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B-spline approximation of discontinuous functions defined on a closed contour in the complex plane

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Abstract. In this paper we propose an efficient algorithm for approximating piecewise continuous functions, defined on a closed contour Γ in the complex plane. The function, defined numerically on a finite set of points of Γ, is approximated by a linear combination of B-spline functions and Heaviside step functions, defined on Γ. Theoretical and practical aspects of the convergence of the algorithm are presented, including the vicinity of the discontinuity points.

Mathematics subject classification: 65D07, 41A15.
Keywords and phrases: piecewise continuous function, closed contour, complex plane, approximation, B-spline, step function, convergence.

1 Introduction and problem formulation

Let Γ be a simple closed contour in the complex plane that includes inside it the origin of coordinates and $f : \Gamma \to C$ is a function defined at the points of this contour. Let the function $f \in PC(\Gamma)$, where $PC(\Gamma)$ is the set of all continuous or piecewise continuous functions on Γ. If the function $f \in PC(\Gamma)$ is discontinuous on Γ, we consider that it has finite jump discontinuities, being left continuous at the discontinuity points.

In multiple practical situations the function f is not defined analytically, but by its values on a finite set of points. In this paper we aim to develop an efficient algorithm for approximating the function $f \in PC(\Gamma)$, defined numerically on the set $\{t_j\}$ of points belonging to the contour Γ.

The proposed approximation algorithm is based on the concept of B-spline functions, defined on the contour Γ. The spline functions, defined on the Jordan curve Γ in the complex plane, have been introduced in the paper [1] and the B-spline functions – in [2]. For B-spline functions, some properties analogous to those that occur for B-splines defined on a segment of the real axis have been proved.

For two points $t_1, t_2 \in \Gamma$ we use the notation $t_1 \prec t_2$ if when traversing the contour Γ in counterclockwise direction we meet first the point t_1, and then t_2 (see Figure 1). Let $t_1 \prec t_2 \prec \ldots \prec t_n \ (\prec t_1)$ be a set of distinct points of the contour Γ. We denote by $\Gamma_j := arc\,[t_j, t_{j+1}]$ the set of points of the contour Γ, located between the points t_j and t_{j+1}.

Let the positive integers $m, n \geq 2$. The spline function $s(t)$ of order m, defined on the contour Γ, satisfies the following properties:

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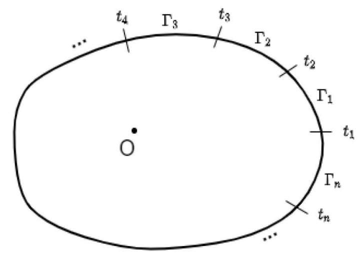

Figure 1: The type of contour and notations used

a) $s \in C^{m-2}(\Gamma)$;

b) the restriction of s on Γ_j for $j = 1, ..., n$ is a polynomial of degree $m - 1$.

The set of all spline functions of the order m forms the linear space $S_{m,n}$. In [1] it is shown that any continuous function on Γ can be approximated uniformly on Γ with a linear ($m = 2$) or cubic ($m = 4$) spline function.

In [2] the B-spline functions of order $m \geq 2$ on the contour Γ are defined, based on the recursive formula

$$B_{m,j}(t) := \frac{m}{m-1}\left(\frac{t-t_j}{t_{j+m}-t_j}B_{m-1,j}(t) + \frac{t_{j+m}-t}{t_{j+m}-t_j}B_{m-1,j+1}(t)\right), \ j = 1, ..., n, \tag{1}$$

where $B_{1,j}(t) = \begin{cases} \frac{1}{t_{j+1}-t_j} & \text{if } t \in arc\,[t_j, t_{j+1}) \\ 0 & \text{otherwise} \end{cases}$. Also, it is shown that the set of B-splines $\{B_{m,1}, ..., B_{m,n}\}$ forms a basis of the space $S_{m,n}$ of spline functions on Γ. It follows that any continuous function on Γ can be uniformly approximated on Γ by a linear combination of B-spline functions. Now we intend to study what happens when the approximated function f has discontinuities on Γ.

