this method is the resistivity of blood (p). In recent studies of Demeter *et al* it was found that the best results are obtained when p is calculated dependent on the patient's haematocrit. No consensus can be reached on the accuracy of impedance cardiography in the measurement of stroke volume based on the present studies. In some studies, the method is evaluated as highly accurate, in others more dispersion between the two methods is found. In most studies however, the mean difference between the two methods and its standard deviation are not shown. In order to establish the validity of impedance cardiography, more research is needed on the latter, and more studies need to be performed comparing both impedance cardiographic methods with each other and other non-impedance methods. In this study we validated the Sramek-Bernstein method against ultra-sonography method, which in the last decade is considered to be the gold standard.

2.3 Data Analysis

All signals were acquired at a sample rate of 1000 Hz using Data Acquisition Software ("Windaq Pro 2.59, DI-7x0 USB0, "DATAQ Instruments"). Subsequent analysis was performed using "Windaq Browser", "Physiowave 6.21, Visual Numerics" and Microsoft Excel 2003 software installed on a "Dell Latitude D-600" laptop. A mixed effects model ANOVA (one between factor, gender, and two within factors, treatment model (AG vs HUT), and treatment level, four levels of gravity stress) was performed using SAS Enterprize Guide 4.2 (SAS Institute). A p value of < 0.05 was considered statistically significant. Data are presented as means $\pm$ S.E.M.

3 Results

Data were collected and results obtained for cardiovascular variables: SV, HR, CO, SBP, DBP, THX, ABD, UL and LL impedances. The Stroke Volume Results are shown in Fig.2.

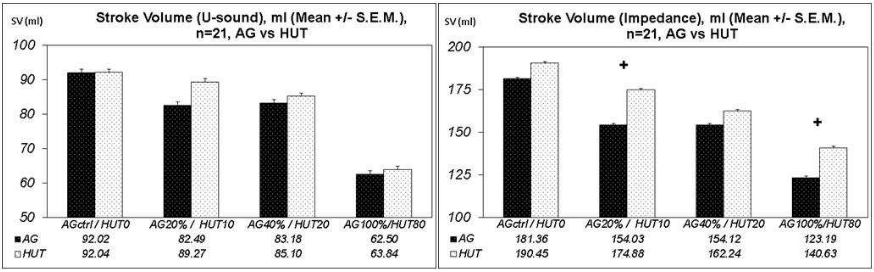

Figure 2. Mean Stroke Volume from ultrasound/sonography (SV1, left) and from impedance equation (SV2, right)

During HUT, SV determined by ultrasound, declined at 80 degrees, Fig. 2, left panel, solid line. During Alter-G, SV was decreased (from supine) at Alter-G 20% BW and again at 100% BW, left panel, dashed line. Tilt-induced changes in SV calculated from impedance, Fig. 2, right panel, were similar to ultrasound results, SV was smaller at 80 degrees and Alter-G values declined at 20% BW and again at 100% BW. Percentage changes in response to matched levels of orthostatic stress were similar for both SV techniques: Δ (HUT0 vs. HUT80) $\sim$ 30% difference in relative terms. As far as other cardiovascular variables, compared to supine: fluid shifts from the chest to the abdomen, increases in HR and decreases in stroke volume were greater at 100% body weight than at reduced weights, in response to both LBPP and HUT. Differences between the two models (Alter-G and HUT) were found for systolic BP, diastolic BP, mean arterial BP, stroke volume, total peripheral resistance, thorax and abdomen impedances, while HR, cardiac output, upper and lower leg impedances were similar.

4 Discussion / Conclusion

Stroke volume comparisons between ultrasound and impedance techniques indicate that impedance cardiography [3] provides a legitimate estimate of this important variable. Even though the impedance formula generated higher absolute values for SV than did the ultrasound technique, percentage changes in response to matched levels of orthostatic stress were similar for both techniques: Δ (HUT0 vs. HUT80) $\sim$ 30% difference in relative terms for both (Fig. 2). That implies that our initial hypothesis about similarities between the two stroke volume techniques should be rejected, however, due to similarities in relative terms, the stroke volume obtained from impedance cardiography could be used as a cheap, fairly accurate (in percentage change) and non-invasive alternative to ultra-sonography.

In addition to SV changes, when compared to normal weight standing, HR was lower with unloading (HUT80° vs. HUT0° or Alt-G 100% vs. Alt-G ctrl), CO and SV were greater, and fluids were shifted from lower leg to upper leg, to abdomen and less fluid was shifted from the thoracic cavity. From this study we found that body weight unloading via both LBPP and HUT resulted in cardiovascular changes similar to those anticipated in actual reduced gravity environments. The LBPP model/Alter-G had the advantage of providing an environment that allowed dynamic activity at reduced body weight, however, the significant increase in blood pressures in the Alter-G may favor the HUT model.

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A Fuzzy Ontology for Walking

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Abstract

We discuss the usefulness of and suggest several elements for a fuzzy ontology for describing the human walk.

Keywords: fuzzy ontology, web ontology language, human movements, robotics

1 Introduction

Human movement may be described with linguistic terms that are difficult to interpret because their values are imprecise or vague, as in the linguistic representation of the parameters in the ontology proposed in [1], based on [2]. Fuzzy ontologies were introduced by [3] to represent knowledge in domains in which the concepts have imprecise definitions. In this paper, we propose a fuzzy extension for the ontology of human movements, from [1], [2]. After recalling in the next section the main concepts and axioms defined in the ontology [1], we discuss in the third section fuzzy concepts and fuzzy datatypes of the fuzzy ontology for human movements. A brief conclusive section ends the paper.

2 Ontology for Human Movements

The ontology of human movements described in [1] uses the main concepts recalled below.

- **BodyParts** – contains concepts about joints and segments of the human body, which are used in human action.

- **ReferencePlanes** – are used to describe the relative position of a body part and has the following elements FrontalPlane, SagittalPlane and Transversal - Plane (see [2]).
- **RelativePosition** – is defined by reference planes and contains the following elements: Upper, Lower, Left, Right, Front and Back [1,2].
- **Parameters** – describe a human action; include Postural (Angle, StepLength, Size), Frequential, and Cinematic parameters [2].
- **ParameterRepresentation** – representation with linguistic terms (Small, Medium and Large), or numerical values (MultipleValues, SingleValues in Ptrotegé [4] terminology).
- **HumanAction** – describes the actions: Standing, Walking, Running.
- **HumanCondition** – describes the condition of the human: physical status (normal, disabled) and age (child, young, mature, old) [2].

The ontology was implemented in Protégé 4.3 [4]. An example of linguistic representation of a parameter is small/medium/large KneeAngle. As reference values, we consider 120° as the maximum value for small, 150° the maximum value for medium and 180° the maximum value for large. The KneeAngle parameter is used in axioms to define some HumanAction concepts. For example, the description for Running class is [1] presented in Table 1.

A fuzzy extension of the proposed ontology for human movement is described in the next section.

3 Fuzzy Parameters and Rules in the Proposed Ontology

Some concepts and parameters in the human movement ontology [1] are imprecisely and vague defined. For these, we propose fuzzy concepts and fuzzy datatypes using Protégé FuzzyOWL Plugin [5]. Aggressive

Table 1.

