BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(99), 2022, Pages 76–102
ISSN 1024–7696, E-ISSN 2587–4322

Limits of solutions to the semilinear plate equation with small parameter

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Abstract. We study the existence of the limits of solutions to the semilinear plate equation with boundary Dirichlet condition with a small parameter coefficient of the second order derivative in time. We establish the convergence of solutions to the perturbed problem and their derivatives in spacial variables to the corresponding solutions to the unperturbed problem as the small parameter tends to zero.

Mathematics subject classification: 35B25, 35K15, 35L15, 34G10.
Keywords and phrases: a priory estimate, boundary layer, semilinear plate equation, singular perturbation.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with the smooth boundary $\partial\Omega$. Consider the following initial boundary value problem for the plate equation:

$$\begin{cases} \varepsilon\, u_{tt}(x,t) + u_t(x,t) + \Delta^2 u(x,t) + B(u(t)) = f(x,t), \;\; (x,t) \in \Omega \times (0,T), \\ u\big|_{t=0} = u_0(x), \quad u_t\big|_{t=0} = u_1(x), \quad x \in \Omega, \\ u\big|_{x\in\partial\Omega} = \dfrac{\partial u}{\partial \bar{\nu}}\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{P_ε}$$

where $\bar{\nu}$ is the outer normal vector to $\partial\Omega$ and ε is a small positive parameter.

We study the behaviour of the solutions to the problem (P_ε) as $\varepsilon \to 0$. It is natural to expect that the solutions to the problem (P_ε) tend to the corresponding solutions to the following unperturbed problem:

$$\begin{cases} v_t(x,t) + \Delta^2 v(x,t) + B(v(t)) = f(x,t), \;\; (x,t) \in \Omega \times (0,T), \\ v\big|_{t=0} = u_0(x), \quad x \in \Omega, \\ v\big|_{x\in\partial\Omega} = \dfrac{\partial v}{\partial \bar{\nu}}\Big|_{x\in\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{P_0}$$

as $\varepsilon \to 0$.

We investigate two cases: the first case when the operator B is Lipschitzian and the second case when the operator B is monotone.

DOI: https://doi.org/10.56415/basm.y2022.i2.p76

The main results are contained in Theorems 8 and 9. Under some conditions on u_0, u_1 and f we prove that

$$u \to v \quad \text{in} \quad C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0. \tag{1}$$

This means that the perturbation (P_ε) of the system (P_0) is regular in the indicated norms. At the same time, we prove that

$$u' - v' - \alpha\, e^{-t/\varepsilon} \to 0 \text{ in } C([0,T]; L^2(\Omega)) \cap L^\infty(0,T; H^2(\Omega)),\ \alpha \neq 0,\ \text{as } \varepsilon \to 0. \tag{2}$$

It means that the derivatives of the solutions to the problem (P_ε) do not converge to the derivatives of the corresponding solutions to the problem (P_0), as $\varepsilon \to 0$. The relation (2) shows that the derivative u' has a singular behaviour, as $\varepsilon \to 0$, in the neighborhood of $t = 0$. This singular behaviour is determined by the function $\alpha\, e^{-\tau/\varepsilon}$, which is *the boundary layer function* and the neighborhood of $t = 0$ is *the boundary layer* for u'.

The proofs of the relations (1) and (2) are based on two key points. The first one is the relationship between the solutions to the problem (P_0) and (P_ε) in the linear case (see Lemma 3 and Theorem 7). The second key point is the *a priori* estimates of the solutions to the problem (P_ε), which are uniform relative to the small parameter ε (see Lemmas 1 and 2).

The singularly perturbed nonlinear problems of hyperbolic-parabolic type were studied by many authors. Without pretending to a complete list of the papers in this area, we mention the works [4–11] containing a wide list of references.

In what follows, we use some notations. For $m \in [1,\infty)$ denote by

$$L^m(\Omega) = \{f : \text{a.e.}\, \Omega \to \mathbb{C};\ \int_\Omega |f(x)|^m\, dx < \infty\},$$

the Banach space, endowed with the norm

$$||f||_{L^m(\Omega)} = \Big(\int_\Omega |f(x)|^m\, dx\Big)^{1/m}$$

and for $m = \infty$ denote by

$$L^\infty(\Omega) = \{f : \text{a.e.}\, \Omega \to \mathbb{C};\ \text{ess sup}_\Omega\, |f(x)| < \infty\}$$

the Banach space, endowed with the norm

$$||f||_{L^\infty(\Omega)} = \text{ess sup}_\Omega\, |f(x)|.$$

By $L^m_{loc}(\Omega)$ denote the space of integrable functions on each compact $K \subset\subset \Omega$. Denote by $W^{l,m}(\Omega)$ the Banach space of all elements of $L^m(\Omega)$ whose derivatives $\partial^\alpha u$ in the sense of distributions up to the order l belong to $L^m(\Omega)$. The norm in $W^{l,m}(\Omega)$ is defined as

$$||u||_{W^{l,m}(\Omega)} = \Big(\int_\Omega \sum_{|\alpha|\le l} |\partial^\alpha u|^m\, dx\Big)^{1/m}.$$

By $W^{l,m}_{loc}(\Omega)$ denote the local Sobolev space, i.e. a function $u \in W^{l,m}_{loc}(\Omega)$ if $u \in W^{l,m}(K)$ for every compact $K \subset\subset \Omega$.

For $k \in \mathbb{N}$ we denote by $H^k(\Omega)$ ($H^0(\Omega) := L^2(\Omega)$) the usual real Hilbert spaces equipped with the following scalar product and norm:

$$\Big(u, v\Big)_{H^k(\Omega)} = \int\limits_\Omega \sum_{|\alpha|\leq k} \partial^\alpha u(x) \partial^\alpha v(x) dx, \quad ||u||_{H^k(\Omega)} = \Big(\int\limits_\Omega \sum_{|\alpha|\leq k} |\partial^\alpha u(x)|^2 \, dx\Big)^{1/2}.$$

Denote by $H^k_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in the norm of the space $H^k(\Omega)$. By $H^{-k}(\Omega)$ denote the dual space of $H^k_0(\Omega)$, i.e. $H^{-k}(\Omega) = (H^k_0(\Omega))'$.

Denote by V the space $V = \{u \in H^2(\Omega); u|_{\partial\Omega} = \dfrac{\partial u}{\partial \bar\nu}|_{\partial\Omega} = 0\}$, endowed with the norm of the space $H^2(\Omega)$, and by V' the dual space of the space V. We will write $\langle \cdot, \cdot \rangle$ to denote the pairing between V' and V. Also denote by

$$(u, v) = \int\limits_\Omega u(x) v(x) dx, \quad |u| = ||u||_{L^2(\Omega)}, \quad ||u|| = ||u||_{H^2(\Omega)}.$$

Let X be a Banach space. For $k \in \mathbb{N}$, $p \in [1, \infty)$ and $(a, b) \subset (-\infty, +\infty)$ we denote by $W^{k,p}(a, b; X)$ the usual Sobolev space of the vectorial distributions $W^{k,p}(a, b; X) = \{f \in D'(a, b, X); f^{(l)} \in L^p(a, b; X), l = 0, 1, \dots, k\}$ equipped with the norm

$$||f||_{W^{k,p}(a,b;X)} = \Big(\sum_{l=0}^k ||f^{(l)}||^p_{L^p(a,b;X)}\Big)^{1/p}.$$

For each $k \in \mathbb{N}, W^{k,\infty}(a, b; X)$ is the Banach space equipped with the norm

$$||f||_{W^{k,\infty}(a,b;X)} = \max_{0\leq l\leq k} ||f^{(l)}||_{L^\infty(a,b;X)}.$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and $p \in [1, \infty]$ we also denote by

$$W^{k,p}_s(a, b; H) = \{f : (a, b) \mapsto H; f^{(l)}(\cdot)\, e^{s\,t} \in L^p(a, b; X), l = 0, \dots, k\}$$

the Banach space, endowed with norms $||f||_{W^{k,p}_s}(a, b; X) = ||f e^{s\,t}||_{W^{k,p}(a,b;X)}$.

2 Solvability of the problems (P_ε) and (P_0)

The framework of our investigations will be determined by the following conditions:

(B1) *The operator* $B : D(B) \subseteq L^2(\Omega) \mapsto L^2(\Omega)$ *verifies the condition:* $V \subset D(D)$ *and there exists a constant* $L > 0$ *such that*

$$|B(u_1) - B(u_2)| \leq L\, ||u_1 - u_2||_{H^2(\Omega)}, \quad \forall u_1, u_2 \in V;$$

(B2) *The operator* B *possesses the Fréchet derivative* B' *in* V*, so that there exist some constants* $L_0 \geq 0$ *and* $L_1 \geq 0$ *such that*

$$\big|\big(B'(u_1) - B'(u_2)\big)\, v\big| \leq L_1\, ||u_1 - u_2||_{H^2(\Omega)}\, ||v||_{H^2(\Omega)}, \quad \forall u_1, u_2, v \in V,$$

$$\big|B'(u)\,v\big| \le L_0\,|v|, \quad \forall u \in V, \quad \forall v \in L^2(\Omega);$$

(B3) *The operator $B : D(B) \subseteq L^2(\Omega) \to L^2(\Omega)$ is $H^2(\Omega)$ local lipschitzian, i.e. $V \subset D(B)$ and for every $R > 0$ there exists $L(R) \ge 0$ such that*

$$|B(u_1) - B(u_2)| \le L(R)\,||u_1 - u_2||_{H^2(\Omega)}, \quad \forall u_i \in V,\ ||u_i||_{H^2(\Omega)} \le R,\ i = 1, 2,$$

and B is Fréchet derivative of some convex and positive functional $\mathcal{B}$ with $V \subset D(\mathcal{B})$.

The hypothesis that operator B is Fréchet derivative of some convex and positive functional implies, in particular, that the operator B is monotone and verifies the condition

$$\frac{d}{dt}\mathcal{B}(u(t)) = \big(B(u(t)), u'(t)\big), \quad t \in [a, b] \subset \mathbb{R},$$

for $u \in C([a, b], V) \cap C^1([a, b], L^2(\Omega))$ (see [13]).

(B4) *The operator B possesses the Fréchet derivative B' in V and for every $R > 0$ there exists a constant $L_1(R) \ge 0$ such that*

$$\big|\big(B'(u_1) - B'(u_2)\big)v\big| \le L_1(R)\,||u_1 - u_2||_{H^2(\Omega)}\,||v||_{H^2(\Omega)}, \quad \forall u_1,\, u_2,\, v \in V,$$
$$||u_i||_{H^2(\Omega)} \le R, \quad i = 1, 2.$$

Firstly we remind the definitions of solutions to the problems (P_ε) and (P_0) and the existence theorems for solutions to the considered problems.