The case when the piecewise continuous function is defined on a closed interval $[a, b]$ of the real axis is examined in [3]. In this paper it is shown that at the approximation of the discontinuous function $f \in PC([a, b])$ with spline functions of order $m \geq 2$, we do not have uniform convergence, because in the vicinity of the discontinuity points we have strong oscillations of the spline values around the values of the function f. When amplifying the number of nodes on which the spline is built, the amplitude of the oscillations does not tend to zero. When we move away from the points of discontinuity, the approximation becomes uniform and the error can be evaluated based on the relationships established at the approximation of continuous functions. Also, in [3] it is shown that the oscillating effect in the vicinity of discontinuity points can be annihilated if the approximation is constructed as a linear combination of m-order B-spline functions. Moreover, in order to construct a piecewise continuous approximation, which converges uniformly to the function f, a linear combination of B-spline functions and Heaviside step functions is considered.

Next, we apply the approach proposed in [3] and study the convergence of the linear combination of B-splines on Γ to the function $f \in PC(\Gamma)$, and as a result we present an algorithm for approximating the function f. The algorithm is efficient in the sense that it achieves a uniform approximation of the function f on the whole contour Γ, but also due to the fact that it consumes a limited amount of computational resources.

2 Approximation of function by a linear combination of B-splines

Let a closed and piecewise smooth contour Γ be the boundary of the simply connected domain $\Omega^+ \subset \mathrm{C}$. Let the point $z = 0 \in \Omega^+$. Consider the Riemann function $z = \psi(w)$, that performs the conformal map of the domain D^- from the outside of the circle $\Gamma_0 := \{w \in \mathrm{C} : \ |w| = 1\}$ onto the domain Ω^- from the outside of the contour Γ, such that $\psi(\infty) = \infty$, $\psi'(\infty) > 0$. The function $\psi(w)$ transforms the circle Γ_0 onto the contour Γ. Next, we consider that the points of the contour Γ are defined by means of the function $\psi(w)$.

Let $\{t_j\}_{j=1}^{n_B}$ be the set of distinct points of the contour Γ where the values of the function $f \in PC(\Gamma)$ are defined. We consider that the points t_j are generated based on the relation

$$t_j = \psi(w_j), \ w_j = e^{i\theta_j}, \ \theta_j = 2\pi(j-1)/n_B, \ j = 1, ..., n_B.$$

Thus, the variation of the parameter θ ensures a uniform coverage of the interval $[0, 2\pi]$ and the points t_j are distributed over the entire contour Γ.

As a set of nodes on which the B-spline functions of order m $(m \leq n_B)$ are constructed (see formula (1)), we consider the set $\left\{t_j^B\right\}_{j=1}^{n_B+m}$, where $t_j^B = t_j$, $j = 1, ..., n_B$, and $t_{n_B+1}^B = t_1^B, t_{n_B+2}^B = t_2^B, ..., t_{n_B+m}^B = t_m^B$.

We construct the approximation of the function $f(t)$ in the form $\varphi_{n_B}(t) := \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t)$, where the coefficients $\alpha_k \in \mathrm{C}$, $k = 1, ..., n_B$, are determined imposing the interpolation conditions

$$f\left(t_j^C\right) = \varphi_{n_B}\left(t_j^C\right), \ \ j = 1, ..., n_B. \tag{2}$$

The set of nodes of the B-spline, arranged in a certain order, is considered as interpolation points t_j^C.

The system of equations (2) can be written as $B\bar{x} = \bar{f}$, where

$$B = \{m_{j,k}\}_{j,k=1}^{n_B}, \ m_{j,k} = B_{m,k}\left(t_j^C\right), \ \bar{x} = \{\alpha_k\}_{k=1}^{n_B}, \ \bar{f} = \left\{f\left(t_j^C\right)\right\}_{j=1}^{n_B}.$$

To approximate the function $f(t)$ we use the B-spline functions of order $m \in \{2, 3, 4\}$. Based on formula (1) one can deduce the following explicit representations for the B-splines $B_{m,k}(t)$ $(k = 1, ..., n_B)$:

For $m = 2$:

$$B_{2,k}(t) = \begin{cases} \dfrac{2\left(t - t_k^B\right)}{\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)} & \text{if } t \in arc\left[t_k^B, t_{k+1}^B\right) \\ \dfrac{2\left(t_{k+2}^B - t\right)}{\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)} & \text{if } t \in arc\left[t_{k+1}^B, t_{k+2}^B\right) \\ 0 & \text{otherwise} \end{cases}$$

For $m = 3$:

$$B_{3,k}(t) = \begin{cases} \dfrac{3\left(t - t_k^B\right)^2}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)} & \text{if } t \in arc\left[t_k^B, t_{k+1}^B\right) \\ 3\left(I_1 + I_2\right) \quad \text{if } t \in arc\left[t_{k+1}^B, t_{k+2}^B\right) \\ \dfrac{3\left(t_{k+3}^B - t\right)^2}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+2}^B\right)} & \text{if } t \in arc\left[t_{k+2}^B, t_{k+3}^B\right) \\ 0 & \text{otherwise} \end{cases}$$

where

$$I_1 := \frac{\left(t - t_k^B\right)\left(t_{k+2}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_2 := \frac{\left(t - t_{k+1}^B\right)\left(t_{k+3}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)}.$$

For $m = 4$:

$$B_{4,k}(t) = \begin{cases} \dfrac{4\left(t - t_k^B\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+1}^B - t_k^B\right)} & \text{if } t \in arc\left[t_k^B, t_{k+1}^B\right) \\ 4\left(I_3 + I_4\right) \quad \text{if } t \in arc\left[t_{k+1}^B, t_{k+2}^B\right) \\ 4\left(I_5 + I_6\right) \quad \text{if } t \in arc\left[t_{k+2}^B, t_{k+3}^B\right) \\ \dfrac{4\left(t_{k+4}^B - t\right)^3}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+4}^B - t_{k+2}^B\right)\left(t_{k+4}^B - t_{k+3}^B\right)} & \text{if } t \in arc\left[t_{k+3}^B, t_{k+4}^B\right) \\ 0 & \text{otherwise} \end{cases}$$

where

$$I_3 := \frac{t - t_k^B}{t_{k+4}^B - t_k^B}\left(I_{3,1} + \frac{\left(t - t_{k+1}^B\right)\left(t_{k+3}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)}\right),$$

$$I_{3,1} := \frac{\left(t - t_k^B\right)\left(t_{k+2}^B - t\right)}{\left(t_{k+3}^B - t_k^B\right)\left(t_{k+2}^B - t_k^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_4 := \frac{\left(t_{k+4}^B - t\right)\left(t - t_{k+1}^B\right)^2}{\left(t_{k+4}^B - t_k^B\right)\left(t_{k+4}^B - t_{k+1}^B\right)\left(t_{k+3}^B - t_{k+1}^B\right)\left(t_{k+2}^B - t_{k+1}^B\right)},$$

$$I_5 := \frac{\left(t^B_{k+3}-t\right)^2\left(t-t^B_k\right)}{\left(t^B_{k+4}-t^B_k\right)\left(t^B_{k+3}-t^B_k\right)\left(t^B_{k+3}-t^B_{k+1}\right)\left(t^B_{k+3}-t^B_{k+2}\right)},$$

$$I_6 := \frac{t^B_{k+4}-t}{t^B_{k+4}-t^B_k}\left(I_{6,1}+\frac{\left(t-t^B_{k+2}\right)\left(t^B_{k+4}-t\right)}{\left(t^B_{k+4}-t^B_{k+1}\right)\left(t^B_{k+4}-t^B_{k+2}\right)\left(t^B_{k+3}-t^B_{k+2}\right)}\right),$$

$$I_{6,1} := \frac{\left(t-t^B_{k+1}\right)\left(t^B_{k+3}-t\right)}{\left(t^B_{k+4}-t^B_{k+1}\right)\left(t^B_{k+3}-t^B_{k+1}\right)\left(t^B_{k+3}-t^B_{k+2}\right)}.$$

It can be seen that the B-spline functions $B_{m,k}(t)$ have the support on the curve arc $\left[t^B_k, t^B_{k+m}\right)$. This leads to a sparse matrix $B = \left\{B_{m,k}\left(t^C_j\right)\right\}_{j,k=1}^{n_B}$ in the system of equations (2). On the one hand, it can be considered as an advantage because small computational resources can be involved when calculating the solution to the system (2). On the other hand, it is possible that the determinant of the matrix B to be equal to zero.