Class: Running
SubClassOf:
HumanAction,
((hasAngle some Angle)
and (hasJointAngle value KneeAngle)
and (hasLinguisticRepresentation some Medium))
and ((hasFrequency some Frequency)
and (hasLinguisticRepresentation some Large))
and ((hasStepLength some StepLength)
and (hasLinguisticRepresentation some (Medium OR Large)))

and slow walking are examples of fuzzy concepts that can be defined based on velocity and postural parameters. These, in turn, are frequently defined using linguistic attributes. For example, for velocity, we define Small, Medium and HighVelocity as fuzzy datatypes using triangular membership functions. In this setting, HighVelocity fuzzy datatype has the membership function (m.f.) everywhere null, except

$$\mu(v) = \begin{cases} v-2 & \text{for } v \in [2,3] \\ 1-(v-3) & \text{for } v \in [3,4]. \end{cases}$$

Postural parameters are imprecise because of the limits of technical process of determining them from images (visually) – limited image resolution, limited ability of correct segmentation of the image, and imperfect shape boundary detection; also, errors due to the change of the shape of the clothes and changing orientation of the person with respect to the camera limit the precision. Hence, a fuzzy representation may be more suitable. KneeAngle is a parameter we define by a fuzzy datatype. For interval-type m.f.s, we use trapezoidal m.f.s, e.g., (0°, 0°, 120°, 120°) for SmallKneeAngle, (120°, 120°, 150°, 150°) for MediumKneeAngle, and (150°, 150°, 180°, 180°) for HighKneeAngle, as in Table 2.

Table 2.

Declaration(Datatype(:HighKneeAngle))
AnnotationAssertion(:fuzzyLabel :HighKneeAngle
“ <fuzzyOwl2 fuzzyType= \“datatype\“ >
<Datatype type= \“trapezoidal\“a = \“150\“ b = \“150\“
c = \“180\“d = \“180\“/ >
< /fuzzyOwl2> “)
DatatypeDefinition
(:HighKneeAngle DataIntersectionOf(DatatypeRestriction
(xsd:double xsd:maxInclusive “180.0“^^xsd:double)
DatatypeRestriction(xsd:double xsd:minInclusive
“0.0“^^xsd:double)))

The following rule, which describes HurriedWalk [1], is completed with fuzzy sets (denoted by $\tilde{\square}$), $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{B}_1$, $\tilde{C}_1$, $\tilde{D}_1$, and $\tilde{E}_1$, as in Table 3.

Therefore, the HurriedWalk rule reads as:

$$((\tilde{A}_1) \cup (\tilde{A}_2)) \cap \neg\tilde{A}_3 \cap \tilde{B}_1 \cap \tilde{C}_1 \cap \tilde{D}_1 \cap \tilde{E}_1$$

and the membership function is computed (in min-max fuzzy logic) as:

$$\mu_{\tilde{H_w}} = \min(\mu_{\tilde{E_1}}, \min(\mu_{\tilde{D_1}}, \min(\mu_{\tilde{C_1}}, \min(\mu_{\tilde{B_1}}, \min(1 - \mu_{\tilde{A_3}},$$

$$\max(\mu_{\tilde{A_1}}, \mu_{\tilde{A_2}}))))))$$

The fuzzy concept assertion of HurriedWalk is represented by annotating the concept assertion with a degree ≥ 0.5. Notice that a membership value less than 0.5 still means that the concept cannot be firmly asserted and the related rules must be dealt with as fuzzy rules.

Table 3.

If
[postural]
(average length of step medium ($\tilde{A}_1$) or high ($\tilde{A}_2$) but NOT very high ($\tilde{A}_3$)) **and**
(maximum angle at knee medium ($\tilde{B}_1$)) **and**
(foot angle medium ($\tilde{C}_1$)) **and**
(height of leg hoisting average ($\tilde{D}_1$)) **and**
[frequential]
(step frequency average ($\tilde{E}_1$))
Then
Hurried walk

4 Conclusions

Because there are numerous sources of imprecision and ambiguity (uncertainty) in visually determining the postural and cinematic parameters of the human movement, we suggested a fuzzy extension of the ontology of human movement to define fuzzy datatype for linguistic representation of parameters. An implementation based on Protegé was sketched. This ontology is still a work in progress.

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System Strategy of Survivability and Safety of Complex Engineering Objects Operation

Nataliya Pankratova

Abstract

A system strategy to estimation of guaranteed survivability and safety for operation of complex engineering objects (CEO) is proposed. The principles that underlie the strategy of the guaranteed safety of CEO operation provide a flexible approach to timely detection, recognition, forecast, and system diagnostic of risk factors and situations, to formulation and implementation of a rational decision in a practicable time within an unremovable time constraint. Implementation of the proposed strategy is shown on example of diagnostics of electromobile-refrigerator functioning in real mode.

Keywords: risks, abnormal mode, safety, information platform for engineering diagnostics.

1 Introduction

Creation of modern technology defines a new requirement to ensure their technological and environmental safety. To improve the quality control of complex objects the reasons and the factors should be found out, that can not provide the agreed level of survivability and safety of complex engineering objects operation. One of these reasons is the peculiarities of diagnostic systems, focused on identifying failures and malfunctions. This approach to security eliminates the possibility of a priori prevent abnormal regime and, as a consequence, there is a possibility of its subsequent transition into an accident or disaster.

Here the system strategy to estimation of guarantee of survivable and safe operation of complex engineering objects on the basis of multifactor risks and principle of timely detection of reasons of abnormal situations and prevention of transition of normal situations into abnormal is proposed.

2 Mathematical Formulation of Complex Object System Control Problem

Let us show the mathematical formulation of this problem with a priori set variation intervals of main indicators of the system in the normal mode and predefined permissible boundes of the influence of external factors [1].

It is known that system functioning is characterized by the following sequence of complex system states: $E_1, E_2, ..., E_k$. Every state E is characterized by specified indicators of system function processes (Y_k, X_k, U_k) and specified indicators of external environmental influence and risk factors Ξ:

$$E_k = \{(Y_k \in Y) \wedge (X_k \in X) \wedge (U_k \in U) \wedge (\Xi_k \in \Xi)\},$$

where the meaning of indicators at the moment $T_k \in T^{\pm}$ is defined by the following relations:

$$Y_k = \hat{Y}[T_k]; \quad X_k = \hat{X}[T_k]; \quad U_k = \hat{U}[T_k]; \quad \Xi_k = \hat{\Xi}[T_k];$$

$$T_k = \{t_k | t_k > t_{k-1}\}; \quad T_k \in T^{\pm};$$

$$T^{\pm} = \{t | t^- \le t \le t^+\}; \quad Y = (Y_i | i = \overline{1, m});$$

$$X = (X_j | j = \overline{1, n}); \quad U = (U_q | q = \overline{1, Q}); \quad \Xi = (\Xi_p | p = \overline{1, P}).$$

Here Y is a set of external parameters Y_i that includes technical, economic, and other indicators of system-function quality; X is a set of internal parameters X_j that includes constructional, technological, and other indicators; U is a set of control parameters U_q; Ξ is a set of external environmental influence parameters and parameters of risk factor influence Ξ_p; $\hat{Y}[T_k], \hat{X}[T_k], \hat{U}[T_k]$ and $\hat{\Xi}[T_k]$ are sets of meanings of appropriate parameters at the moment T_k; and $T^{\pm}$ is a specified or predicted complex object functioning period.