Definition 1. *Let $T > 0$, $f \in L^2(0, T; V')$ and $B : D(B) \subseteq L^2(\Omega) \to V'$. A function $u \in L^2(0, T; V \bigcap D(B))$ with $u' \in L^2(0, T; L^2(\Omega))$ and $u'' \in L^2(0, T : V')$ is called solution to the problem (P_ε) if u satisfies the equality*

$$\begin{cases} \varepsilon\langle u''(t), \eta\rangle + (u'(t), \eta) + \big(\Delta u(t), \Delta\eta\big) + (B\big(u(t)\big), \eta) = (f(t), \eta), \\ \forall \eta \in V,\ \text{a.e. } t \in [0, T], \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{3}$$

Definition 2. *Let $T > 0$, $f \in L^2(0, T; V')$ and $B : D(B) \subseteq L^2(\Omega) \to V'$. A function $v \in L^2(0, T; V \cap D(B))$ with $v' \in L^2(0, T; V')$ is called solution to the problem (P_0) if v satisfies the equality*

$$\begin{cases} \langle v'(t), \eta\rangle + \big(\Delta v(t), \Delta\eta\big) + (B\big(v(t)\big), \eta) = (f(t), \eta), \forall \eta \in V, \quad \text{a.e. } t \in [0, T], \\ v(0) = u_0. \end{cases} \tag{4}$$

Remark 1. *For $u \in L^2(0, T; V), u' \in L^2(0, T; L^2(\Omega))$, and $u'' \in L^2(0, T; V')$ it follows that $u \in C([0, T]; L^2(\Omega))$ and $u' \in C([0, T]; V')$. Consequently, the initial conditions from (3) are understood in the following sense:*

$$|u(t) - u_0| \to 0, \quad ||u'(t) - u_1||_{V'} \to 0, \ \textit{as}\ t \to 0.$$

Similarly, for $v \in L^2(0,T;V)$ *with* $v' \in L^2(0,T:V')$, *it follows that* $v \in C([0,T];V)$, *consequently, the initial conditions from* (4) *are understood in the following sense* $|v(t)-u_0| \to 0$ *as* $t \to 0$.

Using the methods developed in [2] and [3], in [8] the following theorems are proved.

Theorem 1. *Let* $T > 0$. *Suppose that condition* **(B1)** *is fulfilled. If* $u_0 \in H^4(\Omega) \cap V$, $u_1 \in V$, *and* $f \in W^{1,1}(0,T;L^2(\Omega))$ *then there exits a unique solution to the problem* (P_ε) *such that* $u \in W^{2,\infty}(0,T;L^2(\Omega))$, $\Delta\, u \in W^{1,\infty}(0,T;L^2(\Omega))$, $\Delta^2 u \in L^\infty(0,T;L^2(\Omega))$.

The function $t \in [0,T) \mapsto u'(t) \in L^2(\Omega)$ *is derivable to the right and the equalitiy*

$$\frac{d^+u'}{dt}(t) = f(t_0) - \Delta^2 u(t) - B(u(t)) - u'(t), \quad t \in [0,T),$$

is true. The function $t \in [0,T] \mapsto \Delta^2 u(t)$ *is weakly continuous in* $L^2(\Omega)$ *and the equality*

$$\frac{d}{dt}\big(\Delta^2 u(t), u(t)\big) = 2\left(\Delta^2 u(t), u'(t)\right), \quad t \in [0,T),$$

is true.

If, in addition, $u_1 \in H^4(\Omega) \cap V$, $f(0) - B(u_0) - \Delta^2 u_0 - u_1 \in V$, $f \in W^{2,1}(0,T;L^2(\Omega))$ *and condition* **(B2)** *is fulfilled, then* $u \in W^{3,\infty}(0,T;L^2(\Omega))$ *and* $\Delta u \in W^{2,\infty}(0,T;L^2(\Omega))$.

Theorem 2. *Let* $T > 0$. *Suppose that condition* **(B3)** *is fulfilled. If* $u_0 \in H^4(\Omega) \cap V, u_1 \in V$ *and* $f \in W^{1,1}(0,T;L^2(\Omega)$, *then there exists a unique solution to the problem* (P_ε) *such that* $u \in C^2([0,T];L^2(\Omega))$, $u' \in C^1([0,T];V)$, $\Delta^2 u \in C([0,T];L^2(\Omega))$.

If, in addition, $u_1 \in H^4(\Omega) \cap V$, $f(0) - B(u_0) - \Delta^2 u_0 - u_1 \in V$, $f \in W^{2,1}(0,T;L^2(\Omega))$ *and condition* **(B4)** *is fulfilled, then* $u \in W^{3,\infty}(0,T;L^2(\Omega))$, $\Delta u \in W^{2,\infty}(0,T;L^2(\Omega))$.

Theorem 3. *Let* $T > 0$. *Suppose that condition* **(B1)** *is fulfilled. If* $u_0 \in H^4(\Omega) \cap V$ *and* $f \in W^{1,1}(0,T;L^2(\Omega))$, *then there exits a unique solution to the problem* (P_0). *The function* $t \in [0,T) \mapsto v(t) \in L^2(\Omega)$ *is derivable to the right, verifies the equality*

$$\frac{d^+v}{dt}(t) + \Delta^2 v(t) + B(v(t)) = f(t), \quad t \in [0,T),$$

and the estimates

$$||v(t)||_{C([0,t];L^2(\Omega))} + ||v||_{L^2(0,t;V)} + ||v'||_{L^\infty(0,t;L^2(\Omega))} + ||v'||_{L^2(0,t;V)} \leq$$

$$\leq C\, \widetilde{M}_0(t)\, e^{\gamma\, t}, \ \forall t \in [0,T],$$

are true with C *and* γ *depending on* L, n, Ω, *and*

$$\widetilde{M}_0(t) = |u_0| + |B(u_0)| + |\Delta^2 u_0| + ||f||_{W^{1,2}(0,t;L^2(\Omega))}.$$

Remark 2. *In the conditions of Theorem 3,* $v \in C([0,T];L^2(\Omega))$, $v' \in L^\infty(0,T;L^2(\Omega))$, *the term* $\langle v'(t),\eta\rangle$ *in* (4) *can be expressed in the form* $(v'(t),\eta)$.

Theorem 4. *Let* $T>0$. *Suppose that condition* **(B3)** *is fulfilled. If* $u_0 \in H^4(\Omega)\cap V$ *and* $f \in W^{1,1}(0,T;L^2(\Omega))$, *then there exists a unique solution to the problem* (P_0) *such that* $v \in C^1([0,T];L^2(\Omega))\cap C([0,T];V)$ *and the following estimates*

$$||v||_{C^1([0,t];L^2(\Omega))} + ||v||_{C([0,t];V)} + ||v'||_{L^2(0,t;V)} \le C\,\widetilde{M}_1(t), \quad \forall t\in[0,T],$$

hold, where $\widetilde{M}_1(t) = |u_0| + |\Delta^2 u_0| + ||f||_{W^{1,1}(0,t;H)} + |B(0)|\,t$.

3 A priori estimates for the solutions to the problem (P_ε)

In this section we prove some *a priori* estimates for the solutions to the problem (P_ε), which are uniform relative to the small values of the parameter ε.

Firstly we remind the following theorems.

Theorem 5. [14] *Let* $\Omega\subset\mathbb{R}^n$ *be an open and bounded set with the compact boundary of class* C^2. *If* $u,\Delta u\in L^2(\Omega)$, *then* $u\in H^2(\Omega)$ *and there exists a constant* $C_0(n,\Omega)$ *such that*

$$||u||_{H^2(\Omega)} \le C_0\big(||\Delta u||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}\big). \tag{5}$$

Theorem 6. [1] *Let* $\Omega\subset\mathbb{R}^n$ *be an open and bounded set. For* $n>m\,l$ *if* $q\le \dfrac{m\,n}{n-m\,l}$ *and for* $n=m\,l$, $\forall q$, *the following inequality*

$$||u||_{L^q(\Omega)} \le C(q,m,n,\Omega)\,||u||_{W^{l,m}(\Omega)}, \quad \forall u\in W^{l,m}(\Omega)$$

is true.

For $n<m\,l$ *we have*

$$\max_{x\in\overline{\Omega}} |u(x)| \le C(q,m,n,\Omega)\,||u||_{W^{l,m}(\Omega)}, \quad \forall u\in W^{l,m}(\Omega).$$

In what follows, denote by $u(t)=u(t,\cdot)$, $u'(t)=u_t(t,\cdot)$.

Lemma 1. *Let* $u_0\in H^4(\Omega)\cap V$, $u_1\in V$, $f\in W^{1,2}(0,\infty;L^2(\Omega))$ *and condition* **(B1)** *is fulfilled. Then there exist some positive constants* $C=C(n,\Omega,L)$ *and* $\gamma(n,\Omega,L)$ *such that for every solution* u *to the problem* (P_ε) *the following estimates*

$$||u||_{C^1([0,t];L^2(\Omega))} + ||\Delta u||_{W^{1,\infty}(0,t;L^2(\Omega))} + ||u||_{W^{2,2}(0,t;L^2(\Omega))} \le$$

$$\le C M(t)\,e^{\gamma t}, \quad t\ge 0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big], \tag{6}$$

hold, where

$$M(t) = |\Delta^2 u_0| + |u_1| + |B(u_0)| + ||f||_{W^{1,2}(0,t;L^2(\Omega))}. \tag{7}$$

If, in addition, condition **(B2)** *is fulfilled and* $u_0, u_1, \alpha \in H^4 \cap V$, $f \in W^{2,2}(0,\infty; L^2(\Omega))$, *then there exist some positive constants* $\gamma = \gamma(n, \Omega, L, L_0, L_1)$, $C = C(n, \Omega, L, L_0, L_1)$ *such that for the function* z, *defined by*

$$z(t) = u'(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = f(0) - u_1 - \Delta^2 u_0 - B(u_0), \tag{8}$$

the following estimates

$$||z||_{W^{1,\infty}(0,t;L^2(\Omega))} + ||z||_{W^{1,\infty}(0,t;V)} + ||z||_{W^{2,2}(0,t;L^2(\Omega))} \leq$$

$$\leq C\, M_0(t), \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big], \tag{9}$$

are true with

$$M_0(t) = \big|\Delta^2 u_0\big| + \big|\Delta^2 u_1\big| + \big|\Delta^2 \alpha\big| + \big|\big|f\big|\big|_{W^{2,2}(0,\,t;L^2(\Omega))} + M^2(t)\, e^{2\,\gamma\, t}. \tag{10}$$

If $B = 0$, *then* $\gamma = 0$ *in* (6) *and in* (9).

Proof. *Proof of the estimate* (6). In what follows let us agree to denote all constants depending on n, Ω, L, L_0 and L_1 by the same constant C. Due to Theorem 1 we have that $u \in W^{2,\infty}(0,t;L^2(\Omega))$, $\Delta u \in W^{1,\infty}(0,t;L^2(\Omega))$, $\Delta^2 u \in L^\infty(0,t;L^2(\Omega))$ for every $t > 0$.