The location of the interpolation points t^C_j on the contour Γ has a direct influence on the conditioning of the matrix $B = \{m_{j,k}\}_{j,k=1}^{n_B}$ in the system (2). In order to ensure the good conditioning of the matrix B, it is proposed the interpolation points t^C_j to be selected as follows.

For $m = 2$ we consider $t^C_j = t^B_{j+1},\ j = 1, ..., n_B$, and in this case the matrix B has a diagonal structure with non-zero elements on the main diagonal, that means most often in practice that it is a well-conditioned matrix.

For $m = 3$ and $m = 4$ we consider $t^C_j = t^B_{j+2},\ j = 1, ..., n_B$, and in this case, for $m = 3$, the matrix B has a bidiagonal structure, and for $m = 4$ it has a tridiagonal structure. Matrix B has non-zero diagonal and codiagonal elements and, as a rule, it is well conditioned.

After determining the solution $\alpha_k \in$ C, $k = 1, ..., n_B$ to the system (2), we construct the approximation $\varphi_{n_B}(t) := \sum_{k=1}^{n_B}\alpha_k B_{m,k}(t)$ of the function $f(t)$ and calculate its values at points $t \in \Gamma$. In the presented approximation algorithm there are two problems:

1. The graph of the function $\varphi_{n_B}(t)$ passes through the origin of coordinates, even if $f(t) \neq 0,\ \forall t \in \Gamma$. To overcome this problem, we proceed as follows. If $f(t_0) \neq 0$, where $t_0 = \psi\left(e^{i\theta_0}\right),\ \theta_0 = 0$, then from the table with generated values of the approximation $\varphi_{n_B}(t)$ (calculated for the parameter $\theta \in [0, 2\pi)$, starting with $\theta_0 = 0$), we eliminate the first values $\varphi_{n_B}\left(\tilde{t}\right)$ for which $\left|\varphi_{n_B}\left(\tilde{t}\right) - f(t_0)\right| \geq \varepsilon_1$, where ε_1 is a small value, for example, $\varepsilon_1 = 0.01$.

2. The approximation curve $\varphi_{n_B}(t)$ is continuous, being generated as a linear combination of continuous B-spline functions. Therefore, at the points of discontinuity of the function $f(t)$, we have no "breaks" of the graph of the function $\varphi_{n_B}(t)$, but continuous connections of its values. Thus, often the graph of the function $\varphi_{n_B}(t)$ has a distorted aspect compared to the graph of the approximated function $f(t)$. Next, we present an algorithm that allows to overcome the mentioned difficulty.

3 Approximation of function through a linear combination of B-spline and Heaviside functions

We admit that the values of the function f are known at the discontinuity points t_r^d, $r = 1, ..., npd$, on the contour Γ. For the function f, defined numerically, in [4] and [5] several algorithms have been proposed for establishing the locations of the discontinuity points on Γ.

We construct the approximation φ_{n_B} in the form

$$\varphi_{n_B}^H(t) := \sum_{k=1}^{n_B} \alpha_k B_{m,k}(t) + \sum_{r=1}^{npd} \beta_r H\left(t - t_r^d\right), \tag{3}$$

where H is the Heaviside function on the contour Γ, defined as follows

$$H\left(t - t_r^d\right) := \begin{cases} 0 & \text{if } t \in \Gamma_1 \cup ... \cup \Gamma_{s-1} \cup arc\left[t_s^B, t_r^d\right) \\ 1 & \text{if } t \in arc\left[t_r^d, t_{s+1}^B\right) \cup \Gamma_{s+1} \cup ... \cup \Gamma_{n_B} \end{cases},$$

$$\Gamma_s = arc\left[t_s^B, t_{s+1}^B\right], \ t_r^d \in \Gamma_s.$$

We determine the coefficients $\alpha_k, \ k = 1, ..., n_B$, and $\beta_r, \ r = 1, ..., npd$, from the interpolation conditions

$$f\left(t_j^C\right) = \varphi_{n_B}^H\left(t_j^C\right), \ j = 1, ..., n,$$

where $n := n_B + npd$, and the interpolation points $t_j^C, \ j = 1, ..., n$, are chosen as follows:

- the first n_B points $t_j^C, \ j = 1, ..., n_B$, are identical to those used to determine the solution to the system (2);
- the remaining npd points are considered as discontinuity points of the function f.