Required: determine in the moment $T_i \in T^{\pm}$ such values of degrees η_i and levels W_i of risk, as well as a margin of permissible risk T_{ar}, which provides, during the abnormal mode, the possibility of transition from the mode $\overline{\tilde{R}}^{+}_{tr}$ during the period $\check{T}^{\pm}_{tr}$ to the normal mode till the critical moment T_{cr} of transition of abnormal mode becomes an accident or catastrophe.

3 Strategy for Solving the Problem of System Control of Complex Objects

The main goal of the proposed strategy is to guarantee a rationally justified reserve of survivability of a complex system in real conditions of fundamentally irremovable information and time restrictions.

The main idea of the strategy is to ensure the timely and credible detection, recognition, and estimation of risk factors, forecasting their development during a definite period of operation in real conditions of a complex objects operation, and on this basis ensuring timely elimination of risk causes before the occurrence of failures and other undesirable consequences.

The main approaches and principles of the strategy for providing guaranteed safety of complex systems should be formed on the basis of the following principles [2]:

- timely detection, guaranteed recognition, and system diagnosis of factors and situations of risk;

- efficient forecasting and credible estimation of abnormal and critical situations;

- timely formation and efficient realization of decisions of safety control in the process of prevention of abnormal and critical situations.

The diagnostic unit, which is the basis of a safety control algorithm for complex objects in abnormal situations, is developed as an information platform.

Diagnostics of electromobile-refrigerator functioning in real mode is considered.

4 Conclusion

A system strategy to estimation of guaranteed survivability and safety for operation of CEO allows preventing the inoperativeness and abnormal situations. The principles that underlie the strategy of the guaranteed safety of CEO operation provide a flexible approach to timely detection, recognition, prediction, and system diagnostic of risk factors and situations, to formulation and implementation of a rational decision in a practicable time within an unremovable time constraint.

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On combinatorial properties of elementary intramolecular operations

Vladimir Rogojin

Abstract

Here we tackle a problem from biology in terms of discrete mathematics. We are interested in a complex DNA manipulation process happening in eukariotic organisms of a subclass of ciliate species called *Stichotrichia* during so-called gene assembly. This process is in particular interesting since one can interpret gene assembly in ciliates as sorting of permutations. We survey here results related to studies on sorting permutations with some specific rewriting rules that formalize elementary intramolecular gene assembly operations. The research question is "what permutation may be sorted with our operations?"

Keywords: ciliates, gene assembly, elementary operations, combinatorics, molecular computing

1 Introduction

Ciliates posses two types of nuclei called *micronucleus* and *macronucleus*. A macronucleus contains short gene-sized DNA molecules, where each molecule stores a single gene represented as a contiguous sequence of nucleotides. Meanwhile, a micronucleus contains long DNA organized on chromosomes, where each DNA contains many genes, each gene is broken into fragments (called MDSs), those fragments are separated by non-genetic nucleotide sequences (called IESs), MDSs are shuffled throughout the molecule, and some of the MDSs may be even inverted (for example we refer to Fig.1).

During gene assembly, the micronuclear DNA molecules get transformed into the macronuclear form. During this process IESs get removed from DNA and MDSs are spliced together so that to form mature macronuclear genes (for mode details see [1]).

(a) (b)

Figure 1. From [4]. Micronuclear and macronuclear versions of *actin I* gene from *Stylonychia lemnae*. *(a) Micronuclear gene pattern:* the gene is broken into 8 MDSs. Each MDS (blue rectangle) is separated from each other by an IES (white rectangle). *MDS2* is inverted. *(b) Assembled macronuclear gene:* all IESs are removed, all MDSs are properly ordered and linked to each other to form the contiguous gene. The orientation of *MDS2* is restored.

The *intramolecular model* explains gene assembly in terms of three molecular folding-recombination operations Ld, Hi and Dlad that operate within a single molecule and splice together two or mode MDSs. Operation Ld removes IESs, Hi inverts a piece of DNA containing MDSs and IESs, and Dlad exchanges two pieces of DNA with places in such a way that two or more MDSs get spliced together.

Our focus is at the *elementary intramolecular operations*. Those are intramolecular operations that are "allowed" to invert/relocate piece of DNA containing only one micronuclear (i.e., non-composite) MDS. As the result of such a restriction, unlike the general model, elementary operations may not assemble all the micronuclear gene patterns. Moreover, for the same micronuclear gene pattern there may exist both successful (assemble the gene) as well as non-successful (fail to assemble the gene) strategies. In this way, it is not easy to answer the question what micronuclear gene patterns can be assembled into macronuclear one by elementary intramolecular operations. The goal is to find an efficient method to decide whether a micronuclear gene pattern can be assembled by elementary operations. The problem is translated into a permutation sorting problem. Here we present the recent results related to this problem.

We refer to a recent review on general, simple (less restrictive than elementary) and elementary model here [3].

2 Elementary gene assembly as permutation sorting

A gene pattern with n MDSs is formalized as a signed permutation π over set of integers $\Pi_n = \{1, 2, \dots, n\}$. Here, an integer i represents the ith MDS, while signed integer $\bar{i}$ represents the inverted ith MDS in the pattern. For instance, the micronuclear gene pattern from Fig. 1 is represented as permutation $\pi = 3\,4\,5\,7\,\bar{2}\,1\,6\,8$.

Operations eh and ed formalize Hi and Dlad respectively as rewriting rules over Π_n^* [2]

orthodox eh_i**:**	**inverted** eh_i**:**	**orthodox** ed_i**:**	**inverted** ed_i**:**
$ui(\overline{i+1})v \to$	$u(\overline{i+1})iv \to$	$uiv(i-1)(i+1)w \to$	$u\bar{i}v(\overline{i+1})(\overline{i-1})w \to$
$\to ui(i+1)v,$	$\to u(\overline{i+1})\bar{i}v,$	$\to uv(i-1)i(i+1)w,$	$\to uv(\overline{i+1})\bar{i}(\overline{i-1})w,$
$u\bar{i}(i+1)v \to$	$u(i+1)\bar{i}v \to$	$u(i-1)(i+1)viw \to$	$u(\overline{i+1})(\overline{i-1})v\bar{i}w) \to$
$\to ui(i+1)v,$	$\to u(\overline{i+1})\bar{i}v,$	$\to u(i-1)i(i+1)vw,$	$\to u(\overline{i+1})\bar{i}(\overline{i-1})vw,$

where $i \geq 1$ and $u, v, w \in \Pi_n^*$.

The process of gene assembly is represented as a permutation sorting process by eh and ed operations.

3 Deciding eh, ed-sortability

The decision procedure is in terms of directed graphs and permutations. In [2, 4] we have considered so-called dependency graphs associated to signed permutations, where a directed edge (j, i) means that a eh, ed operation on integer i is preceded by a corresponding operation on integer j in any strategy applicable to the given permutation. In this way, the dependency graphs "tell" us in which order eh, ed operations are used and which operations cannot be used at all in any strategy applicable to π.