Let us denote by

$$E(u;t) = \varepsilon\,|u'(t)|^2 + |u(t)|^2 + 2\,\varepsilon\big(u(t), u'(t)\big) + |\Delta u(t)|^2 +$$

$$+2\,(1-\varepsilon)\int_0^t |u'(s)|^2\, ds + 2\int_0^t |\Delta u(s)|^2\, ds, \quad t \geq 0. \tag{11}$$

The direct computations show that for every solution to the problem (P_ε) the following equality

$$\frac{d}{dt}E(u;t) = 2\,\Big(f(t) - B(u), u(t) + u'(t)\Big), \quad a.e. \quad t \in [0,\infty), \tag{12}$$

is fulfilled. According to the condition **(B1)** and (5), we have

$$|B(u)| \leq |B(0)| + L\,||u(t)|| \leq |B(0)| + L\,C_0\,\big(|u(t)| + |\Delta u(t)|\big)$$

and

$$|u(t)|^2 + |\Delta u(t)|^2 \leq$$

$$\leq 2\,\big[\varepsilon\,|u'(t)|^2 + |u(t)|^2 + 2\,\varepsilon\big(u(t), u'(t)\big)\big] + |\Delta u(t)|^2 \leq 2\,E(u;t), \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Then, we get

$$\Big|\big(f(t) - B(u), u(t) + u'(t)\big)\Big| \leq \big(|f(t)| + |B(0)| + L\,||u(t)||_{H^2(\Omega)}\big)\,\big(|u(t)| + |u'(t)|\big) \leq$$

$$\leq \Big[|f(t)| + |B(0)| + 2\sqrt{2}\, L\, C_0\, E^{1/2}(u;t)\Big] \left(\sqrt{2}\, E^{1/2}(u;t) + |u'(t)|\right) \leq$$

$$\leq \frac{1-\varepsilon}{2}\, |u'(t)|^2 + \frac{4}{1-\varepsilon} \Big[2\, E(u;t) + \Big(|f(t)| + |B(0)| + 2\sqrt{2}\, L\, C_0\, E^{1/2}(u;t)\Big)^2\Big] \leq$$

$$\leq \frac{1-\varepsilon}{2}\, |u'(t)|^2 + \frac{4}{1-\varepsilon} \left[1 + 8\, L^2\, C_0^2\right] E(u;t) + C\, (|f(t)| + |B(0)|)^2 \leq$$

$$\leq \gamma\, E(u;t) + C\, (|f(t)| + |B(0)|)^2 +$$

$$+ \frac{1-\varepsilon}{2}\, \frac{d}{dt} \int\limits_0^t |u'(s)|^2\, ds, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big], \tag{13}$$

where $\gamma = 8\,(1 + 8\, L^2\, C_0^2)$.

Therefore, from (12) it follows that

$$\frac{d}{dt} \Big[E(u;t) - (1-\varepsilon) \int\limits_0^t |u'(s)|^2\, ds\Big] \leq$$

$$\leq \gamma\, E(u;t) + C\, (|f(t)| + |B(0)|)^2, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{14}$$

As

$$E(u;t) \leq 2\, E_0(u;t), \quad \text{where} \quad E_0(u;t) = E(u;t) - (1-\varepsilon) \int\limits_0^t |u'(s)|^2\, ds, \tag{15}$$

then from (14) we obtain

$$\frac{d}{dt} \Big[e^{-2\gamma t}\, E_0(u;t)\Big] \leq C\, (|f(t)| + |B(0)|)^2\, e^{-2\gamma t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Integrating this inequality, we get

$$E_0(u;t) \leq E_0(u;0)\, e^{2\gamma t} + C \int\limits_0^t (|f(s)| + |B(0)|)^2\, e^{2\gamma (t-s)}\, ds, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

From the last inequality it follows that

$$|u(t)| + |\Delta u(t)| + ||u'||_{L^2(0,t;L^2(\Omega))} + ||\Delta u||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\, M(t)\, e^{\gamma t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{16}$$

To prove the estimate (6) let us denote by $u_h(t) = h^{-1}(u(t+h) - u(t))$, $h > 0$. For every solution to the problem (P_ε) the equality

$$\frac{d}{dt} E(u_h;t) = 2\Big(F_h(t), u_h'(t) + u_h(t)\Big), \quad a.e. \quad t \in [0, \infty), \tag{17}$$

is true, where

$$F_h(t) = f_h(t) - h^{-1}\Big((Bu)(t+h) + (Bu)(t)\Big). \tag{18}$$

Due to the condition (5), proceeding as in the proof of the estimate (13), we get

$$\Big|\Big(F_h(t), u_h'(t) + u_h(t)\Big)\Big| \leq \big(|u_h(t)| + |u_h'(t)|\big)\,\big(|f_h(t)| + L\,||u_h(t)||_{H^2(\Omega)}\big) \leq$$

$$\leq \big(|u_h(t)| + |u_h'(t)|\big)\,\Big(|f_h(t)| + L\,C_0\,\big(|u_h(t)| + |\Delta u_h(t)|\big)\Big) \leq$$

$$\leq \gamma\, E(u_h;t) + C\,|f_h(t)|^2 + \frac{1-\varepsilon}{2}\,\frac{d}{dt}\int\limits_0^t |u_h'(s)|^2\,ds, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{19}$$

Consequently,

$$\frac{d}{dt}\Big[e^{-2\gamma\, t}\,E_0(u_h;t)\Big] \leq C\,|f_h(t)|^2\,e^{-2\,\gamma\, t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Integrating the last equality on $(0,t)$, we get

$$E_0(u_h;t) \leq E_0(u_h;0)\,e^{2\gamma\, t} + C\int\limits_0^t |f_h(s)|^2\,e^{2\,\gamma\,(t-s)}\,ds, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{20}$$

Since for $1 \leq p < \infty, k \in \mathbb{N}$ and $u \in W^{1,p}(0,T;H^k(\Omega))$ the inequality

$$\int_0^t ||u_h(\tau)||^p_{H^k(\Omega)}d\tau \leq \int_0^t ||u'(\tau)||^p_{H^k(\Omega)}d\tau, \quad t \in [0,\infty), \tag{21}$$

is true (see [2]), then

$$\int_0^t |f_h(s)|^2\,ds \leq \int_0^t |f'(s)|^2\,ds, \quad t \in [0,\infty). \tag{22}$$

As $u'(0) = u_1$, $\varepsilon u''(0) = f(0) - u_1 - \Delta^2 u_0 - B(u_0)$, then

$$E_0(u',0) \leq C\,M(t). \tag{23}$$

Using the estimates (22), (23) and passing to the limit in the inequality (20) as $h \to 0$ we obtain the estimate

$$|u'(t)| + |\Delta u'(t)| + ||u''||_{L^2(0,t;L^2(\Omega))} + ||\Delta u'||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\,M(t)\,e^{\gamma\, t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{24}$$

Finally, from (16) and (24) the inequality (6) follows.

It is easy to see from the proof, that in the case of $B = 0$, $\gamma = 0$.

Proof of the estimate (9). Under the conditions of the Lemma, if u is a solution to the problem (P_ε), then $\big(B(u)\big)' \in W^{1,1}(0,t;L^2(\Omega))$ for every $t>0$ and $\varepsilon \in \Big(0, \dfrac{1}{2}\Big]$. Indeed, due to the conditions **(B2)** and (5), we have

$$\big|\big(B(u(t))\big)'\big| = \big|B'((u(t))u'(t)\big| \le L_0 \big|u'(t)\big|, \quad t \ge 0, \tag{25}$$

and for $u_h(t) = h^{-1}(u(t+h)-u(t))$, $h>0$ and $t>0$, the estimate

$$\Big|h^{-1}\Big(\big(B'(u(t))\big)\,u'(t)\Big)_h\Big| \le$$

$$\le \Big|h^{-1}\Big(B'\big(u(t+h)\big) - B'\big(u(t)\big)\Big)\,u'(t+h)\Big| + \Big|B'(u(t))\,u_h'(t)\Big| \le$$

$$\le L_1\, C_0^2\,\Big(\big|\Delta u_h(t)\big| + \big|u_h(t)\big|\Big)\,\Big(|\Delta u'(t+h)| + |u'(t+h)|\Big) + L_0\,|u_h'(t)|, \quad t \ge 0, \tag{26}$$

is valid.

Using the estimate (6) and inequality (21), from (25) and (26) we deduce that $\big(B(u)\big)' \in W^{1,2}(0,t;L^2(\Omega))$ and

$$\Big|\Big|\Big(\big(B'(u(t))\big)\Big)'\Big|\Big|_{L^2(0,T;L^2(\Omega))} \le$$

$$\le C\ M(t)\,e^{\gamma t}\,\Big(||\Delta u'||_{L^2(0,t;L^2(\Omega))} + ||u'||_{L^2(0,t;L^2(\Omega))}\Big) + L_0\,||u''||_{L^2(0,t;L^2(\Omega))},$$

$$\le C\,M^2(t)\,e^{2\gamma t}, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

Therefore, $\big(B(u)\big)' \in W^{1,1}(0,t;L^2(\Omega))$ for $\varepsilon \in \Big(0, \dfrac{1}{2}\Big]$ and every $t>0$. If $u_1 + \alpha \in H^4(\Omega) \cap V$ and $f \in W^{2,1}(0,t;L^2(\Omega))$, then, in virtue of Theorem 1, the function z, defined by (8), is the solution in $L^2(\Omega)$ to the problem

$$\begin{cases} \varepsilon\, z''(t) + z'(t) + \Delta^2 z(t) = \mathcal{F}(t,\varepsilon), & \text{a. e.} \quad t \ge 0, \\ z(0) = u_1 + \alpha, \quad z'(0) = 0, \end{cases} \tag{27}$$

with

$$\mathcal{F}(t,\varepsilon) = f'(t) - \Big(B\big(u(t)\big)\Big)' + e^{-t/\varepsilon}\,\Delta^2\alpha \tag{28}$$

and z possesses the properties:

$$z \in W^{2,\infty}(0,T;L^2(\Omega)), \quad \Delta z \in W^{1,\infty}(0,T;L^2(\Omega)), \quad \Delta^2 z \in L^\infty(0,T;L^2(\Omega)).$$

Furthermore

$$||\mathcal{F}(t,\varepsilon)||_{L^2(0,t;L^2(\Omega))} \le C\big(||f||_{W^{2,2}(0,t;L^2(\Omega))} + M^2(t)\,e^{2\gamma t}\big), \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

In the same way, as the estimate (16) was obtained in the case $B=0$, we get the estimate

$$|z(t)|+|\Delta z(t)|+||z'||_{L^2(0,t;L^2(\Omega))}+||\Delta z||_{L^2(0,t;L^2(\Omega))}\leq$$
$$\leq C\,M_0(t),\quad t\geq 0,\quad \varepsilon\in\Big(0,\frac{1}{2}\Big]. \tag{29}$$

Also, similarly as the estimate (24) was proved in the case $B=0$, we prove the estimate

$$|z'(t)|+|\Delta z'(t)|+||z''||_{L^2(0,t;L^2(\Omega))}+||\Delta z'||_{L^2(0,t;L^2(\Omega))}\leq$$
$$\leq C\,M_0(t),\quad t\geq 0,\quad \varepsilon\in\Big(0,\frac{1}{2}\Big]. \tag{30}$$

Finally, from (29) and (30) the inequality (9) follows. Lemma 1 is proved.