If among the points $t_j^C, \ j = 1, ..., n_B$, there are points of discontinuity $t_j^d = \psi\left(e^{i\theta_j^d}\right)$ of the function f on Γ, then instead of them we consider the points $\tilde{t}_j^d = \psi\left(e^{i\left(\theta_j^d - \varepsilon_2\right)}\right)$, where $\varepsilon_2 > 0$ is a small value, for example, $\varepsilon_2 = 0.01$. Since the function is left continuous, for a sufficiently small ε_2, it can be considered that the value of the function f at point $\tilde{t}_j^d$ coincides with its value at point t_j^d.

The term $\sum_{k=1}^{n_B} \alpha_k B_{m,k}(t)$ in relation (3) defines a continuous function on Γ, that approximates the aspect of the pieces of the graph of the function f corresponding to the arcs of the contour Γ between the points of discontinuity. The coefficients $\beta_r, \ r = 1, ..., npd$, define the "jumps" of the pieces of the graph at the discontinuity points t_r^d, so that each term $\beta_r H\left(t - t_r^d\right)$ determines the displacement of the piece of the graph $\sum_{k=1}^{n_B} \alpha_k B_{m,k}(t)$, corresponding to the points of Γ, which are located after the discontinuity point t_r^d when traversing the contour Γ in a positive direction.

4 Numerical example

Consider the Riemann function $z = \psi(w)$ that performs the conformal transformation of the set $\{w \in \mathrm{C} : \ |w| > 1\}$ on the domain Ω^- from the outside of the contour Γ as $\psi(w) = w + 1/\left(3w^3\right)$. Thus, $\psi(w)$ transforms the unit circle Γ_0 onto the astroid Γ (see Figure 2).

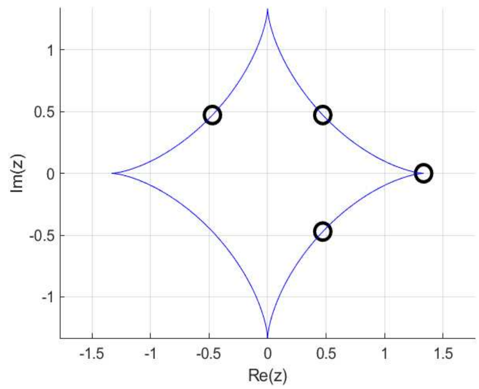

Figure 2: The contour and discontinuity points

For testing purposes, we consider the function of a complex variable f given analytically on Γ:

$$f(t) = \begin{cases} t^3 \ \text{ if } \ \theta \in (0, \zeta_1] \\ -\cos(t) \ \text{ if } \ \theta \in (\zeta_1, \zeta_2] \\ t^2 e^t \ \text{ if } \ \theta \in (\zeta_2, \zeta_3] \\ t^2 Re(2t) \ \text{ if } \ \theta \in (\zeta_3, \zeta_4] \\ t^2 Re(2t) \ \text{ if } \ \theta = 0 \end{cases} ,$$

where $\zeta_1 = \pi/4, \ \zeta_2 = 3\pi/4, \ \zeta_3 = 7\pi/4, \ \zeta_4 = 2\pi$. The function f has $npd = 4$ jump discontinuity points on the contour Γ corresponding to the points $t_j^d = \psi\left(e^{i\zeta_j}\right)$, $j = 1, ..., 4$ (see Figure 2 and Figure 3).