Currently, there exists an efficient method (cubic time complexity) to decide the ed sortability for permutations with no signed element [5, 6]. The situation with eh, ed sortability is more complex, currently the search for an efficient decision method is in progress [4].

A characterization method for eh, ed-sortable permutations that yields though no efficient solution one can find at [2].

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On ontology of research, development and innovation in Republic of Moldova

Andrei Rusu

Abstract

Building information systems supposes an unambiguous understanding of key terms and relations between them in the domain of interest. We consider the process of evaluating the research, development and innovation project proposals in Republic of Moldova and build an initial ontology of it with the goal to be useful to make available the use of software agents in the process of evaluation of proposals and their results.

Keywords: research, development, innovation, ontology.

1 Introduction

An ontology is a formal specification of the concepts and the relationships that can exist between concepts. According to Guarino's definition, *"an ontology is a logical theory accounting for the intended meaning of a formal vocabulary, i.e. its ontological commitment to a particular conceptualization of the world"* [1].

2 Method of ontology construction

The methodology used for ontology construction is based on existing methodologies, like ontology development 101 [2] and other ones. The proposed method translates the knowledge semantics described semi-formally in the already developed information system *EXPERT Online* https://expert.idsi.md/ developed at IDSI, http://idsi.asm.md/. The method consists in the following steps: i) define the classes and their hierarchy, ii) define relations between classes; iii) descibe the business rules in ontological manner; iv) make other adjustments; v) verify the ontology. We have to mention also here the use of Protégé ontology editor [3] and the web ontology language OWL DL [4].

3 Ontology construction

Today there are some top-level ontologies which describe general concepts like space, time, object, event which are independent concepts from a particular domain or a concrete problem. Among these we use DOLCE ontology, Temporal Relations and other ones. Other top-level ontologies might be used.

3.1 Identifying classes and creating taxonomy

To identify OWL classes used in our ontology we start with the glossary of terms developed by InfoScientic group at Information Society Development Institute [5] since OWL classes represent individuals that form an extension of the concept mapped by the class. Many attributes of the classes have been transformed in corresponding qualities of time and other abstract characteristics of the DOLCE ontology and other top-level ones.

The classes are organized in a taxonomy created on the basis of the subsumtion relation. Classes A and B are linked by subsumption relation if and only if every instance (individual) of A is also an instance of B and in this case we say that B subsumes A or A is a subclass of B. For example, the taxonomy of **Activitate** is shown in the Fig. 3.1.

3.2 Defining relations and business rules

Most of the relations find in our ontology map to relations found in DOLCE ontology and its submodules. Business rules were identified during analysis of the proposed methodology for evaluating research, innovation and development project in Republic of Moldova proposed

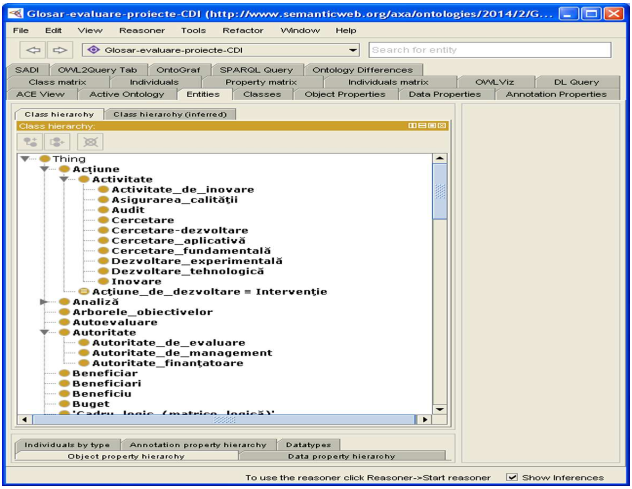

Figure 1. Initial ontology based on [5]

in [6] and taking in consideration the information system *EXPERT Online*.

3.3 Ontology verification

The ontology consistency is checked with the help of the Protégé tool [3] and the Pellet reasoner system [7].

4 Conclusion

The proposed ontology can be used for information exchange between humans and software agents as well as it can be extended to better support the controlled natural language support in information systems developed to support research, development and innovation in Republic of Moldova.

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SonaRes methodology enhancement using knowledge discovery technique

Iulian Secrieru

Abstract

The general goal of the research described in the article is to discover and extract "nontrivial" professional knowledge from the knowledge base and data sets, previously unknown and potentially useful for the development of intelligent informational systems. "Nontrivial" not in the proper sense of the professional reasoning logic for specific problem domain, but in understanding that this knowledge can be distinguished only through supervision of the actions or analyzing knowledge and data used in the decision making process of the experts in this area. The proposed new knowledge discovery technique was applied in the domain of medical ultrasound diagnostics.

Keywords: knowledge base, taxonomy, ontology, decision making process, knowledge discovery technique, SonaRes methodology.

1 Introduction

Professional knowledge in the domain of medical ultrasound examination (as a part of medical knowledge in general) can be divided into declarative and procedural. Declarative knowledge is professional statements that can be true or false. Procedural knowledge is the knowledge that describes the decision making process or explains the drawn conclusion.

At the knowledge acquisition stage the developers of the SonaRes methodology have decided that in the domain of ultrasound examination experts are the best source of knowledge. It has been also proved that the best strategy for the experts' knowledge acquisition in this area is direct involvement of the knowledge engineer [1].

The expert group was led by Dr. Turcanu Vasile and Dr. Puiu Sergiu, well-recognized in medical community physicians, and functions of the knowledge engineer were performed by Popcova Olga, a researcher at the IMCS.

At the formalization stage, declarative knowledge was represented as the fact base, and procedural knowledge – as the rule base.

Decision tree for the fact base and productions (symbolic semantics) for the rule base were chosen as models of knowledge representation.

Thus, the development team mentioned above for the first time has created a primary formalized knowledge base of the domain of medical ultrasound diagnostics. The fact base represented as decision tree is, in fact, the taxonomy of problem domain. The whole knowledge base, which provides possibility to develop intelligent information systems, is the first formalized ontology in this area [2].

The rule base and the fact base form the basis of the knowledge base of the SonaRes methodology. It is the object of analysis in this article. The main objective of this study is to identify new "nontrivial" professional knowledge using intellectual analysis of the SonaRes knowledge base.

2 SonaRes methodology and technology

SonaRes methodology is a comprehensive and integrated approach for design and development of clinical decision support systems. It provides clinicians with information support in all decision making stages, and currently has no analogy in the world.

SonaRes methodology combines new advanced methods for acquisition and management of medical professional knowledge with effective algorithms of ultrasound images processing.

SonaRes methodology:

- Offers principles to select the acquisition method and mode, corresponding to the application domain;

- Proposes an original alternative form for representation of the acquired knowledge;

- Gives new original methods and algorithms for inference, ultrasound images processing, quick search, encryption of personal data about patients, and creation of adaptive interfaces.

SonaRes technology provides effective algorithms for storage and documentation of specific cases, corresponding to normal/pathological states and anomalies of organs from the hepato-pancreato-biliary region, detected by ultrasound diagnostics. This technology allows in reasonable terms to formalize the same organs with another degree of particularization, other organs, or some other types of medical diagnostics.