Lemma 2. *Suppose the condition* **(B3)** *is fulfilled. If* $u_0\in H^4(\Omega)\cap V$, $u_1\in V$ *and* $f\in W^{1,2}(0,\infty;L^2(\Omega))$, *then for every solution* u *to the problem* (P_ε) *the following estimates*

$$||u||_{C^1([0,\,t];L^2(\Omega))}+||\Delta u||_{C^1([0,\,t];L^2(\Omega))}+\big|\big|\Delta u'\big|\big|_{L^2(0,t;L^2(\Omega))}+|\mathcal{B}(u)|^{1/2}\leq$$
$$\leq C(\mathbf{m})\,M_1(t)\,e^{\gamma(\mathbf{m})\,t},\quad t\geq 0,\quad \varepsilon\in\Big(0,\frac{1}{2}\Big]. \tag{31}$$

are true, where

$$M_1(t)=\big|\Delta^2u_0\big|+\big|\Delta u_1\big|+\big|\big|f\big|\big|_{W^{1,2}(0,\,t;L^2(\Omega))}+|\mathcal{B}(u_0)|^{1/2} \tag{32}$$

and

$$\mathbf{m}=|\Delta u_0|+|u_1|+|\mathcal{B}(u_0)|^{1/2}+||f||_{L^2(0,\infty;L^2(\Omega))}.$$

If, in addition, condition **(B4)** *is fulfilled and* $u_0,u_1,\alpha\in H^4(\Omega)\cap V$ *and* $f\in W^{2,2}(0,\infty;L^2(\Omega))$, *then for the function* z, *defined by* (8), *the estimates*

$$||\Delta z||_{C([0,\,t];L^2(\Omega))}+||z'||_{C([0,\,t];L^2(\Omega))}+\big|\big|\Delta z'\big|\big|_{L^2(0,\,t;\,L^2(\Omega))}\leq$$
$$\leq C\,M_2(t)\,e^{\gamma(\mathbf{m})\,t},\quad t\geq 0,\quad \varepsilon\in\Big(0,\frac{1}{2}\Big]. \tag{33}$$

are true, where $C=C(\mathbf{m},||B'(0||)$ *and*

$$M_2(t)=M_1^2(t)\,e^{2\,\gamma(\mathbf{m})\,t}+||f||_{W^{2,2}(0,t;L^2(\Omega))}+|\Delta^2\alpha|. \tag{34}$$

Proof. *Proof of the estimate* (31). Due to Theorem 2 we have that $u\in C^2([0,T];L^2(\Omega))$, $u'\in C^1([0,t];V)$, $\Delta^2u\in C([0,t];L^2(\Omega))$ for every $t>0$.

Denote by

$$E_1(u;t)=\varepsilon\,|u'(t)|^2+|\Delta u(t)|^2+2\int_0^t|u'(s)|^2ds+2\,\mathcal{B}(u(t)).$$

Then for every solution u to the problem (P_ε), we have

$$\frac{d}{dt}\,E_1(u;t) = 2\Big(f(t),u'(t)\Big), \quad t\geq 0.$$

Integrating this inequality, we obtain

$$E_1(u;t)\leq E_1(u;0)+2\int\limits_0^t |f(s)|\,|u'(s)|\,ds \leq \int\limits_0^t |f(s)|^2\,ds + \int\limits_0^t |u'(s)|^2\,ds, \quad t\geq 0.$$

Therefore, we get the estimate

$$\big|\big|\Delta u\big|\big|_{C([0,t;L^2(\Omega))} + ||u'||_{L^2(0,t;\,L^2(\Omega))} + \Big(\mathcal{B}\big(u(t)\big)\Big)^{1/2} \leq$$

$$\leq C\,\big(E_1^{1/2}(u,0) + ||f||_{L^2(0,t;L^2(\Omega))} + |\mathcal{B}(u_0)|^{1/2}\big), \quad t\geq 0, \quad \varepsilon\in(0,1].$$

As $||u||_{L^2(\Omega)} \leq C(n,\Omega)\,||\Delta u||_{L^2(\Omega)}$ for $u\in V$, then from the last inequality the estimate

$$\big|\big|u\big|\big|_{C([0,t];L^2(\Omega))} + \big|\big|\Delta u\big|\big|_{C([0,t];L^2(\Omega))} + ||u'||_{L^2(0,t;L^2(\Omega))} + \Big(\mathcal{B}\big(u(t)\big)\Big)^{1/2} \leq$$

$$\leq C\,\mathbf{m}, \quad t\geq 0, \quad \varepsilon\in(0,1), \tag{35}$$

follows.

Let $u_h(t) = h^{-1}\big(u(t+h)-u(t)\big)$, $h>0$, $t\geq 0$ and the functional $E(u,t)$ is defined by (11). For every solution u to the problem (P_ε) the equality (17) is true with $F_h(t)$ defined by (18).

Due to (5), conditions **(B3)** and the estimate (35), proceeding as in the proof of the estimate (19), we obtain

$$\Big|\Big(F_h(t),u_h'(t)+u_h(t)\Big)\Big| \leq \big(|u_h(t)|+|u_h'(t)|\big)\,\big(|f_h(t)|+L(\mathbf{m})\,||u_h(t)||_{H^2(\Omega)}\big) \leq$$

$$\leq \big(|u_h(t)|+|u_h'(t)|\big)\,\Big(|f_h(t)|+L(\mathbf{m})\,\big(|u_h(t)|+|\Delta u_h(t)|\big)\Big) \leq$$

$$\leq \gamma(\mathbf{m})\,E(u_h;t) + C(\mathbf{m})\,|f_h(t)|^2 + \frac{1-\varepsilon}{2}\,\frac{d}{dt}\int\limits_0^t |u_h'(s)|^2\,ds, \quad t\geq 0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big].$$

Consequently, for $E_0(u;t)$, defined by (15), we have

$$\frac{d}{dt}\Big[e^{-2\gamma(\mathbf{m})\,t}\,E_0(u_h;t)\Big] \leq C(\mathbf{m})\,|f_h(t)|^2\,e^{-2\,\gamma\,t}, \quad t\geq 0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big].$$

Integrating the last equality on $(0,t)$, we get

$$E_0(u_h;t) \le E_0(u_h;0)\, e^{2\,\gamma(\mathbf{m})\,t} + C(\mathbf{m}) \int_0^t |f_h(s)|^2\, e^{2\,\gamma(\mathbf{m})\,(t-s)}\, ds, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big].$$

In what follows, proceeding as in the proof of the estimate (24), we get the estimate

$$||u'||_{C([0,t];L^2(\Omega))} + ||\Delta u'||_{C([0,t];L^2(\Omega))} + ||\Delta u'||_{L^2(0,t;L^2(\Omega))} \le$$
$$\le C(\mathbf{m})\, M_1(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big], \tag{36}$$

with $M_1(t)$ from (32). Finally, from (35) and (36) the inequality (31) follows.

Proof of the estimate (33). Under the conditions of Lemma we have $\big(B(u)\big)' \in W^{1,1}(0,t;L^2(\Omega))$ for every $t > 0$. Indeed, due to Thorem 2, $u \in W^{3,\infty}(0,t;L^2(\Omega))$ and $\Delta u \in W^{2,\infty}(0,t;L^2(\Omega))$ for every $t > 0$. Therefore, using the condition **(B4)** and the estimate (31), we deduce

$$\big|\big(B(u(t))\big)'\big| = \big|B'(u(t))\, u'(t)\big| \le C\big(L_1(\mathbf{m}) + ||B'(0)||\big)\, ||u'(t)||_{H^2(\Omega)}, \quad t > 0.$$

For $h > 0$, $t > 0$ and $u_h(t) = h^{-1}\left(u(t+h) - u(t)\right)$ we have

$$\left|h^{-1}\left(\big(B(u(t))\big)'\right)_h\right| \le$$
$$\le \left|h^{-1}\Big(B'\big(u(t+h)\big) - B'\big(u(t)\big)\Big)\, u'(t+h)\right| + \left|B'\big(u(t)\big)\, u_h'(t)\right| \le$$
$$\le L_1(\mathbf{m})\, M_1(t)\, e^{\gamma(\mathbf{m})\,t}\, ||u_h(t)||_{H^2(\Omega)} + C\left(L_1(\mathbf{m}) + ||B'(0)||\right) ||u_h'||_{H^2(\Omega)} \le$$
$$\le C\left(L_1(\mathbf{m})\, M_1(t)\, e^{\gamma(\mathbf{m})\,t} + ||B'(0)||\right)\left(||u_h(t)||_{H^2(\Omega)} + ||u_h'||_{H^2(\Omega)}\right), \quad t > 0. \tag{37}$$

In virtue of (22), (31) and (37), we conclude that $\Big(\big(B'(u)\big)\Big)' \in W^{1,2}(0,t;L^2(\Omega))$ for every $t > 0$ and

$$\Big|\Big|\Big(\big(B'(u(t))\big)\Big)'\Big|\Big|_{L^2(0,t;L^2(\Omega))} \le$$
$$\le C(\mathbf{m}, ||B'(0||)\, M_1^2(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t > 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{38}$$

From (38) it follows that the function $\mathcal{F}$, which is defined by (28), belongs to $W^{1,1}(0,t;L^2(\Omega))$, for every $t > 0$, and

$$||\mathcal{F}(t,\varepsilon)||_{L^2(0,t;L^2(\Omega))} \le C(\mathbf{m}, ||B'(0||)\, M_2(t)\, e^{\gamma(\mathbf{m})\,t}, \quad t \ge 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big]. \tag{39}$$

According to Theorem 2, for every $t > 0$, the function z possesses the following properties: $z \in W^{2,\infty}(0,t;L^2(\Omega))$, $\Delta z \in W^{1,\infty}(0,t;L^2(\Omega))$, $\Delta^2 z \in L^\infty(0,t;L^2(\Omega))$. The estimate (33) is obtained in the same way as the estimate (9) was obtained, using (31) and (39). Lemma 2 is proved.

4 Relationship between solutions to the problems (P_ε) and (P_0) in the linear case

In this section we establish the relationship between solutions to the problems (P_ε) and (P_0) in the linear case, i.e. in the case when the term $B(u)$ in the problems (P_ε) and (P_0) is missing. This relationship was inspired by the work [12]. Firstly we give some properties of the kernel $K(t,\tau,\varepsilon)$ of the transformation which realizes this connection.