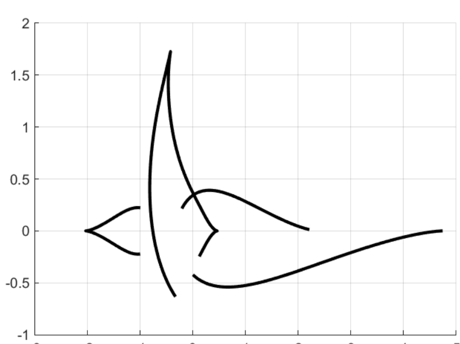

Figure 3: Graph of the function

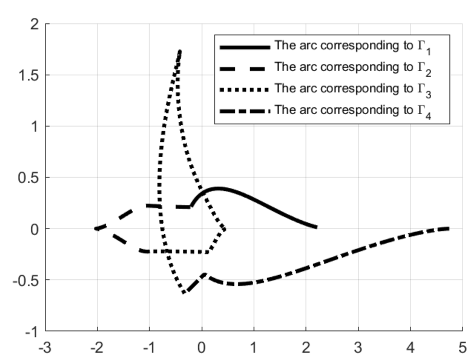

Figure 4: Combination of B-splines

The approximation algorithm takes as initial data the values f_j of the function f at the points

$$t_j = \psi\left(e^{i\theta_j}\right) \in \Gamma,\ \theta_j = 2\pi\left(j-1\right)/n_B,\ n_B \in \mathrm{N},\ k = 1, ..., n_B.$$

Let the number of points where the values of the function f on Γ are given be $n_B = 320$. Considering the approximation by linear combination of the form (3), where B-spline functions of order $m = 4$ are involved, we determine the solution to the system of equations $B\bar{x} = \bar{g}$, where $\bar{x} = \left(\alpha_1, ..., \alpha_{n_B}, \beta_1, ..., \beta_{npd}\right)^T$, $\bar{g} = \left(f\left(t_1^c\right), ..., f\left(t_n^c\right)\right)^T$, $n = n_B + npd$, and

$$B = \begin{pmatrix} B_{m,1}\left(t_1^c\right) & \cdots & B_{m,n_B}\left(t_1^c\right) & H\left(t_1^c - t_1^d\right) & \cdots & H\left(t_1^c - t_{npd}^d\right) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ B_{m,1}\left(t_n^c\right) & \cdots & B_{m,n_B}\left(t_n^c\right) & H\left(t_n^c - t_1^d\right) & \cdots & H\left(t_n^c - t_{npd}^d\right) \end{pmatrix}.$$

The coefficients $\alpha_1, ..., \alpha_{n_B}$ specify the linear combination of B-splines (see the graph in Figure 4), and the coefficients

$$\beta_1 = -0.7744 + 0.0439i,\ \beta_2 = 1.1119 - 0.0126i,$$

$$\beta_3 = 0.2962 + 0.2465i,\ \beta_4 = -2.2529 - 0.0033i,$$

establish approximations of displacements of the pieces of the graph $\sum\limits_{k=1}^{n_B} \alpha_k B_{m,k}\left(t\right)$, corresponding to the arcs between the discontinuity points (compare the data in Figure 2 and Figure 3).

For values $n_B = 160$ and $n_B = 320$ in Figure 5 and Figure 6 the error obtained at the approximation of the function f by $\varphi_{n_B}^H$ is presented. It can be seen that the maximum error decreases significantly for $n_B = 320$.

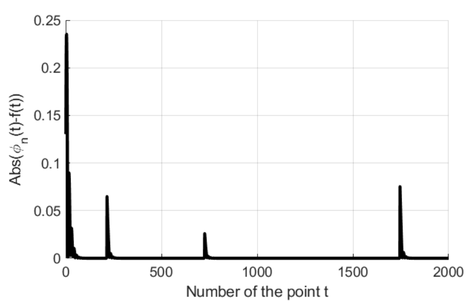

Figure 5: The approximation error for nB=160

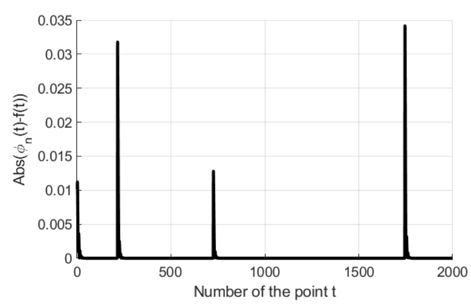

Figure 6: The approximation error for nB=320

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