Under the presented research, SonaRes methodology will be used to choose the method for representation of the new discovered knowledge [3] and SonaRes technology will be used to incorporate this knowledge into the kernel of the SonaRes knowledge base of four abdominal organs (gallbladder, pancreas, liver and bile ducts).

The SonaRes knowledge base, which will be examined to search for new "nontrivial" knowledge, includes the following data and expert knowledge:

- for gallbladder – 335 facts, 54 decision rules, 166 model images annotated by the expert group, 226 images with regions of interest (ROIs) marked;

- for pancreas – 231 facts, 52 decision rules, 106 model images, 137 images with ROIs marked;

- for liver – 167 facts, 31 decision rules, 87 model images, 111 images with ROIs marked;

- for bile ducts – 257 facts, 15 decision rules, 30 model images, 37 images with ROIs marked.

3 Knowledge discovery technique in domain of medical ultrasound diagnostics

Statistical and logic analysis form the basis of knowledge discovery technique [4]. Usually classification, modeling and predictions methods, used in domain of knowledge discovery, are based on decision trees, neural networks, genetic algorithms, fuzzy logic, associative memory, cognitive representation, etc.

As the main statistical methods used for new knowledge discovery we can mention the following: descriptive analysis, correlation and regression analysis, factor analysis, componential analysis, discriminant analysis, time series analysis, survival analysis, relational analysis, etc.

Given the specific of the knowledge base of the SonaRes methodology (existence of two different forms of its presentation and new knowledge obtained during the transition from the ontology representation into a cognitive modular matrix [3]), there were selected classification and modeling methods based on decision trees and cognitive representation, and correlation and relational analysis as statistical methods.

Use of logical analysis of the knowledge base of the SonaRes methodology represented as a decision tree and its cognitive form allowed us to reformulate specific pieces of some rules in order to improve inference. For example, for the knowledge base that describes gallbladder 133 fragments in 42 rules have been reformulated.

Use of correlation analysis and relational analysis of the knowledge base of the SonaRes methodology, represented as a cognitive modular matrix allowed us to formulate new hypotheses. For example, for the knowledge base that describes gallbladder 27 new hypotheses were formulated. Fig. 1 shows a fragment of the cognitive modular matrix, served as a basis for formulation of 2 hypotheses:

1. IF "Perivesicular area has modifications, the collection shape being striplike" THEN "The collection is homogeneous".

2. IF "Perivesicular area has modifications, the collection shape being abnormal" then "The collection is inhomogeneous".

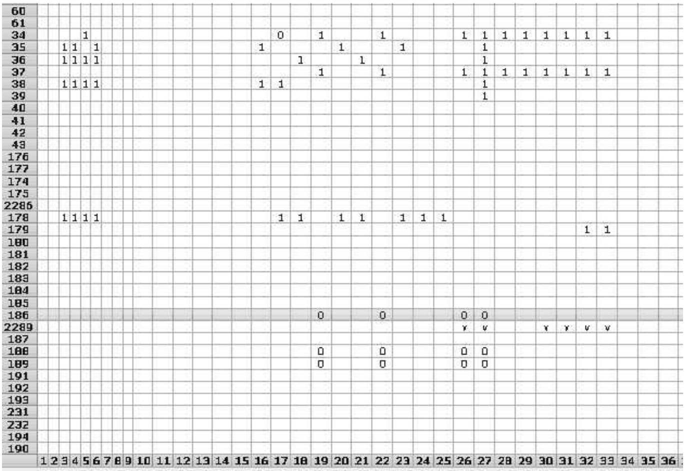

Figure 1. Fragment of the SonaRes knowledge base represented as a cognitive modular matrix

The identified hypothesis will be given to the expert group for validation and addition to the knowledge base as new "nontrivial" knowledge.

4 Conclusion and future work

Reformulation of specific fragments in some rules of the knowledge base of the SonaRes methodology allowed us to create a faster, simpler and clearer logical inference.

New "nontrivial" knowledge obtained in the form of hypotheses, being validated and accepted by the expert group, will increase the SonaRes knowledge base and will reduce the required number of queries to the user. It is extremely important when diagnostics is made in emergency situations.

The described results were obtained in semi-automatic mode. Currently, a computer-aided version of the used algorithms is under development.

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Opportunities for Object Detection using Haar Feature-based Cascade Classifier algorithm

Inga Titchiev, Mircea Petic, Grigorii Horos

Abstract

The main goal of this paper is the evaluation of the usual methods of processing and analysis of images for the purpose of object detection. Limitations of the processing and image analysis were determined using Haar Feature-based Cascade Classifier algorithm.

Keywords: Haar Feature-based Cascade Classifier, image processing, open CV.

1 Introduction

Image processing and analysis includes all the techniques and methods of acquisition, storage, display, modification and operation of visual information contained in images. In particular, image analysis refers to the ability to describe, understand and recognize scenes, scene objects and links between them. From the functional point of view, image analysis transforms the input image into a description.

Using information technologies for acquisition of the images (including medical one) leads to the increased quality of life by applying fast and effective treatment. Also to further increase the effect of their use the problem related to time optimization in the process of searching for an image (object) is being investigated. For this purpose Haar Feature-based Cascade Classifier algorithm will be used.

2 Haar Feature-based Cascade Classifier

Initially the object detector described below has been proposed by Paul Viola [1] and improved by Rainer Lienhart [2, 3]. For the purpose of objects detection there was developed an application that provides the following opportunities:

- Loading of a new image.
- Deleting of an image.
- Verification of the searched object. To perform this, the Haar Feature-based Cascade Classifier algorithm is used. This algorithm describes how to train and use a cascade of boosted classifiers for rapid object detection. The use of this algorithm implies several stages.
  - At the first stage a template file with specific features of searched object is created. This requires two types of sample images: negative samples and positive samples images.

    Negative samples correspond to non-object images. Positive samples correspond to object images. Negative samples are taken from arbitrary images. These images must not contain object representations.

    Positive samples are created by the create samples utility. They may be created from single object image or from collection of previously marked up images. For our application, template file [5] has been used.
  - The next stage after samples creation is training the classifier. It is performed by the haartraining utility. In the following, OpenCV cvHaarDetectObjects() function (in particular haarFaceDetect demo) is used for detection.
  - Also DetectandWrite() function is used to determine the number of found objects of the same type, it is returned as a result in a file containing the image description with .json extension.

3 Obtained results after applying the algorithm

After using this algorithm for a set of 200 images its efficiency is 98 percent.

There were detected the following problems that reduce its effectiveness and which require the improvements:

- there are not detected images that contain the sought object with a deviation of more than 30 percent.
- there are not detected very small objects.
- there are not detected images that contain black faces.

4 Future work

At the next step we want to use this algorithm in parallel [4] on multiple processes. For this, a set of images will be divided into multiple partitions in order to determine the time effectiveness of the Haar Feature-based Cascade Classifier algorithm application. In order to use multiprocessor advantages, a compiler that supports OpenMP 4.0 standard will be used.

5 Conclusion

Image processing is a current research direction, useful in various fields. In particular, we are interested in efficient methods and algorithms used in the processing of medical images obtained by ultrasound investigation.

In this paper the opportunities for object detection with Haar Feature-based Cascade Classifier algorithm were described. There were identified limitations of this algorithm. Further research will be directed to adapt this algorithm for the purpose of Parallel Object Detection.