For $\varepsilon > 0$ denote by

$$K(t,\tau,\varepsilon) = \frac{1}{2\sqrt{\pi}\,\varepsilon}\Big(K_1(t,\tau,\varepsilon) + 3\,K_2(t,\tau,\varepsilon) - 2\,K_3(t,\tau,\varepsilon)\Big),$$

where

$$K_1(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t-2\,\tau}{4\,\varepsilon}\Big\}\lambda\Big(\frac{2\,t-\tau}{2\sqrt{\varepsilon\,t}}\Big), \quad K_2(t,\tau,\varepsilon) = \exp\Big\{\frac{3\,t+6\,\tau}{4\,\varepsilon}\Big\}\lambda\Big(\frac{2\,t+\tau}{2\sqrt{\varepsilon\,t}}\Big),$$

$$K_3(t,\tau,\varepsilon) = \exp\Big\{\frac{\tau}{\varepsilon}\Big\}\lambda\Big(\frac{t+\tau}{2\sqrt{\varepsilon\,t}}\Big), \quad \lambda(s) = \int\limits_s^\infty e^{-\eta^2}\,d\eta.$$

The properties of the kernel $K(t,\tau,\varepsilon)$ are collected in the following lemma.

Lemma 3 [9] *The function $K(t,\tau,\varepsilon)$ is the solution to the problem*

$$\begin{cases} K_t(t,\tau,\varepsilon) = \varepsilon K_{\tau\tau}(t,\tau,\varepsilon) - K_\tau(t,\tau,\varepsilon), \quad \forall t>0, \quad \forall \tau > 0,\\ \varepsilon\,K_\tau(t,0,\varepsilon) - K(t,0,\varepsilon) = 0, \quad \forall t \ge 0\\ K(0,\tau,\varepsilon = \dfrac{1}{2\,\varepsilon}\exp\Big\{-\dfrac{\tau}{2\,\varepsilon}\Big\}, \quad \forall \tau \ge 0, \end{cases}$$

from $C([0,\infty)\times[0,\infty))\cap C^2((0,\infty)\times(0,\infty))$ and possesses the following properties:

(i) $K(t,\tau,\varepsilon) > 0, \quad \forall t \ge 0, \quad \forall \tau \ge 0,$ *and* $\displaystyle\int_0^\infty K(t,\tau,\varepsilon)\,d\tau = 1, \quad \forall t \ge 0;$

(ii) *Let $q\in[0,1]$. Then* $\displaystyle\int_0^\infty K(t,\tau,\varepsilon)\,|t-\tau|^q\,d\tau \le C\,\big(\varepsilon+\sqrt{\varepsilon\,t}\big)^q$, $\forall\varepsilon>0$, $\forall t\ge 0$;

(iii) *Let $\gamma > 0$ and $q\in[0,1]$. There exist C_1, C_2 and ε_0, all of them positive and depending on γ and q, such that the following estimates are fulfilled:*

$$\int_0^\infty K(t,\tau,\varepsilon)\,e^{\gamma\tau}|t-\tau|^q\,d\tau \le C_1\,e^{C_2 t}\,\varepsilon^{q/2}, \quad \forall\varepsilon\in(0,\varepsilon_0],\ \forall t>0;$$

(iv) *Let $p\in(1,\infty]$ and $f:\,[0,\,\infty)\to H$, $f(t)\in W^{1,p}(0,\infty;H)$. Then*

$$\Big|f(t) - \int_0^\infty K(t,\tau,\varepsilon)f(\tau)d\tau\Big| \le C(p)\,\|f'\|_{L^p(0,\infty;H)}\,\big(\varepsilon+\sqrt{\varepsilon\,t}\big)^{\frac{p-1}{p}},\ \forall\varepsilon>0,\ \forall t\ge 0.$$

Theorem 7.[9] *Suppose that* $f \in L^\infty_\gamma(0,\infty;L^2(\Omega))$, $u \in W^{2,\infty}_\gamma(0,\infty;L^2(\Omega))$ $\cap L^\infty_\gamma(0,\infty;V)$ *and* $\Delta^2 u \in L^{2,\infty}_\gamma(0,\infty:V')$ *is the solution to the problem*

$$\begin{cases}\varepsilon(u''(t),\eta)+(u'(t),\eta)+\big(\Delta u(t),\Delta\eta\big)=(f(t),\eta),\ \forall\eta\in V, \quad a.\ e.\ t\in[0,\infty),\\ u(0)=u_0, \quad u'(0)=u_1,\end{cases}$$

then for $0<\varepsilon<(4\,\gamma)^{-1}$ *the function*

$$w_0(t)=\int\limits_0^\infty K(t,\tau,\varepsilon)u(\tau)d\tau$$

is solution to the problem

$$\begin{cases}(w_0'(t),\eta)+\big(\Delta w_0(t),\Delta\eta\big)=(F_0(t,\varepsilon)\,u_1,\eta),\ \forall\eta\in V, \qquad a.e.\ t\in[0,\infty),\\ w_0=\varphi_\varepsilon,\end{cases}$$

where

$$F_0(t,\varepsilon)=f_0(t,\varepsilon)\,u_1+\int\limits_0^\infty K(t,\tau,\varepsilon)\,f(\tau)d\tau,$$

$$f_0(t,\varepsilon)=\frac{1}{\sqrt{\pi}}\Big[2\ \exp\Big\{\frac{3\,t}{4\,\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{t}{\varepsilon}}\Big)-\lambda\Big(\frac{1}{2}\sqrt{\frac{t}{\varepsilon}}\Big)\Big], \quad \varphi_\varepsilon=\int\limits_0^\infty e^{-\tau}u(2\,\varepsilon\,\tau)d\tau.$$

Moreover, $w_0\in W^{2,\infty}_{\rm loc}(0,\infty;L^2(\Omega))\cap L^\infty_{\rm loc}(0,\infty;V)$.

5 Behaviour of solutions to the problem (P_ε)

In this section we prove the main results concerning the behavior of the solutions to the problem (P_ε) as $\varepsilon\to 0$ relative to solution to the corresponding unperturbed problem (P_0).

Theorem 8. *Let* $T>0$ *and* $p\in[2,\infty]$. *Assume that* **(B1)** *is fulfilled. If* $u_0\in H^4(\Omega)\cap V$, $u_1\in V$ *and* $f\in W^{1,p}(0,T;L^2(\Omega))$, *then there exist constants* $C=C(L,T,p,\Omega,n)>0$ *and* $\varepsilon_0=\varepsilon_0(L,p,\Omega,n)$ *such that*

$$||u-v||_{C([0,T];L^2(\Omega))}\le C\,M(T)\,\varepsilon^\beta, \quad t\in[0,T], \quad \varepsilon\in(0,\varepsilon_0], \tag{40}$$

$$||u-v||_{L^\infty(0,T;V)}\le C\,M(T)\,\varepsilon^\beta, \quad t\in[0,T], \quad \varepsilon\in(0,\varepsilon_0], \tag{41}$$

where u *and* v *are solutions to the problems* (P_ε) *and* (P_0), *respectively,*

$$M(T)=|\Delta^2u_0|+||u_1||+|B(u_0)|+||f||_{W^{1,p}(0,T;L^2(\Omega))}, \tag{42}$$

$$\beta = \begin{cases} 1/2 \text{ if } f = 0, \\ (p-1)/(2\,p) \text{ if } f \neq 0. \end{cases}$$

If, in addition, condition **(B2)** *is fulfilled and* $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ *and* $f \in W^{2,p}(0,T; L^2(\Omega))$, *then there exist constants* $\varepsilon_0 = \varepsilon_0(L, L_0)$, $\varepsilon_0 \in (0,1)$, $\gamma = \gamma(L, L_0, L_1)$, $C = C(p, L, L_0, L_1)$ *such that*

$$||u' - v' + \alpha\, e^{-t/\varepsilon}||_{C([0,T]\,;L^2(\Omega))} + ||u' - v' + \alpha\, e^{-t/\varepsilon}||_{L^2(0,T\,;H^2(\Omega))} \leq$$

$$\leq C\, M_0(T)\, e^{\gamma\, t}\, \varepsilon^{\beta}, \quad t \geq 0, \quad \varepsilon \in (0, \varepsilon_0], \tag{43}$$

with $M_0(T)$ *defined by* (10).

Proof. In this section, we agree to denote by C all constants depending on T, p, Ω, n, L, L_0 and L_1. For every $f \in W^{k,p}(0,T; L^2(\Omega))$ then there exists the extension $\widetilde{f} : [0, \infty) \mapsto L^2(\Omega)$ such that

$$||\widetilde{f}||_{W^{k,p}(0,\infty;L^2(\Omega))} \leq C(T,p)\, ||f||_{W^{k,p}(0,T;L^2(\Omega))}. \tag{44}$$

If we denote by $\widetilde{U}$ the unique solution to the problem (P_ε), defined on $(0,\infty)$ instead of $(0,T)$ and $\widetilde{f}$ instead of f, then, from Theorem 1 and Lemma 1, it follows that $\widetilde{U} \in W^{2,\infty}(0,\infty; L^2(\Omega))$, $\widetilde{U}' \in L^2(0,\infty; L^2(\Omega))$, $\Delta^2 \widetilde{U} \in L^\infty(0,\infty; L^2(\Omega))$. Due to the estimates (24), for $\widetilde{U}$ we obtain the following estimates

$$\big|\big|\tilde{U}'\big|\big|_{C([0,t];L^2(\Omega))} + \big|\big|\Delta\tilde{U}'\big|\big|_{L^\infty(0,t;L^2(\Omega))} \leq C\, M(T)\, e^{\gamma t}, \quad t \geq 0, \quad \varepsilon \in \Big(0, \frac{1}{2}\Big], \tag{45}$$

with $M(T)$ from (42) and γ from (13).

By Theorem 7, the function W defined by $W(t) = \displaystyle\int_0^\infty K(t,\tau,\mu)\, \widetilde{U}(\tau)\, d\tau$, is a solution to the problem

$$\begin{cases} W'(t) + \Delta^2 W(t) = F(t,\varepsilon), \quad \text{a.e.} \quad t > 0, \quad \text{in} \quad L^2(\Omega), \\ W(0) = \varphi_\varepsilon, \end{cases} \tag{46}$$

where

$$F(t,\varepsilon) = f_0(t,\varepsilon)\, u_1 + \int_0^\infty K(t,\tau,\varepsilon)\, \widetilde{f}(\tau)\, d\tau - \int_0^\infty K(t,\tau,\varepsilon)\, B(\widetilde{U}(\tau))\, d\tau,$$

$$\varphi_\varepsilon = \int_0^\infty e^{-\tau}\, \widetilde{U}(2\,\varepsilon\tau)\, d\tau.$$

Denote by $R(t,\varepsilon) = \widetilde{V}(t) - W(t)$, where $\widetilde{V}$ is the solution to the problem (P_0) with $\tilde{f}$ instead of f, $T = \infty$ and W is the solution to the problem (46). Then, due to

Theorem 2, $R(\cdot,\varepsilon)\in W^{2,\infty}_{\rm loc}(0,\infty;\,L^2(\Omega))\cap L^2(0,\infty;V)$ and R is a solution in $L^2(\Omega)$ to the problem

$$\begin{cases} R'(t,\varepsilon)+\Delta^2 R(t,\varepsilon)+B\big(\widetilde V(t)\big)-B\big(W(t)\big)=\mathcal{F}(t,\varepsilon), \quad \text{a.e.} \quad t>0,\\ R(0,\varepsilon)=u_0-W(0),\end{cases} \tag{47}$$

where

$$\mathcal{F}(t,\varepsilon)=\tilde f(t)-\int_0^\infty K(t,\tau,\varepsilon)\tilde f(\tau)\,d\tau-f_0(t,\varepsilon)\,u_1+$$
$$+B\big(\widetilde U(t)\big)-B\big(W(t)\big)+\int_0^\infty K(t,\tau,\varepsilon)\Big[B\big(\widetilde U(\tau)\big)-B\big(\widetilde U(t)\big)\Big]\,d\tau. \tag{48}$$

In what follows, we need the following two Lemmas, which will be proved after the proof of the estimates (40) and (41).

Lemma 4. *Assume the conditions of Theorem 8 are fulfilled. Then there exist constants $C=C(L,\Omega,n)$, $C_0=C_0(L,\Omega,n)$ and $\varepsilon_0=\varepsilon_0(L,\Omega,n)$ such that following estimates*

$$|\widetilde U(t)-W(t)|\le C\,M(T)\,\varepsilon^{1/2}\,e^{C_0\,t},\quad t\ge 0,\quad \varepsilon\in(0,\varepsilon_0], \tag{49}$$

$$||\widetilde U(t)-W(t)||_{L^\infty(0,t;V)}\le C\,M(T)\,\varepsilon^{1/2}\,e^{C_0\,t},\quad t\ge 0,\quad \varepsilon\in(0,\varepsilon_0], \tag{50}$$

are true with $M(T)$ from (42).

Lemma 5. *Assume the conditions of Theorem 8 are fulfilled. Then there exist constants $C=C(L,\Omega,n)$, $c_0=c_0(L,\Omega,n)$ and $\varepsilon_0=\varepsilon_0(L,\Omega,n)$ such that for the solution to the problem* (47) *the following estimates*

$$||R||_{C([0,\,t];\,L^2(\Omega))}+||\Delta R||_{L^2(0,t;L^2(\Omega))}\le$$
$$\le C\,M(T)\,e^{c_0\,t}\,\varepsilon^{(p-1)/(2\,p)},\quad t\ge 0,\quad \varepsilon\in(0,\varepsilon_0], \tag{51}$$

$$||R||_{L^\infty(0,\,t;H^2(\Omega))}\le C\,M(T)\,e^{c_0\,t}\,\varepsilon^{(p-1)/(2\,p)},\quad t\ge 0,\quad \varepsilon\in(0,\varepsilon_0], \tag{52}$$

are true with $M(T)$ from (42).

From the last two lemmas we deduce that

$$||\tilde U-\tilde V||_{C([0,t];L^2(\Omega))}\le ||\tilde U-W||_{C([0,t];L^2(\Omega))}+||R||_{C([0,t];L^2(\Omega))}\le$$
$$\le C\,M(T)\,e^{C_0\,t}\,\varepsilon^{\beta},\quad t\ge 0,\quad \varepsilon\in(0,\varepsilon_0].$$

Since $u(t)=\tilde U(t)$, $v(t)=\tilde V(t)$, for all $t\in[0,T]$, then we have

$$|u(t)-v(t)|=\big|\tilde U(t)-\tilde V(t)\big|\le C\,M(T)\,\varepsilon^{\beta},\quad t\in[0,T],\quad \varepsilon\in(0,\varepsilon_0]. \tag{53}$$

Concequently, from (53) the estimate (40) follows. Similarly, using (50) and (52), we obtain the estimate (41).

Proof of Lemma 4. Using the properties (**i**), (**ii**) and (**iii**) from Lemma 3, the estimate (45) and the Hölder's inequality, we get

$$|\widetilde{U}(t) - W(t)| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\big|\widetilde{U}(t) - \widetilde{U}(\tau)\big|\,d\tau \leq$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^s \big|\widetilde{U}'(\xi)\big|\,d\xi\Big|\,d\tau \leq C\,M(T)\int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^t e^{\gamma\,\xi}\,d\xi\Big|\,d\tau \leq$$

$$\leq C\,M(T)\int_0^\infty K(t,\tau,\varepsilon)\,|\tau - t|\,\big[e^{\gamma\,t} + e^{\gamma\,\tau}\big]\,d\tau \leq$$

$$\leq C\,M(T)\Big[e^{\gamma\,t}\int_0^\infty K(t,\tau,\varepsilon)\,|\tau - t|\,d\tau + \int_0^\infty K(t,\tau,\varepsilon)\,|\tau - t|\,e^{\gamma\,\tau}\,d\tau\Big] \leq$$

$$\leq C\,M(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{54}$$

Thus, the estimate (49) is proved.

In the same way, using properties (**i**), (**i**) and (**iii**) from Lemma 3, the estimate (45) and the Hölder's inequality, we get

$$\big|\Delta\widetilde{U}(t) - \Delta W(t)\big| \leq \int_0^\infty K(t,\tau,\varepsilon)\,\big|\Delta\widetilde{U}(t) - \Delta\widetilde{U}(\tau)\big|\,d\tau \leq$$

$$\leq \int_0^\infty K(t,\tau,\varepsilon)\Big|\int_\tau^s \big|\Delta\widetilde{U}'(\xi)\big|\,d\xi\Big|\,d\tau \leq C\,M(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{55}$$

Due to Theorem 5, we have that

$$||\widetilde{U} - W||_{L^\infty(0,t;V)} = ||\widetilde{U} - W||_{L^\infty(0,t;H^2(\Omega))} \leq$$

$$\leq C\,\big[||\widetilde{U} - W||_{L^\infty(0,t;L^2(\Omega))} + ||\Delta\widetilde{U} - \Delta W||_{L^\infty(0,t;L^2(\Omega))}\big].$$

From the last inequality, using (49) and (55), we get (50). Lemma 4 is proved.

Proof of Lemma 5. *Proof of the estimate* (51). Multiplying scalarly in $L^2(\Omega)$ the equation (47) by R and using the condition **(B1)** and Theorem 5 we obtain the inequality

$$\frac{d}{dt}|R(t,\varepsilon)|^2 + 2\,|\Delta R(t,\varepsilon)|^2 \leq 2\,|\mathcal{F}(t,\varepsilon)|\,|R(t,\varepsilon)| + 2\,L\,||R(t,\varepsilon)||_{H^2(\Omega)}\,|R(t,\varepsilon)| \leq$$

$$\leq 2\,|\mathcal{F}(t,\varepsilon)|\,|R(t,\varepsilon)| + C_0\,L\,\big(|R(t,\varepsilon)| + |\Delta R(t,\varepsilon)|\big)\,|R(t,\varepsilon)|, \quad t \geq 0,$$

from which it follows that

$$\frac{d}{dt}|R(t,\varepsilon)|^2 + |\Delta R(t,\varepsilon)|^2 \leq 2\,|\mathcal{F}(t,\varepsilon)|^2 + 2\,\gamma_1\,|R(t,\varepsilon)|^2, \quad t \geq 0,$$

or

$$\frac{d}{dt}\Big[|R(t,\varepsilon)|^2\,e^{-2\,\gamma_1\,t}\Big] + |\Delta R(t,\varepsilon)|^2\,e^{-2\,\gamma_1\,t} \leq 2\,|\mathcal{F}(t,\varepsilon)|^2\,e^{-2\,\gamma_1\,t}, \quad t \geq 0,$$

with some γ_1 depending on L and constant C_0 from Theorem 5. Integrating on $(0,t)$ the last equality, we deduce

$$|R(t,\varepsilon)| + ||\Delta R(\cdot,\varepsilon)||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\,\Big[|R(0,\varepsilon)| + ||\mathcal{F}(\cdot,\varepsilon)|_{L^2(0,t;L^2(\Omega))}\Big]\,e^{\gamma_1\,t}, \quad \forall t \geq 0, \tag{56}$$

where $\mathcal{F}(t,\varepsilon)$ is defined by (48). In what follows, we will estimate the right side of (56). Using (45), we get

$$\big|R(0,\varepsilon)\big| \leq \int_0^\infty e^{-\tau}\Big|\tilde{U}(2\varepsilon\,\tau) - u_0\Big|\,d\tau \leq \int_0^\infty e^{-\tau}\int_0^{2\,\varepsilon\,\tau}\big|\tilde{U}'(\xi)\big|\,d\xi\,d\tau \leq$$

$$\leq C\,M(T)\,\varepsilon\int_0^\infty \tau\,e^{-\tau}\,d\tau = C\,M(T)\,\varepsilon, \quad \varepsilon \in \Big(0,\frac{1}{2}\Big]. \tag{57}$$

Using the property (**iv**) from Lemma 3 and (44), we deduce

$$\Big|\tilde{f}(t) - \int_0^\infty K(t,\tau,\varepsilon)\tilde{f}(\tau)\,d\tau\Big| \leq C\,\|\tilde{f}'\|_{L^p(0,\infty\,;\,L^2(\Omega))}\,(\varepsilon + \sqrt{\varepsilon\,t})^{(p-1)/p} \leq$$

$$\leq C\,\|f'\|_{L^p(0,T\,;\,L^2(\Omega))}\,\big(\varepsilon + \sqrt{\varepsilon\,t})\big)^{(p-1)/p}, \quad t \geq 0, \quad \varepsilon > 0. \tag{58}$$

Since $e^{\xi}\lambda(\sqrt{\xi}) \leq C, \quad \forall \xi \geq 0$, then the following estimates

$$\int_0^t \exp\Big\{\frac{3\xi}{4\,\varepsilon}\Big\}\lambda\Big(\sqrt{\frac{\xi}{\varepsilon}}\Big)d\xi \leq C\,\varepsilon\int_0^\infty e^{-\xi/4}\,d\xi \leq C\,\varepsilon, \quad t \geq 0, \quad \varepsilon > 0,$$

$$\int_0^s \lambda\Big(\frac{1}{2}\sqrt{\frac{\xi}{\varepsilon}}\Big)\,d\xi \leq \varepsilon\int_0^\infty \lambda\Big(\frac{1}{2}\sqrt{\xi}\Big)\,d\xi \leq C\,\varepsilon, \quad t \geq 0, \quad \varepsilon > 0$$

hold. Consequently

$$\Big|\int_0^t f_0(\xi,\varepsilon)\,u_1 d\xi\Big| \leq C\,\varepsilon|u_1|, \quad t \geq 0, \quad \varepsilon > 0. \tag{59}$$

Using **(B1)**, (5) and the estimates (49) and (50), we get the following estimates

$$\left|B(\widetilde{U}(t)) - B(W(t))\right| \le$$

$$\le L\,\|\widetilde{U}(t) - W(t)\|_{H^2(\Omega)} \le C\,M(T)\,\varepsilon^{1/2}\,e^{c_0\,t}, \quad t \ge 0, \varepsilon \in (0,\varepsilon_0]. \tag{60}$$

Similarly as the estimate (54) was obtained, we get

$$\int\limits_0^\infty K(t,\tau,\varepsilon)\left|B(\widetilde{U}(\tau)) - B(\widetilde{U}(t))\right| d\tau \le C\,M(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{61}$$

Using (58), (59), (60) and (61), from (48) we get

$$\left|\mathcal{F}(\tau,\varepsilon)\right| \le C\,M(T)\,e^{C_2\,t}\,\varepsilon^{(p-1)/(2\,p)}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0].$$

Consequently,

$$\Big(\int\limits_0^t \left|\mathcal{F}(\tau,\varepsilon)\right|^2 d\tau\Big)^{1/2} \le C\,M(T)\,e^{C_2\,t}\,\varepsilon^{(p-1)/(2\,p)}, \quad t \ge 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{62}$$

From (56), using (57) and (62) we get the estimate (51).