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Metaheuristic Optimization in Image Processing

Milan Tuba

Abstract

Nondeterministic metaheuristic optimization and digital image processing are two very different research fields, both extremely active and applicable. They touch in a very limited area, but that narrow interaction opens new very promising applications for digital image processing and new and different deployment of metaheuristic optimization. Multilevel image thresholding is very important for image segmentation, which in turn is crucial for higher level image analysis, while JPEG quantization table selection is a newly proposed method that adjusts JPEG algorithm for many specific applications. Both problems include exponential combinatorial optimization with complex objective functions which are solvable only by nondeterministic methods. This lecture presents successful applications of the recent seeker optimization, firefly algorithm, bat algorithm and coocko search metaheuristics to multilevel thresholding and JPEG quantization table selection problems.

Keywords: digital image processing, multilevel thresholding, quantization matrix optimization, JPEG compression, swarm intelligence, metaheuristic optimization.

1 Introduction

Digital image processing is one of the most applicable research areas used in medicine, security, quality control, astronomy etc. It consists of very different techniques belonging to low level signal processing, medium level morphological processing and segmentation for feature detection and high level artificial intelligence algorithms for object recognition, information extraction, representation and understanding. At different stages of image processing some hard optimization problems occur. For example, multilevel image thresholding is a step in segmentation, but even though this problem at first sight seems to be simple, to determine optimal n numbers in the range [0-255] is NP-hard combinatorial problem. JPEG quantization matrix optimization is also an exponential combinatorial problem since the number of coefficients, after DCT, is 64 and each can be an integer between 0 and 255.

Such problems cannot be solved in reasonable time by standard mathematical deterministic methods. Nature inspired metaheuristic algorithms have recently been successfully used for this type of hard optimization problems. They try to guide random Monte-Carlo search by simulating some successful systems from the nature. Swarm intelligence is an important branch of nature inspired algorithms where collective intelligence of different species like ants, bees, cuckoos, fireflies, bats, fish, birds, krill etc. is simulated.

2 Multilevel thresholding

The criteria for selecting thresholds can be different, but most often used are Kapur's entropy criterion based on elements of the form $H_k = -\sum_{i=t_k}^{t_{k+1}-1} \frac{P_i}{w_k} \ln \frac{P_i}{w_k},\ where\ w_k = \sum_{i=t_k}^{t_{k+1}-1} P_i$ and between-class variance Otsu's criterion based on the elements of the form $\sigma_k = w_k(\mu_k - \mu_t)^2,\ \mu_k = \sum_{i=t_k}^{t_{k+1}-1} \frac{iP_i}{w_k}$, where $\mu_t = \sum_{i=0}^{L-1} iP_i$ is the total mean intensity of the original image.

Both, Kapur's and Otsu's criteria were used and exhaustive search was performed for 2, 3, 4, and 5 thresholds. For 5 thresholds exhaustive search required almost an hour of computational time and for each new threshold that time increases 255-fold so for 6 thresholds the computation would last one week, for 7 threshods 5 years etc. In [1] seeker optimization algorithm was used on 4 standard benchmark images, while in [2] cuckoo search and firefly are tested on 6 images. To the same 6 images bat algorithm was applied in [3]. All these swarm intelligence algorithms required less that 0.1 sec of computational time for 5 thresholds and could easily be applied to any size problem, with result almost always equal to results of exhaustive search i.e. global optimum. The mechanism for selecting coefficients that will give desired compression ratio includes variable number of non-zero coefficients, but with the fixed total length of bits for representation.

3 JPEG quantization

For different applications various measures of compression quality or image similarity can be devised. However, for many applications simple metrics like sum of the squares of differences (or sum of absolute values of differences) of intensities of the corresponding pixels can be used since in the compression and decompression there is no spatial movement. Firefly algorithm was applied to quantization matrix optimization problem in [4]. Extremely complex objective function includes for each generated candidate matrix inverse DCT and distance to the original image calculation. JPEQ recommended Q_{10} table was used for considerable compression of the benchmark image. Measure of the similarity of images was 5.9 (average distance). Firefly algorithm optimization generated table that achieved distance of 5.1 which is equivalent to Q_{18} standard JPEG table, with apparently better looking image.

4 Conclusion

In this lecture successful application of swarm intelligence metaheuristics to hard optimization problems that occur in image processing was illustrated by seeker optimization algorithm, cuckoo search, firefly algorithm and bat algorithm for multilevel thresholding and firefly algorithm for JPEG quantization table optimization.

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Intelligent Robust Control System based on Quantum KB-Self-organization: Quantum Soft Computing and Kansei / Affective Engineering Technologies

Sergey Ulyanov, Veacheslav Albu, Irina Barchatova

Abstract

New results in robust intelligent cognitive control are introduced based on unconventional computational intelligence as quantum soft computing technology. Synergetic effect of integrated IT of Kansei / Affective and System of Systems Engineering as intelligent cognitive robust control on Benchmarks is considered. An example of designing integrated fuzzy intelligent control systems (IFICS) in unpredicted situations using Kansei / Affective Engineering is described. The background of applied unconventional computational intelligence is soft and quantum computing technologies.

Keywords: Intelligent robust control, Kansei / Affective engineering, toolkit of quantum computational intelligence, quantum soft computing, quantum fuzzy inference.

1 Introduction

This report presents an example of designing integrated fuzzy intelligent control systems (IFICS) in unpredicted situations using hybrid technology of computational intelligence, cognitive processes and Kansei / Affective Engineering. The background of applied unconventional computational intelligence is soft and quantum computing technologies. All researches are supported by relevant publications and patents (see http: //www.qcoptimizer.com/) [1–7].

New approach to cognitive intelligent robust fuzzy control that includes the human factor risk is considered. With the developed toolkit it can design intelligent control systems that guarantee the goal control achievement in unpredicted control situations.

2 IT design of IFICS and Kansei / Affective Engineering toolkit

Design processes of IFICS in unpredicted situations were constructed of two approaches [3–5]:

- system of systems engineering technology describes the possibility of complex ill-defined (autonomous or hierarchically connected) dynamic control system's design that includes human decision making and risk factors in unpredicted (unforeseen) control situations;
- Kansei / Affective Engineering technology and its toolkit include qualitative description of human being emotion, instinct and intuition that are used effectively in design processes of smart / wise cognitive robotics and intelligent mechatronics.

Kansei process gathers the functions related to emotions, sensitivity, feelings, experience, intuition (i.e. sensory qualities related functions (Clark 1996)), including interactions between them; Kansei means are all the senses (sight, hearing, taste, smell, touch, balance, recognition...) and – probably – other internal factors (such as personality, mood, experience, and so on); Kansei result is the fruit of Kansei process (i.e. of these function processes and of their interactions) [8, 9].

Therefore, Kansei result is a synthesis of sensory brain cognitive qualities. For example, it has been argued that emotion, pain and cognitive control are functionally segregated in distinct subdivisions of the cingulate cortex of brain. However, recent observations encourage a fundamentally different view [10]. In humans and other primates, the cingulate – a thick belt of cortex encircling the corpus callosum – is one of the most prominent features on the mesial surface of the brain [11]. Early research suggested that the rostral cingulate cortex (Brodmann's "precingulate"; architectonic areas) plays a key part in affect and motivation [12].