Proof of the estimate (52). From Theorem 3 it follows that $R \in W^{1,2}_{\rm loc}(0,t;V) \cap W^{1,\infty}_{\rm loc}(0,t;L^2(\Omega))$ and $\Delta^2 R \in L^2_{\rm loc}(0,\infty;L^2(\Omega))$. Moreover the function $t \mapsto (\Delta^2 R(t,\varepsilon), R(t,\varepsilon))$ is an absolutely continuous function on $[0,T]$ for every $T > 0$ and

$$\frac{d}{dt}(\Delta^2 R(t,\varepsilon), R(t,\varepsilon)) = 2(\Delta^2 R(t,\varepsilon), R'(t,\varepsilon)), \qquad \text{a. e.} \quad t > 0.$$

Multiply the equation (47) by $\Delta^2 R(t,\varepsilon)$ and then integrate on $(0,t)$ to get

$$\left|\Delta R(t,\varepsilon\right|^2 + 2\int\limits_0^t \left|\Delta^2 R(s,\varepsilon)\right|^2 ds =$$

$$= \left|\Delta R(0,\varepsilon)\right|^2 + 2\int\limits_0^t \big(\mathcal{F}(s,\varepsilon) - B\big(\widetilde{V}(s)\big) + B\big(W(s)\big), \Delta^2 R(s,\varepsilon)\big)\,ds, \quad t \ge 0.$$

Therefore,

$$|\Delta R(t,\varepsilon)|^2 + \int\limits_0^t \left|\Delta^2 R(s,\varepsilon)\right|^2 ds \le$$

$$\le \left|\Delta R(0,\varepsilon)\right|^2 + \int\limits_0^t \Big[\left|\mathcal{F}(s,\varepsilon)\right|^2 + \left|B\big(\widetilde{V}(s)\big) - B\big(W(s)\big)\right|^2\Big]\,ds, \quad t \ge 0.$$

From the last inequality, using (62) and (51), we obtain

$$\big|\Delta R(t,\varepsilon\big| + ||\Delta^2 R||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\Big[|\Delta R(0,\varepsilon)| + ||\mathcal{F}||_{L^2(0,t;L^2(\Omega))} + L\,||R||_{L^2(0,t;L^2(\Omega))}\Big] \leq$$

$$\leq C\Big[|\Delta R(0,\varepsilon)| + M(T)\,e^{C_2\,t}\,e^{(p-1)/(2\,p)}\Big], \quad t>0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{63}$$

Using (45), we get

$$|\Delta R(0,\varepsilon)| \leq \int_0^\infty e^{-s}\,|\Delta(\tilde{U}(2\,\varepsilon\,s)-u_0))|\,ds \leq$$

$$\leq \int_0^\infty e^{-s}\int_0^{2\,\varepsilon\,s}|\Delta\tilde{U}'(\tau)|\,d\tau\,ds \leq C\,M(T)\,\varepsilon, \quad \varepsilon\leq\frac{\gamma}{4}. \tag{64}$$

From (63) and (64) it follows that

$$\big|\Delta R(t,\varepsilon\big| \leq C\,M(T)\,e^{C_2\,t}\,e^{(p-1)/(2\,p)}, \quad t>0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{65}$$

As, due to Theorem 5, we have that

$$||R||_{L^\infty(0,t;V)} = ||R||_{L^\infty(0,t;H^2(\Omega))} \leq C_0\left[||R||_{L^\infty(0,t;L^2(\Omega))} + ||\Delta R||_{L^\infty(0,t;L^2(\Omega))}\right],$$

then using (51) and (65) we get (52). Lemma 5 is proved.

Proof of the estimate (43). According to Lemma 1, the function $\tilde{z}$, defined as

$$\tilde{z}(t) = \tilde{U}'(t) + \alpha e^{-t/\varepsilon}, \quad \alpha = \tilde{f}(0) - u_1 - \Delta^2 u_0 - B(u_0),$$

is solution to the problem (27) with

$$\mathcal{F}(t,\varepsilon) = \tilde{f}'(t) - \Big(B\big(\tilde{U}(t)\big)\Big)' + e^{-t/\varepsilon}\,\Delta^2\alpha$$

and $\tilde{z}$ satisfies the following estimate

$$||\tilde{z}||_{W^{1,\infty}(0,t;L^2(\Omega))} + ||\tilde{z}||_{W^{1,\infty}(0,t;V)} + ||\tilde{z}||_{W^{2,2}(0,t;L^2(\Omega))} \leq$$

$$\leq C\,M_0(t), \quad t\geq 0, \quad \varepsilon\in\Big(0,\frac{1}{2}\Big], \tag{66}$$

wherein, due to inequality (44), with the same $M_0(t)$ from (10).

As $\tilde{z}'(0)=0$, then according to Theorem 7, the function

$$w_1(t) = \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{z}(\tau)\,d\tau,$$

is solution to the following problem:

$$\begin{cases} w_1'(t) + \Delta^2 w_1(t) = F_1(t,\varepsilon), \quad \text{a. e.} \quad t>0, \quad \text{in} \quad L^2(\Omega), \\ w_1(0) = \varphi_{1\varepsilon}, \end{cases}$$

for $0<\varepsilon\leq\varepsilon_0$, where

$$F_1(t,\varepsilon) = \int_0^\infty K(t,\tau,\varepsilon)\tilde{f}\,'(\tau)\,d\tau - \int_0^\infty K(t,\tau,\varepsilon)\left(B(\tilde{U})\right)'(\tau)\,d\tau +$$

$$+\int_0^\infty K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\,\Delta^2\alpha, \quad \varphi_{1\varepsilon} = \int_0^\infty e^{-\tau}\tilde{z}(2\varepsilon\tau)\,d\tau.$$

Using the properties (**i**), (**ii**) and (**iii**) from Lemma 3 and the estimate (66) and proceeding as in the proof of estimate (54), we get

$$\left|\tilde{z}(t) - w_1(t)\right| \leq \int_0^\infty K(t,\tau,\varepsilon)\left|\tilde{z}(t)-\tilde{z}(\tau)\right|d\tau \leq \int_0^\infty K(t,\tau,\varepsilon)\left|\int_\tau^t \left|\tilde{z}\,'(s)\right|ds\right|d\tau \leq$$

$$\leq C\,M_0(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t\geq 0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{67}$$

In the same way, using (66), we obtain the estimate

$$\left|\left|\tilde{z} - w_1\right|\right|_{L^\infty(0,t;H^2(\Omega))} \leq C\,M_0(T)\,e^{C_2\,t}\,\varepsilon^{1/2}, \quad t\geq 0, \quad \varepsilon\in(0,\varepsilon_0]. \tag{68}$$

Let $v_1(t)=v'(t)$, where v is solution to the problem (P_0) with $\tilde{f}$ instead of f, $T=\infty$.

Denote by $R_1(t,\varepsilon)=v_1(t)-w_1(t)$. Then the function $R_1(t,\varepsilon)$ is solution to the problem

$$\begin{cases} R_1'(t,\varepsilon) + \Delta^2 R_1(t,\varepsilon) = \mathcal{F}_1(t,\varepsilon), \quad \text{a. e.} \quad t>0, \quad \hat{\text{i}}\text{n} \quad L^2(\Omega) \\ R_1(0,\varepsilon) = R_{10} =: f(0) - \Delta^2 u_0 - B(u_0) - \varphi_{1\varepsilon}, \end{cases}$$

where

$$\mathcal{F}_1(t,\varepsilon) = \tilde{f}\,'(t) - \int_0^\infty K(t,\tau,\varepsilon)\,\tilde{f}'(\tau)\,d\tau + \int_0^\infty K(t,\tau,\varepsilon)\,e^{-\tau/\varepsilon}\,d\tau\,\Delta^2\alpha -$$

$$-\left(B(v)\right)'(t) + \int_0^\infty K(t,\tau,\varepsilon)\left(B(\tilde{U}_\varepsilon)\right)'(\tau)d\tau. \tag{69}$$

Due to the conditions of Theorem 8, similarly as the inequality (56) was obtained, and the estimates (57), (62), we get the inequality

$$||R_1||_{C([0,t];\,H)} + ||\Delta R_1||_{L^2(0,t;L^2(\Omega))} \leq$$

$$\leq C\Big[|R_{10}| + ||\mathcal{F}_1(\cdot,\varepsilon)|_{L^2(0,t;L^2(\Omega))}\Big]\, e^{\gamma_1 t}, \quad \forall t \geq 0. \tag{70}$$

and the estimates

$$|R_{10}| \leq \int\limits_0^\infty e^{-\tau}\, |\tilde{z}(2\,\varepsilon\tau) - \tilde{z}(0)|\, d\tau \leq$$

$$\leq \int\limits_0^\infty e^{-\tau} \int_0^{2\varepsilon\tau} |\tilde{z}_\varepsilon'(s)|\, ds\, d\tau \leq C\, M_0(T)\, \varepsilon, \quad \varepsilon \in (0,\varepsilon_0], \tag{71}$$

$$||\mathcal{F}_1(\cdot,\varepsilon)|_{L^2(0,t;L^2(\Omega))} \leq C\, M_0(T)\, e^{C_2\, t}\, \varepsilon^{(p-1)/(2\,p)}, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]. \tag{72}$$

From (70), using (71) and (72), we get the estimate

$$||R_1||_{C([0,t];L^2(\Omega))} + ||\Delta R_1||_{L^2(0,t;L^2(\Omega))} \leq C\, M_0(T)\, e^{C_2\, t}\, \varepsilon^{(p-1)/(2\,p)}, \tag{73}$$

$t \geq 0, \quad \varepsilon \in (0,\varepsilon_0]$.