The presence of typically quantum effects, namely superposition and interference, in what happens when human concepts are combined, provide a quantum model in complex Hilbert space that represents faithfully experimental data measuring the situation of combining concepts [13].

We considered the humanized technology of intelligent robotic systems design based on Kansei / Affective Engineering and System of Systems Engineering using Quantum / Soft Computing as unconventional computational intelligence toolkit. As it is well known, the subject of humanized technology or human-related systems has been actively researched. With the increasing concern regarding human factors in system development Kansei Engineering and Soft Computing are the most representative research fields on this subject [10]. Soft computing toolkit is developed for emotion, instinct, and intuition recognition and expression generation [14, 15]. In particular with genetic algorithm – GA – (as effective random search of solution) an intuition process (optimization) is modeled. Fuzzy neural network (FNN) is used for description of instinct process (adaptation and learning) that modeled approximation of optimal solution in unpredicted control situation. Fuzzy logic control is used for design of an emotion according to corresponding designed look-up table [14, 16, 17]. Quantum control algorithm of self-organization is the background of wise robotic control system's design. Quantum computing toolkit is used for increasing of robustness in intelligent control systems (especially for unpredicted control situations) [4, 5, 18].

The basis for the implementation of this idea is the research result that opened the new principle: *self-organization with minimization of generalized entropy production* (as the new physical measure of control quality) [3, 4]. Self-organization is a central coordination mechanism exhibited by both natural and artificial collective social-technical systems. Self-organized mechanisms are characterized by nonlinear responses to stimulus intensity, incomplete information, and randomness. Self-organization coexists with guidance from environmental templates, networks of interactions among individuals, and various forms of leadership or preexisting individual specialization. A general characteristic of self-organizing systems is as follows: they are robust or resilient. This means that they are relatively insensitive to perturbations or errors, and have a strong capacity to restore themselves, unlike most human designed systems [4]. One reason for this fault-tolerance is the redundant, distributed organization: the non-damaged regions can usually make up for the damaged ones. Another reason for this intrinsic robustness is that self-organization thrives on randomness, fluctuations or "noise". A certain amount of random perturbations will facilitate rather than hinder self-organization. A third reason for resilience is the stabilizing effect of feedback loops.

Models of self-organization included natural quantum effects and based on the following information-thermodynamic concepts: (i) macro- and micro-level interactions with information exchange (in agent based model (ABM) micro-level is the communication space where the inter-agent messages exchange and are explained by increased entropy on a micro-level); (ii) communication and information transport on micro-level ("quantum mirage" in quantum corrals); (iii) different types of quantum spin correlation that design different structure in self-organization (quantum dot); (iv) coordination control (swam-bot and snake-bot).

Quantum control algorithm of self-organization is based on quantum fuzzy inference QFI model [4]. QFI includes these concepts of self-organization and has been realized by corresponding quantum operators, and can be considered as quantum algorithmic gate [6, 7].

Structure of QFI that realize the self-organization process is developed on the corresponding quantum algorithmic gate [7]. QFI is one of possible realizations of quantum control algorithm of self-organization that includes all of these features: (i) superposition; (ii) selection of quantum correlation types; (iii) information transport and quantum oracle; and (iv) interference.

With superposition the templating operation is realized. Physically, the quantum operation as the templating one is based on macro- and micro-level interactions with information exchange of active agents. The power source of communication and information transport on micro-level is used for selection of quantum correlation type of operation self-assembling, and this is realized based on quantum genetic algorithm toolkit [2]. In this case the type of correlation defines the level of robustness in designed KB of fuzzy controller (FC). Quantum oracle calculates "intelligent quantum state" that includes the most important (value) information transport for coordination control. Interference is used for extraction the results of coordination control and design in on-line robust knowledge base (KB) [4].

The developed QA of self-organization is applied to the design of robust KB of FC in unpredicted control situations. Main goal of quantum control algorithm of self-organization is the support of optimal thermodynamic trade-off between stability, controllability and robustness of control object behavior using robust self-organized KB of intelligent control system [3]. Main operations of developed QA and concrete examples of QFI applications are described in [4, 5].

Information design technology of robust IFICS includes two steps: 1) step 1 based on soft computing optimizer (SCO); and 2) step 2 based on quantum computing optimizer (QCO). Main problem in this technology is the design of robust KB of FC that can include the self-organization of KB in unpredicted control situations. The background of this design processes is KB optimizers based on quantum / soft computing technologies [3–5].

Fig. 1 contains factors that define the control situation and shows the structure of robust intelligent control, consisting of two (or more) fuzzy PID (proportional- integral- differential) controllers (FC PID) and block QFI implementing property of KB self-organization. QFI model uses private individual KB of FC, each of which is obtained by the toolbox "SCO" for fixed (standard) control situations in the external random environment.

The structure of ICS, that is presented in Fig. 1, shows how the self-organization principle is realized on the base of QFI model, and includes the support of the thermodynamic trade-off relations between stability, controllability, robustness properties (Eq. 2). The kernel of the abovementioned FC design toolkit is a SCO implementing advanced soft computing ideas. SCO is considered as a new flexible tool for design of optimal structure and robust KBs of FC based on a chain of genetic algorithms (GAs) with information-thermodynamic criteria for KB optimization and advanced error BP-algorithm for KB refinement. Some measured or simulated data (called as 'teaching signal" (TS)) about the modelling system [3] can be as the input to SCO.

The functional structure model of QFI in Fig. 2 describes the algorithm for coding, searching and extracting the value information from two KB's of fuzzy PID controllers designed by SCO. Applying the QFI in IFICS's structure, additional (hidden) quantum information is extracted and used to design a robust control signals on-line from responses FC that are received in unpredicted situations. Different quantum controllers based on different fuzzy controllers (controllers with different types of the correlation, controllers with different KB (two, three, four, etc.), cognitive controllers etc.) can be designed and tested by QCO.

After testing, we choose the best robust controller for using in real applications.

Concrete industrial Benchmarks (as "cart–pole" system, robotic unicycle, robotic motorcycle, mobile robot for service use, semi-active car suspension system etc.) are tested successfully with the developed design technology [19–22].

We demonstrate the efficiency of application of QFI by the Benchmark.