Finally, due to (5), from (67), (68) and (73) the estimate (43) follows. Theorem 8 is proved.

Similarly, using Theorems 3 and 4 instead of Theorems 1 and 2 and Lemma 2 instead of Lemma 1, the following theorem is proved.

Theorem 9. *Let $T > 0$ and $p \in [2,\infty]$. Assume that* **(B3)** *is fulfilled. If $u_0 \in H^4(\Omega) \cap V$, $u_1 \in V$ and $f \in W^{1,p}(0,T;L^2(\Omega))$, then there exist constants $C = C(\mathbf{m},T,p,\Omega,n) > 0$ and $\varepsilon_0 = \varepsilon_0(\mathbf{m},p,\Omega,n)$, such that*

$$||u - v||_{C([0,T];L^2(\Omega))} \leq C\, M(T)\, \varepsilon^\beta, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0], \tag{74}$$

$$||u - v||_{L^\infty(0,T;V)} \leq C\, M(T)\, \varepsilon^\beta, \quad t \in [0,T], \quad \varepsilon \in (0,\varepsilon_0], \tag{75}$$

where u and v are solutions to the problems (P_ε) and (P_0), respectively,

$$M(T) = |\Delta^2 u_0| + ||u_1|| + |B(u_0)| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{W^{1,p}(0,T;L^2(\Omega))},$$

$$\mathbf{m} = |\Delta u_0| + |u_1| + |\mathcal{B}(u_0)|^{1/2} + ||f||_{L^2(0,T;L^2(\Omega))}, \quad \beta = \begin{cases} 1/2 \text{ if } f = 0, \\ (p-1)/(2\,p) \text{ if } f \neq 0. \end{cases}$$

If, in addition, condition **(B4)** *is fulfilled and $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ and $f \in W^{2,p}(0,T;L^2(\Omega))$, then there exist constants $\varepsilon_0 = \varepsilon_0(L,L_0)$, $\varepsilon_0 \in (0,1)$, $\gamma = \gamma(L,L_0,L_1)$, $C = C(p,L,L_0,L_1)$ such that*

$$||u' - v' + \alpha\, e^{-t/\varepsilon}||_{C([0,T];L^2(\Omega))} + ||u' - v' + \alpha\, e^{-t/\varepsilon}||_{L^2(0,T;H^2(\Omega))} \leq$$

$$\leq C\, M_2(T)\, e^{\gamma\, t}\, \varepsilon^\beta, \quad t \geq 0, \quad \varepsilon \in (0,\varepsilon_0] \tag{76}$$

with $M_2(T)$ defined by (34).

6 An Examples

In this section, we present some applications of Theorems 8 and 9, which are determined by different operators B.

The Lipschitzian case. Let the operator B be one of the following: $B(u) = |u|$, or $B = |\nabla u|$, or $B(u) = \sin u$. In these cases it is easy to check that for the operator B the conditions (**B1**) are fulfilled. Consequently, for every $T > 0$ and every $p \in [2, \infty]$, if $u_0 \in H^4(\Omega) \cap V$, $u_1 \in V$ and $f \in W^{1,2}(0, T; L^2(\Omega))$, then from Theorem 8 the estimates (40) and (41) follow.

For $B(u) = \sin u$, due to Theorem 6, condition **(B2)** is fulfilled if $1 \leq n \leq 12$. Indeed, for $n = 1, 2, 3, 4$, Theorem 6 ensures the fulfillment of the condition (**B2**). For $n > 4$, using the Hölder's inequality and Theorem 6, we have that

$$\int\limits_{\Omega} \big|\big(B'(u_1) - B'(u_2)\big)\, v\big|^2\, dx \leq \int\limits_{\Omega} \big|\big(\cos(u_1) - \cos(u_2)\big)\, v\big|^2\, dx \leq$$

$$\leq 4 \int\limits_{\Omega} \big|\sin\big((u_1 - u_2)/2\big)\, v\big|^2\, dx \leq C \int\limits_{\Omega} \big|u_1 - u_2\big|\, |v|^2\, dx \leq$$

$$\leq C\Big(\int\limits_{\Omega} \big|u_1 - u_2\big|^{2\,n/(n-4)}\, dx\Big)^{(n-4)/(2\,n)} \times \Big(\int\limits_{\Omega} |v|^{4\,n/(n+4)}\, dx\Big)^{(n+4)/(2\,n)} \leq$$

$$\leq C\, ||u_1 - u_2||_{H^2(\Omega)}\, ||v||^2_{L^{4\,n/(n+4)}(\Omega)} \leq C\, ||u_1 - u_2||_{H^2(\Omega)}\, ||v||^2_{H^2(\Omega)}, \quad \text{if} \quad 5 \leq n \leq 12.$$

Therefore, if $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ and $f \in W^{2,p}(0, T; L^2(\Omega))$, then the estimate (43) also holds. It means that

$$u \to v \quad \text{in} \quad C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0. \tag{77}$$

At the same time, the relation (43) shows that in this case the derivative u' of the solution to the problem (P_ε) does not converge to the derivative v' of the solution to the problem (P_0). In this case the derivative u' has a singular behavior in the neighborhood of the point $t = 0$ as $\varepsilon \to 0$. This singular behavior is described by the function $\alpha\, e^{-t/\varepsilon}$, which is *the boundary layer function* for u'. If $\alpha = 0$, then

$$u' \to v' \quad \text{in} \quad C([0, T]; L^2(\Omega)), \quad \text{as} \quad \varepsilon \to 0. \tag{78}$$

The monotone case. Let $B : D(B) = L^2(\Omega) \cap L^{2(q+1)}(\Omega) \mapsto L^2(\Omega)$, $B(u) = b\, |u|^q\, u$, $b > 0$.

Then the operator B is the Fréchet derivative of the convex and positive functional $\mathcal{B}$, defined as follows

$$D(\mathcal{B}) = L^{q+2}(\Omega) \cap L^2(\Omega), \quad \mathcal{B}u = \frac{b}{q+2} \int_\Omega |u(x)|^{q+2}\, dx$$

and the Fréchet derivative of the operator B is defined by the relations

$$D\big(B'(u)\big) = \{v \in L^2(\Omega) : u^q\, v \in L^2(\Omega)\}, \quad B'(u)v = b\,(q+1)|u|^q\, v.$$

In what follows, to check the fulfillment of the condition **(B3)** for the operator B we apply Theorem 6.

If $n>4$ and $q\in[0,4/(n-4)]$, then using the Hölder's inequality and Theorem 6, we get

$$||Bu_1-Bu_2||^2_{L^2(\Omega)}=b^2\int\limits_\Omega \Big||u_1(x)|^q u_1(x)-|u_2(x)|^q u_2(x)\Big|^2\,dx\le$$

$$\le C(q,b)\int\limits_\Omega \big|u_1(x)-u_2(x)\big|^2\Big(\big|u_1(x)\big|^{2q}+\big|u_2(x)\big|^{2\,q}\Big)\,dx\le$$

$$\le C(q,n,b)||u_1-u_2||^2_{L^{2\,n/(n-4)}(\Omega)}\Big(||u_1||^{2\,q}_{L^{q\,n/2}(\Omega)}+||u_2||^{2\,q}_{L^{q\,n/2}(\Omega)}\Big)\le$$

$$\le C(q,b,n,\Omega)||u_1-u_2||^2_{H^2(\Omega)}\Big(||u_1||^{2q}_{H^2(\Omega)}+||u_2||^{2q}_{H^2(\Omega)}\Big),\quad u_1,\,u_2\in V. \qquad (79)$$

Similarly, using the Hölder's inequality and Theorem 6, it is not difficult to prove the estimate (79) in the case $q\in[0,\infty]$ for $n=1,2,3,4$.

Thus, if

$$\begin{cases} b\ge 0, \\ q\in[0,4/(n-4)], \quad \text{if} \quad n>4, \\ q\in[0,\infty], \quad \text{if} \quad n=1,2,3,4 \end{cases} \qquad (80)$$

then the operator B verifies condition **(B3)**.

Finally, if $u_0\in H^4(\Omega)\cap V,\ u_1\in V$ and $f\in W^{1,p}(0,T;L^2(\Omega))$ and conditions (80) are met, then, by virtue of Theorem 9, the estimates (74) and (75) and hence the relations (77) and are also valid.

If $n>4$ and $q\in[1,4/(n-4)]$, then, according to Theorem 6, we have

$$||\big(B'(u_1)-B'(u_2)\big)v||^2_{L^2(\Omega)}=b^2(q+1)^2\int_\Omega \Big||u_1(x)|^q-|u_2(x)|^q\Big|^2|v(x)|^2\,dx\le$$

$$\le C(q,b)\int_\Omega |u_1(x)-u_2(x)|^2\Big(|u_1(x)|^{2(q-1)}+|u_2(x)|^{2(q-1)}\Big)|v(x)|^2\,dx\le$$

$$\le C(q,b)\,||v||^2_{L^{2n/(n-4)}(\Omega)}\,||u_1-u_2||^2_{L^{2n/(n-(n-4)q)}(\Omega)}\times$$

$$\times\Big(||u_1||^{2(q-1)}_{L^{2n/(n-4)}(\Omega)}+||u_2||^{2(q-1)}_{L^{2n/(n-4)}(\Omega)}\Big)\le$$

$$\le C(n,q,b,\Omega,\omega)\,||u_1-u_2||^2_{H^2(\Omega)}\,||v||^2_{H^2(\Omega)}\Big(||u_1||^{2(q-1)}_{H^2(\Omega)}+||u_2||^{2(q-1)}_{H^2(\Omega)}\Big). \qquad (81)$$

Involving the Hölder's inequality and Theorem 6, we get the inequality (81) in the cases $n=1,2,3,4$ and $q\ge 1$. Therefore, if

$$\begin{cases} b\ge 0, \\ q\in[1,4/(n-4)] \quad \text{if} \quad n>4, \\ q\in[1,\infty] \quad \text{if} \quad n=1,2,3,4, \end{cases}$$

then the operator B verifies the condition (**B4**). Therefore, if $u_0, u_1, \alpha \in H^4(\Omega) \cap V$ and $f \in W^{2,p}(0,T; L^2(\Omega))$, then the estimate (76) is fulfilled. Also, as in the Lipschitzian case, this relationship shows that the derivative u' of solution to the problem (P_ε) does not converge to the derivative v' of solution to the problem (P_0). In this case the derivative u' has a singular behavior in the neighborhood of the point $t = 0$ as $\varepsilon \to 0$. This singular behavior is described by the function $\alpha\, e^{-t/\varepsilon}$, which is *the boundary layer function* for u'. If $\alpha = 0$, then as in the Lipschitzian case the relation (78) is true.

Acknowledgement

This research was supported by the State Program of the Republic of Moldova *Multivalued dynamical systems, singular perturbations, integral operators and non-associative algebraic structures* (20.80009.5007.25).

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Received August 13, 2022

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