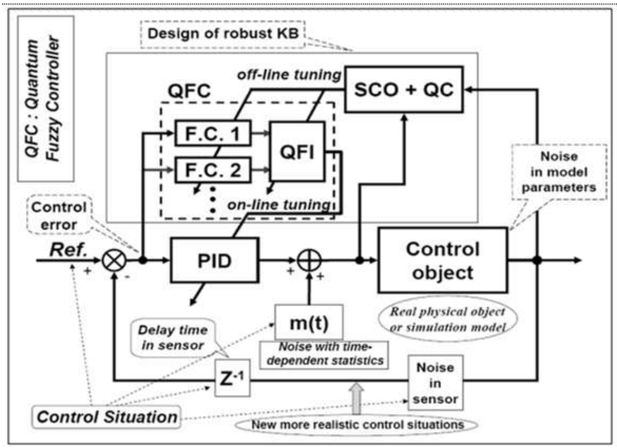


Figure 1. Structure of robust intelligent control system (ICS) in unpredicted control situations

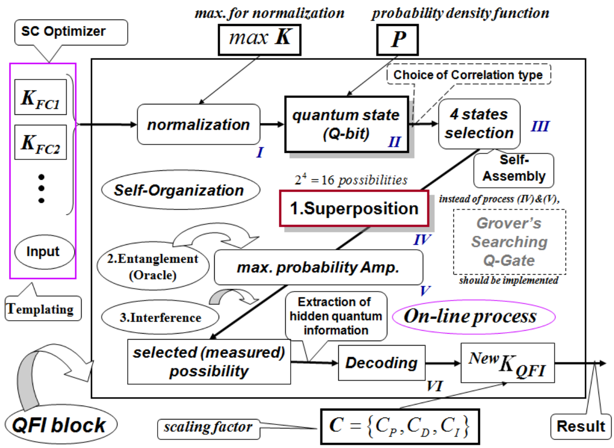
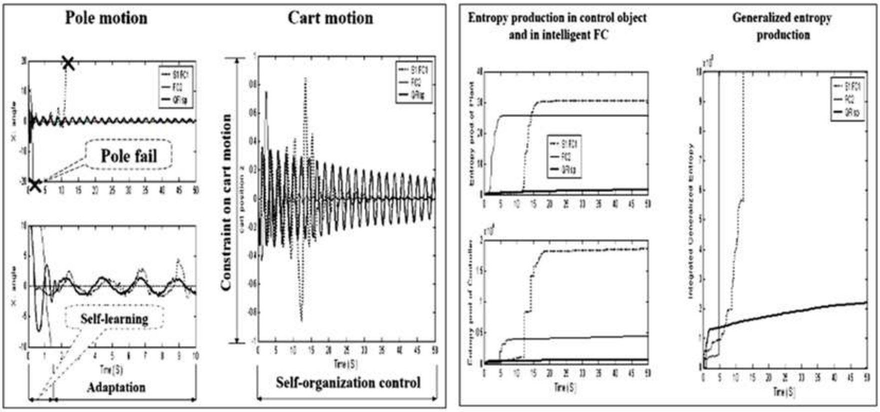


Figure 2. Functional structure model of QFI

3 Examples

3.1 Benchmark of QFI-application: "Cart–Pole system"

Let us consider fuzzy robust control problem of "cart-pole" system as intelligent control Benchmark. This system, as well known, is described by the equation of motion [3, 4]. KB of FC is designed by SCO using Gaussian and Rayleigh noises respectively.



(a) (b)

Figure 3. Dynamic behavior of "cart–pole" system (a); and Thermodynamic behavior of "cart–pole" system (b)

Fig. 3a shows the dynamic behavior of the system in unpredicted control situation. In this case a new time delay in the structure (see Fig. 3a and Fig. 2) in sensor is 0.002 sec; parametric Gaussian noise is with the amplitude 0.01; new initial state $\left[\theta_0, \dot{\theta_0}\right] = [13, 1]\,(deg)$, $[z_0, \dot{z_0}] = [0, 0]$. External noise is Raleigh noise as in the learning situation. Fig. 5b shows the thermodynamic behavior of the system and of FC. Fig. 3b shows that generalized entropy production of the system "control object + fuzzy PID-controller" is minimal (minimum consumption of useful source and power), and with quantum self-organization of KB the required trade-off distribution between stability, controllability and robustness is achieved.

It is a pure quantum effect, and it does not have classical analogy. Thus results of simulation show that winner is quantum fuzzy controller (QFC) designed from two KB controllers with minimum of generalized entropy production. Results of simulations in Fig. 3 show also that from two unstable FC it is possible to design on-line a new robust stable FC (Parrondo paradox).

Therefore, QFI strongly supports optimal thermodynamic trade-off between stability, controllability and robustness in self-organization process (from viewpoint of physical background of global robustness in intelligent control systems).

It is also important the new result for advanced control system: the designed block QFC increases robustness of system control, but all other controllers (FC1, FC2) failed to do this.

3.2 QFI application in cognitive and intelligent FC

Figs 4 and 5 show other situation: control system includes the human factor.

Blok QFI in Fig. 3 includes intelligent FC (FC1) and fuzzy cognitive controller (FCC2). In this case, KB of FC has been designed for the surface roughness of the carriage of Gaussian type. KB of FCC was also adopted, as for FC, but with the ability to change productional rules of output signal proportional gain when a human operator observes the change in the type of surface displacement movement of the carriage.

Unpredicted situation of control system, when roughness of surface obstacles of the carriage with a Gaussian distribution type obstacles comes to the surface with a uniform probability distribution of obstacles, is shown in Fig. 4. In this case roughness of surface obstacles of the carriage has time dependent probability density function. The simulation results show (see Figs. 4 and 5) that the robustness of FC is the loss of control after 30 seconds (after the change of the type of surface). And FCC lost robustness under the same conditions after 80 sec., although time interval robustness of ICS significantly increased compared to FC.

KB quantum PID was designed on-line from the responses of no-robust FC and FCC due to quantum self-organization. Thus the robustness of ICS is increased. In this case all the properties of intelligent and cognitive control systems have been retained.

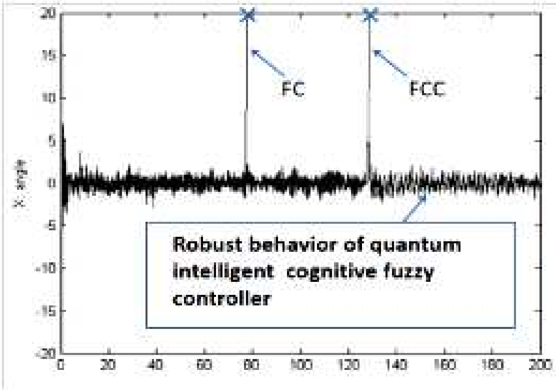


Figure 4. Simulation results of "cart–pole" system

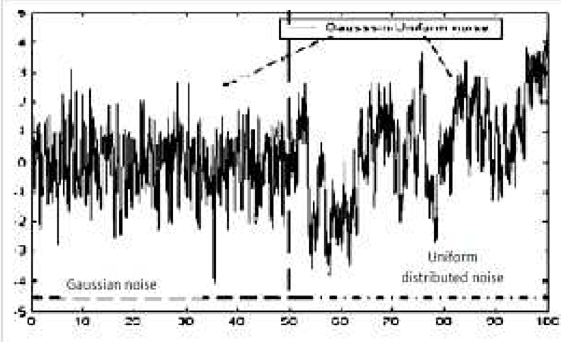


Figure 5. Noise with time dependent probability density function

This approach was applied to other complex commercial industrial robotic systems [19–22].

4 Conclusions

These examples show the possible using of new types of unconventional computational intelligence and quantum algorithm of KBs self-organization, which allows on-line control to achieve the goal in unpredicted situations by improving the robustness of the IFICS in problem-oriented fields. The background of applied unconventional computational intelligence is soft and quantum computing technologies. Creation and implementation of software products (using new types of intelligent computing and software and hardware support, knowledge extraction algorithms, processing and generation of knowledge in intelligent control of quantum nano-technology, etc.) is a special kind of knowledge, which can be considered as a separate item.

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